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ENLARGEMENTS WITHOUT URELEMENTS

BY

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Introduction. This paper was written while the first author was visiting at the University of Houston. Originally planned as a higher order addition to the second author's algebraic investigations of first-order enlargements [14], a separation became necessary because of the much more set-theoretic character of the present investigations.

Let A by any set, $\hat{A}$ the superstructure over A introduced by Robinson and Zakon [12]. In order to construct, within a superstructure $\hat{B}$ over some set B, a higher-order enlargement of $\hat{A}$ in the sense of Robinson [11], one usually assumes that A and B consist of urelements (individuals that are not sets).

"Usually" is stated quite euphemistically here. As a matter of fact, it is hard to find exceptions to that assumption. Points made in favor of urelements are 1. convenience and 2. naturality. While we admit the first, we tend to question the second.

Indeed, where else in mathematics do we find extensions of structures whose construction relies so heavily on that particular set-theoretic assumption? Moreover, what really is that assumption? We are advised that, in order to apply non-standard methods to various structures, we should be able to find for any carrier set A an equivalent set A' consisting of urelements. Shall we understand, then, that for every cardinal number we work in a set theory with just that many urelements, so tayloring as many set theories as needed (as there are cardinal numbers)? Or, preferring to work in one set theory (as many mathematicians would like to), do we admit one in which the number of urelements exceeds each cardinal number, in which, indeed, the urelements form a proper class? Claiming such an unrealistic axiom for the sole purpose of non-standard methods does not appear very "natural".

As far as convenience is concerned, we are advised, over and over again, that there are, of course, several other ways as to get along without urelements. They must be less "convenient", however; for nowhere are these ominous hints substantiated, with the only exception of Robinson and Zakon [12]. Their remarks indicate, indeed, that they had given deeper thought to the matter. Actually, they must have come close to some of our findings (cf. the end of §4).

What now makes urelements convenient? The answer is as simple as that: Each set of urelements is disjoint with each set of sets. In particular, $A \cap A_{n+1} = \emptyset$, where $A_{n+1} = \boldsymbol{P}(A \cup A_n)$ and $A_0 = A$. In Section 2 of this paper, we show that there are, even in a set theory without urelements, arbitrarily large sets with this property, even with the property $A \cap A_\alpha = \emptyset$, for any prescribed ordinal α. Such sets are, in fact, constructed as certain "strong antichains". The latter represent our way to get along without urelements; we would be happy to learn about "several" others. Indeed, no claim is made here that there are no other sets with such disjointness properties, the latter being the only properties, indeed, needed for a set-theoretic construction of higer-order ultrapowers.

Frequently, no clear line is drawn between first-order ultrapowers of higher-order structures (cf. the beginning of §4) and genuine higher-order ultrapowers (§5). In Section 3, we summarize the characteristic features of a first-order ultrapower extension of any set A. The latter is described here as a set E together with an embedding i: $A \to E$ and projections p_t: $E \to A$ $(t \in T)$, where T is the index-domain, i.e., the carrier set of the given ultrafilter U. The whole setting is supposed to satisfy three conditions (axioms) UP1–UP3 (whose flexibility is really put on a test on this paper). One introduces set mappings $_{*k}$: $\boldsymbol{P}(A^k) \to \boldsymbol{P}(E^k)$ $(k \geqslant 1)$ satisfying the first-order part of Robinson's and Zakon's axioms [12] for a higher-order elementary extension. The latter are reinterpreted here in terms of arbitrary transformations of variables, not only permutations.

Applying this in Section 4 to a superstructure $\hat{A}$ over an arbitrary set A, one faces the well-known phenomenon that the relation $\in_A = \{\langle Y, X\rangle \in \hat{A}^2;\ Y \in X \notin A\}$ passes into some exotic relation $\varrho = {}_{*2}\in_A$ in some subset $Z \subset E$ well-founded by ϱ. It is essentially well-known how to pass from ("collapse") the relation ϱ into the relation $\in_B$ for some suitable set B. Usually, of course, B (like A) is assumed to consist of urelements, hence it is crucial here to show that this assumption is not needed. Usually, the "collapsing" procedure is carried out in a – sometimes clumsy – ad hoc fashion (not always free of serious mistakes). We present in Section 1 a unified generalization of the two isomorphism theorems due to Zermelo [19] and Mostowski [10] (the former much less well-known than the latter). Not only are all generalizations we found in the literature special cases of our result (Theorem 1.2); our result is not improvable insofar as we give three necessary and sufficient conditions for the Zermelo–Mostowski function to become an isomorphism. All we have to do then in Section 4 is to show these three conditions to be satisfied. Two being satisfied anyway, the third one is obtained from the disjointness condition $B \cap B_\omega = \emptyset$. Remarkably enough, we need no condition on A whatsoever.

Still no conditions on A are needed yet for a new setup of first-order type (§5): One gets a new injection $j = f \circ i$: $\hat{A} \to f[Z]$ and new projections $q_t = p_t \circ f^{-1}$: $f[Z] \to \hat{A}$. However, this is no longer a full-fledged first-order ultrapower extension, and it is here that the distinction between lower order and higher order becomes very clear: While the axioms UP2 and UP3, with all their nice consequences, still hold, the existence axiom UP1 survives "collapsing" only in a bounded version (Proposition 5.1). This is the price we have to pay for replacing the exotic relation ϱ by the genuine $\in$-relation. Indeed, again introducing higher-order set mappings $*^k$: $\boldsymbol{P}(\hat{A}^k) \to \boldsymbol{P}(f[Z]^k)$ $(k \geq 1)$, in the very same fashion as the lower-order set mappings $_{*k}$: $\boldsymbol{P}(\hat{A}^k) \to \boldsymbol{P}(E^k)$ before, we now have, in particular, ${*^2}{\in_A} = {\in_B} \cap f[Z]^2$. This is an unbounded version of Robinson's and Zakon's only truly higher-order axiom [12], 2.5. Note that almost all authors introduce only $*^1$: $\hat{A} \to f[Z]$. Davis [4] extended this to the set of all definable subsets of $\hat{A}$, so relying heavily on the formal language in the very definition of $*^1$. (We banned the formal language completely from the paper, hoping to give a short algebraic proof of the higher-order ultrapower theorem elsewhere.)

It is in comparing the various mappings so obtained that conditions on A come in with necessity (§§5, 6), namely $A \cap A_\omega = \emptyset$, even $A \cap A_{\omega+1} = \emptyset$. In fact, if the latter holds, all the mappings considered coincide on the intersection of their domains (Theorem 5.3, Proposition 6.3, Theorem 6.6). In particular, $*^k$ is, in fact, the restriction of $*^1$ to $\boldsymbol{P}(\hat{A}^k) \subset \boldsymbol{P}(\hat{A})$, entitling us to write simply $*$, and the restriction of the latter to $\hat{A} \cap \boldsymbol{P}(\hat{A}) = A_\omega$ coincides with the restriction to A_ω of j.

1. A general Zermelo–Mostowski theorem. Let $\langle A, \varrho \rangle$ be a well-founded class. I.e., ϱ is a binary relation in the class A, and for every non-empty subclass $M \subset A$, its *bottom* (class of minimal elements) $M \setminus \varrho[M]$ is non-empty. It suffices to postulate this for every non-empty subset $M \subset A$ once $\langle A, \varrho \rangle$ is *locally small*, i.e., for each element $x \in A$, $\varrho^{-1}[\{x\}] = \{y \in A;\, y \varrho x\}$ is a set.

If the class A is well-founded by ϱ, then it is also well-founded by the transitive closure $<$ of ϱ; in particular, $<$ is irreflexive, $x \nless x$, hence the irreflexive version of a partial ordering $\leq$ of A. The following observation will be used later.

LEMMA 1.1. *Let $\langle A, \varrho \rangle$ be a well-founded class and $x \in M \subset A$. Then there is an element $y \in M \setminus \varrho[M]$ such that $y \leq x$.*

In the sequel, $A_0 = A \setminus \varrho[A]$ denotes the bottom of $\langle A, \varrho \rangle$. For x and

x' in A, we define $x \equiv x'$ iff $x = x' \in A_0$ or $x, x' \notin A_0$ and $\varrho^{-1}[\{x\}] = \varrho^{-1}[\{x'\}]$ ($y\varrho x$ iff $y\varrho x'$). This is an equivalence relation in A; in general, it is not a congruence of $\langle A, \varrho\rangle$, not even a left congruence: It may happen that $y \equiv y' \varrho x$, but not $y\varrho x$.

Let now f_0 be any function from the bottom A_0 into an arbitrary class B. Assuming $\langle A, \varrho\rangle$ well-founded and locally small, we define the *Zermelo–Mostowski extension f of f_0*, with domain A, recursively as follows:

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in A_0, \\ \{f(y);\ y\varrho x\} = f[\varrho^{-1}[\{x\}]] & \text{if } x \notin A_0. \end{cases} \tag{1.1}$$

This function has the following properties:

$$f[A] \subset B \cup \boldsymbol{P}(f[A]), \tag{1.2}$$

i.e., its range (wherever located) is *transitive over the class B*. Moreover, f is a homomorphism from $\langle A, \varrho\rangle$ into the universe V under the $\in$-relation:

$$\text{whenever } y\varrho x, \text{ then } f(y) \in f(x), \tag{1.3}$$

for each $x, y \in A$. Finally,

$$\text{whenever } x \equiv x', \text{ then } f(x) = f(x'), \tag{1.4}$$

for each $x, x' \in A$. Here is the main result of this section:

THEOREM 1.2. *Let $\langle A, \varrho\rangle$ be a locally small well-founded class. Let f_0 be any mapping from the bottom A_0 into some class B, and let f be its Zermelo–Mostowski extension. Then the two conditions*

(i) *$y\varrho x$ iff $f(y) \in f(x) \notin B$, for any $x, y \in A$;*

(ii) *$x \equiv x'$ iff $f(x) = f(x')$, for any $x, x' \in A$;*

are equivalent with the following three conditions:

(iii) *$\equiv$ is a congruence (it suffices, a left congruence) of $\langle A, \varrho\rangle$;*

(iv) *f_0 is one-to-one;*

(v) *$f(x) \in B$ iff $x \in A_0$, for any $x \in A$.*

Proof. (i) implies (iii) and (v), (ii) implies (iv) (iff f_0 is onto B, (ii) also implies (v)).

Conversely, (iii), (iv) and (v) imply (ii). (Davis [4], p. 17, tries, but fails, to prove (i) first.) One shows, by induction on $x \in A$, that $f(x) = f(x')$ implies $x \equiv x'$. If $x \in A_0$, this follows from (iv) and (v). Suppose $x \notin A_0$, and that $f(y) = f(y')$ implies $y \equiv y'$, for each $y\varrho x$. Again by (v), $f(x) = f(x')$ implies $x' \notin A_0$, whence $\{f(y);\ y\varrho x\} = \{f(y');\ y'\varrho x'\}$. By induction hypothesis and the weak version of (iii), $\{y;\ y\varrho x\} = \{y';\ y'\varrho x'\}$, i.e., $x \equiv x'$.

(ii), (iii) and (v) imply (i): If $y\varrho x$, then $f(y) \in f(x) \notin B$ because of (v). Conversely, suppose $f(y) \in f(x) \notin B$. By (v), $x \notin A_0$, whence $f(y) = f(y')$ for some $y'\varrho x$. By (ii), $y \equiv y'$, whence $y\varrho x$ by the weak version of (iii).

Condition (iii) is satisfied if the equivalence $\equiv$ is, in fact, the identity relation i.e., if $\langle A, \varrho\rangle$ is *extensional* inasmuch as it satisfies the axiom of extensionality for a set theory with urelements. In this case, (ii) makes f one-to-one. It is fashionable to call f the "collapsing function" and the whole procedure "Mostowski collapsing". Insofar as "collapsing" might suggest the identification of equivalent elements $x \equiv x'$, this term seems ill-advised in the case above. In that case, (i) makes f an isomorphism between $\langle A, \varrho \rangle$ and a relational system whose binary relation is a restriction of $\in$. The first to prove such an isomorphism theorem was Zermelo 1930 (cf. [19], also [20]): Assuming the axiom of foundation (which was really not necessary because of Proposition 1.5 below), he found the "Mengenbereiche" (cf. below) over two equivalent sets A_0 and $B_0 = B$ of urelements isomorphic under their $\in$-relations. Mostowski 1949 (cf. [10]) was the first to consider an abstract binary relation ϱ. However, he assumed $\langle A, \varrho \rangle$ *strongly extensional* in the sense that $\varrho^{-1}[\{x\}] = \varrho^{-1}[\{x'\}]$ implies $x = x'$ for every $x, x' \in A$, even for minimal elements. Note that $\langle A, \varrho \rangle$ is strongly extensional iff it is extensional and has at most one minimal element. In the well-founded case then, we have exactly one minimal element a_0, which we map onto $\emptyset$. Indeed, $B = \{\emptyset\}$ was what Mostowski took.

The discussion would be incomplete without the following uniqueness result:

THEOREM 1.3. *Let $\langle A, \varrho \rangle$ be a locally small well-founded class. Let f_0 be a mapping from the bottom A_0 into some class B. Let g be an extension of f_0 with domain A and range trasitive over B. Suppose g satisfies* (i) *of Theorem* 1.2. *Then g is the Zermelo–Mostowski extension f of f_0.*

Proof. We show $f(x) = g(x)$ by induction on x. This being trivial for $x \in A_0$, suppose $x \notin A_0$. By induction hypothesis, $f(y) = g(y)$ for each $y \varrho x$. For each such y, we have $f(y) = g(y) \in g(x)$ by (i), whence $f(x) = \{f(y); y \varrho x\} \subset g(x)$. Since g satisfies (i), it satisfies (v): Thus $g(x) \notin B$. On the other hand, $g[A]$ is transitive over B, whence $g(x) \in g[A]$. Hence if $v \in g(x)$, then $v = g(y)$ for some $y \in A$. Since $g(y) \in g(x) \notin B$, $y \varrho x$ by (i). Hence $v = g(y) = f(y) \in f(x)$ and $g(x) \subset f(x)$. Note that in this case (iii) of Theorem 1.2 holds, and (ii) becomes equivalent with (iv).

We neglected the question where $f[A]$ is located. Let U be any class. Its power class $\boldsymbol{P}(U)$ is the class of all subsets of U. In case $\boldsymbol{P}(U) \subset U$, U is called *inductive* (cf. [13]). Such a class is, of course, not a set. Given any class B, there exists the least inductive class containing B, its *inductive closure* $\operatorname{ind} B$. A simple way to construct $\operatorname{ind} B$ is as follows: We define B^α, for each ordinal α, recursively by

$$B^\alpha = B \cup \bigcup_{\beta<\alpha} \boldsymbol{P}(B^\beta). \tag{1.5}$$

In particular,

$$\begin{cases} B^0 = B, \\ B^{\alpha+1} = B \cup \boldsymbol{P}(b^\alpha), \\ B^\alpha = \bigcup_{\beta<\alpha} B^\beta \quad \text{if } \alpha \text{ is a limit number.} \end{cases} \tag{1.6}$$

$$\operatorname{ind} B = B^{\Omega} = \bigcup_{\alpha} B^{\alpha} \tag{1.7}$$

where α ranges over the class of all ordinals (which we denote by Ω). This is, indeed, the "Mengenbereich" (domain of sets) Zermelo [19] erected over a set of urelements B. For an alternative construction, much more in the spirit of the present paper, we define

$$B_0 = B, \quad B_\alpha = \bigcup_{\beta<\alpha} P(B \cup B_\beta) \quad (\alpha \geqslant 1). \tag{1.8}$$

Again,

$$\begin{cases} B_{\alpha+1} = P(B \cup B_\alpha), \\ B_\alpha \;\; = \bigcup_{1 \leqslant \beta < \alpha} B_\beta \quad \text{if } \alpha \text{ is a limit number.} \end{cases} \tag{1.9}$$

We put

$$B_\Omega = \bigcup_{\alpha \geqslant 1} B_\alpha. \tag{1.10}$$

In case B is a set of urelements, B_Ω has been used by Barwise [1] (cf. also [2], p. 45). Note that

$$B^\alpha = B \cup B_\alpha \quad \text{for each ordinal } \alpha, \tag{1.11}$$

so that

$$B_{\alpha+1} = P(B^\alpha), \tag{1.12}$$

also

$$\operatorname{ind} B = B^\Omega = B \cup B_\Omega. \tag{1.13}$$

By definition, B^Ω is the least class U such that $B \cup P(U) \subset U$. On the other hand, each B^α, and with that B^Ω, is transitive over B, $B^\alpha \subset B \cup P(B^\alpha)$, $B^\Omega \subset B \cup P(B^\Omega)$. (The frequently used terminology "transitive in B^Ω" would be absolutely misleading here: $B^\Omega = C^\Omega$ does not imply $B = C$; in fact, B^Ω might be the entire universe.) Hence B^Ω is the least class U such that $B \cup P(U) = U$. Correspondingly, B_Ω is the least class U such that $P(B \cup U) = U$.

Trivially, each class U such that $P(B \cup U) = U$ is an ideal in the Boolean Algebra of classes: $\emptyset \in U$; if X, $Y \in U$, then $X \cup Y \in U$; if $X \in U$ and $Y \subset X$, then $Y \in U$. If $P(B \cup U) = U$, U is also P-closed: That is, $X \in U$ implies $P(X) \in U$. Finally, such a class U satisfies also a strong closure under union:

$$\text{if } X \in P(U), \text{ then } \bigcup X \in U. \tag{1.14}$$

Certainly, B_Ω has these properties. We state without proof:

Proposition 1.4. *If B is a set, then B_Ω is the least P-closed ideal U satisfying* (1.14) *such that $B \in U$.*

PROPOSITION 1.5. *Each B^α ($\alpha \geqslant 1$ or $\alpha = \Omega$) is transitive over B and well-founded by the (restriction of the) relation*

$$\in_B = \{\langle Y, X\rangle;\ Y \in X \notin B\}. \tag{1.15}$$

The bottom of B^α (under this relation) is $B \cup \{\emptyset\}$. $B^\Omega = \operatorname{ind} B$ is the greatest class transitive over B and well-founded by $\in_B$.

COROLLARY 1.6. *The inductive closure* $\operatorname{ind} B$ *is the only inductive class $U \supset B$ which is transitive over B and well-founded by $\in_B$.*

PROPOSITION 1.7. *If f is the Zermelo–Mostowski extension of f_0: $A_0 \to B$, then $f[A] \subset \operatorname{ind} B = B^\Omega$. In case condition* (v) *of Theorem* 1.2 *holds, $f[A \setminus A_0] \subset B_\Omega$.*

One proves $f(x) \in \operatorname{ind} B$ by induction on x. Assuming (v), we get $f[A \setminus A_0] \subset B^\Omega \setminus B \subset B_\Omega$.

2. Superstructures, strong antichains. In this section, we are looking at

$$\hat{B} - B^\omega = B \cup B_\omega, \tag{2.1}$$

introduced by Robinson and Zakon [12] as the *superstructure over B*. Robinson and Zakon did, indeed, consider the classes B_n ($n < \omega$) and B_ω, recursively defined by (1.9), while others (cf. Stroyan and Luxemburg [16], Davis [4], Keisler [5]) consider only the classes B^n ($n < \omega$) and $\hat{B} - B^\omega$, recursively defined by (1.6).

Here is an analogue of Proposition 1.4:

PROPOSITION 2.1. *If B is a set, then B_ω is the least $\boldsymbol{P}$-closed ideal U such that $B \in U$.*

Proof. $B \in \boldsymbol{P}(B) = B_1 \subset B_\omega$. Being the union of a chain of ideals, B_ω is an ideal. If $X \in B_n$ ($n \geqslant 1$), then $\boldsymbol{P}(X) \in B_{n+1}$, whence B_ω is $\boldsymbol{P}$-closed. Let U be a $\boldsymbol{P}$-closed ideal such that $B \in U$. By induction on n, $B_n \in U$ for each finite $n \geqslant 0$. Let $X \in B_{n+1}$, then $X \subset B \cup B_n$. Since U is an ideal, $X \in U$, so $B_{n+1} \subset U$ and $B_\omega \subset U$.

We observe here that $A \subset B$ implies $A_\alpha \subset B_\alpha$ and $A^\alpha \subset B^\alpha$ for each ordinal α and also for $\alpha = \Omega$. In particular, $\emptyset^\alpha = \emptyset_\alpha \subset B_\alpha$ for each ordinal α and $\alpha = \Omega$. So $\hat{\emptyset} = \emptyset^\omega = \emptyset_\omega \subset B_\omega$. $\hat{\emptyset}$, the least $\boldsymbol{P}$-closed ideal, is even the least Grothendieck universe U (if one does not require $\omega \in U$; cf., e.g., Kühnrich [6]). With the axiom of foundation (the nonexistence of Mirimanoff's extraordinary sets) and the nonexistence of urelements, the elements of $\hat{\emptyset}$ are exactly the hereditarily finite sets.

There is a nice description of $\hat{\emptyset}$ valid without foundation. For an arbitrary class X, one defines the *iterated power classes* and *unions* as follows:

$$\begin{cases} \boldsymbol{P}^0(X) = \bigcup^0 X = X; \\ \boldsymbol{P}^{n+1}(X) = \boldsymbol{P}(\boldsymbol{P}^n(X)), \quad \bigcup^{n+1} X = \bigcup \bigcup^n X. \end{cases} \tag{2.2}$$

So $Y \in \bigcup^n X$ iff there are sets $Z_1, \ldots, Z_n$ such that $Y \in Z_n \in Z_{n-1} \in \ldots$ $\ldots \in Z_1 \in X$. For classes X and Y, one has

$$\bigcup X \subset Y \quad \text{iff} \quad X \subset \boldsymbol{P}(Y). \tag{2.3}$$

I.e., the operators $\bigcup$ and $\boldsymbol{P}$ form an adjoint situation (Galois connection) in the Boolean Algebra of classes. This carries over to the iterated operators:

$$\bigcup^n X \subset Y \quad \text{iff} \quad X \subset \boldsymbol{P}^n(Y) \tag{2.4}$$

for each finite $n \geqslant 0$. Note that $\bigcup^n(\emptyset) = \emptyset^n = \emptyset_n$. With that, we get

PROPOSITION 2.2. *$\hat{\emptyset} = \emptyset^\omega = \emptyset_\omega$ is the set of all sets X such that $\bigcup^n X = \emptyset$ for some finite $n \geqslant 0$ (X is "nilpotent").*

Indeed, $X \in \emptyset_\omega$ iff $X \in \boldsymbol{P}^{n+1}(\emptyset)$ for some $n \geqslant 0$. But $X \in \boldsymbol{P}^{n+1}(\emptyset)$ iff $X \subset \boldsymbol{P}^n(\emptyset)$, which is equivalent to $\bigcup^n X \subset \emptyset$.

Skolem [15], 4:0, called sets with this property of first "Stufe" (grade). This description of $\emptyset_\omega$ can be extended to any B_ω. One iterates the operators $B \cup \boldsymbol{P}(X)$ and $\bigcup(X \setminus B)$. We leave the details to the reader.

An ($\in$-)*antichain* would be any class B such that $Y \notin X$ for each X, $Y \in B$. Or to put this way, $Y \in X \in B$ implies $Y \notin B$. Let $<$ denote the transitive closure of the $\in$-relation. B is a *strong antichain* if $Y < X \in B$ implies $Y \notin B$, for each X and Y. $B = \{\emptyset, \{1\}\} = \{\emptyset, \{\{\emptyset\}\}\}$ is a weak antichain, but not a strong one.

THEOREM 2.3. *Let B be a strong antichain. Then*

$$B \cap B_\alpha = B \cap \emptyset_\alpha \tag{2.5}$$

for each ordinal $\alpha \geqslant 1$ and $\alpha = \Omega$.

Proof. Because of (1.10), it suffices to prove this for any ordinal $\alpha \geqslant 1$. For the proof of the nontrivial inclusion, suppose there is a set $X \in B \cap B_\alpha$ such that $X \notin \emptyset_\alpha$. By (1.8), then, $X \subset B \cup B_\beta$ and $X \not\subset \emptyset_\beta$ for some ordinal $\beta < \alpha$. Hence there is a set $Y \in X$ such that $Y \notin \emptyset_\beta$. So $Y \in B \cup B_\beta$. Since $Y \in X \in B$ and B is a weak antichain, $Y \notin B$. So $Y \in B_\beta \setminus \emptyset_\beta$, and $Y < X$. Let γ be the least of the ordinals $\beta < \alpha$ such that $Y < X$ for some set $Y \in B_\beta \setminus \emptyset_\beta$, and let $Y \in B_\gamma \setminus \emptyset_\gamma$ be such that $Y < X$. Since $Y < X \in B$ and B is a strong antichain, $Y \notin B$, whence $\gamma \geqslant 1$. So again by (1.8), $Y \subset B \cup B_\delta$ and $Y \not\subset \emptyset_\delta$ for some ordinal $\delta < \gamma$. Hence there is a set $Z \in Y$ such that $Z \notin \emptyset_\delta$. So $Z \in B \cup B_\delta$. Since $Z \in Y < X \in B$ and B is a strong antichain, $Z \notin B$. So $Z \in B_\delta \setminus \emptyset_\delta$ and $Z < X$, contradicting the minimal property of γ.

Any set of urelements is, of course, a strong antichain. However, there are enough strong antichains in a set theory without urelements, which we will adopt henceforth.

We consider von Neumann's regular universe $\operatorname{ind} \emptyset = \emptyset^\Omega = \emptyset_\Omega$ (which is a Grothendieck universe if one admits universes which are not sets). Its αth layer is defined by

(2.6) $$L_\alpha = \emptyset^{\alpha+1} \setminus \emptyset^\alpha.$$

L_α is the bottom (under $\in$) of $M = \emptyset^\Omega \setminus \emptyset^\alpha$. Its elements are, by definition, the sets $X \in \emptyset^\Omega$ of rank α (α itself among them).

THEOREM 2.4. *L_α is a maximal strong antichain in* $\operatorname{ind}\emptyset$.

Proof. L_α is a strong antichain since $Y < X$ implies $\operatorname{rank} Y < \operatorname{rank} X$. Let now $X \in \operatorname{ind}\emptyset \setminus L_\alpha$. So $\beta = \operatorname{rank} X \neq \alpha$. Suppose $\beta > \alpha$. Then $X \in \operatorname{ind}\emptyset \setminus \emptyset^\alpha$. By Lemma 1.1, there is $Y \in L_\alpha$ such that $Y \leqslant X$. But $Y \neq X$, whence $Y < X$, showing that $L_\alpha \cup \{X\}$ is not a strong antichain. Suppose $\beta < \alpha$. Take any $Y \in L_\alpha$. Then $\operatorname{rank}(\{X\} \cup Y) = \max\{\beta+1, \alpha\} = \alpha$ and $X \in \{X\} \cup Y \in L_\alpha$. Again, $L_\alpha \cup \{X\}$ is not a strong antichain (not even a weak one).

However, not all strong antichains in $\operatorname{ind}\emptyset$ are *homogeneous*, that is, contained in some layer L_α. Here are the simplest counterexamples of lowest possible ranks: Both $\{\{1\}, 3\}$ and $\{2, \{\{1\}\}\}$ are strong antichains, yet $\operatorname{rank}\{1\} = \operatorname{rank} 2 = 2$, $\operatorname{rank} 3 = \operatorname{rank}\{\{1\}\} = 3$. Both examples are *anti-homogeneous* inasmuch as distinct elements are in distinct layers. Here is an impressive example of an anti-homogeneous strong antichain which even is a proper class:

THEOREM 2.5. $B = \{\{\alpha\};\ \alpha \in \Omega,\ \alpha > 0\}$ *is an anti-homogeneous strong antichain.*

Proof. Suppose $\{\alpha\} < \{\beta\}$ where α, $\beta \in \Omega$, α, $\beta > 0$. So we have $\{\alpha\} \leqslant X \in \{\beta\}$ for some set X, i.e., $\{\alpha\} \leqslant \beta$. Since Ω is transitive, $\{\alpha\} \in \Omega$, whence $\{\alpha\} = 1 = \{0\}$ and $\alpha = 0$, making B a strong antichain. For every ordinal α, $\operatorname{rank}\{\alpha\} = \alpha + 1$, making B anti-homogeneous.

THEOREM 2.6. *For every ordinal $\alpha \geqslant 1$, there are sets $B \subset \operatorname{ind}\emptyset$ of arbitrary cardinal number such that $B \cap B_\alpha = \emptyset$. We can choose them as strong antichains, either homogeneous or anti-homogeneous.*

Proof. By (2.6) $L_\beta \cap \emptyset_\alpha = \emptyset$ for each ordinal $\beta \geqslant \alpha$. Likewise, $B = \{\{\beta\};\ \beta \in \Omega,\ \beta \geqslant \alpha\}$ is disjoint with $\emptyset_\alpha$. The cardinal number of L_β becomes as large as one wants while β grows (L_ω has already the power of continuum). So any set is equivalent to some subset of L_β, for large enough β, also with some subset of B as defined in Theorem 2.5. In view of Theorems 2.3, 2.4, 2.5, the proof is complete.

3. Lower order ultrapowers. Let T be an infinite set (the *index-domain*) and U a nonprincipal *ultrafilter* on T (the *index-filter*). For any set $U \subset T$ such that $U \in \boldsymbol{U}$ we will say that $t \in U$ *for almost all* $t \in T$. If $U = \{t \in T;\ P(t)\} \in \boldsymbol{U}$, we will also say that *almost all $t \in T$ have the property $P(t)$* or that *$P(t)$ holds almost everywhere* (abbreviated a.e.). The usual ultrapower extension A^T/U can be conveniently described, up to unique isomorphism, as a set

E together with a mapping i: $A \to E$ and a family of mappings p_t: $E \to A$ $(t \in T)$ such that the following properties hold true:

UP1. For each choice of elements $a_t \in A$ $(t \in T)$, there is some $b \in E$ such that $p_t(b) = a_t$ a.e.;

UP2. for each b, $b' \in E$, if $p_t(b) = p_t(b')$ a.e., then $b = b'$;

UP3. for each $a \in A$, $p_t(i(a)) = a$ a.e.

Note that by UP3 and UP2 the mapping i is one-to-one. One could easily rearrange the situation to the effect that A becomes a subset of E and i the genuine inclusion mapping. For application to higher order ultrapowers (§§4, 5) however, it seems much clearer to keep the unspecified injection i, interrelated with the "projections" p_t by "axiom" UP3. As far as the latter are concerned, one gets them in an obvious way by choosing representatives from the equivalence classes modulo U of A^T. One so uses the full power of the axiom of choice, not only the ultrafilter (prime ideal) theorem guaranteeing the existence of U. However, use of the axiom of choice seems to be unavoidable later (§4) anyway, so we need not worry much here.

For any set X and integer $k \geqslant 1$, X^k will henceforth denote the kth Cartesian power of X (the classes X^α of §1 will no longer be used). By that we mean the set of ordered k-tuples $\langle x_1, \ldots, x_k \rangle$, where $x_1, \ldots, x_k \in X$, those k-tuples defined à la Kuratowski: $\langle x_1 \rangle = x_1$, $\langle x_1, x_2 \rangle = \{\{x_1\}, \{x_1, x_2\}\}$, and $\langle x_1, \ldots, x_{k-1}, x_k \rangle = \langle \langle x_1, \ldots, x_{k-1} \rangle, x_k \rangle$ $(k \geqslant 2)$. This definition has several disadvantages. E.g., an ordered k-tuple $(k > 2)$, being defined as an ordered pair, does certainly not determine its type ("length") k uniquely, nor does an ordered 1-tuple $\langle x_1 \rangle$ in case x_1 happens to be an ordered pair (maybe an ordered k-tuple, for some $k > 2$). An ordered k-tuple is a finite set if $k \geqslant 2$ (we will take advantage of that), whereas an ordered 1-tuple is any set, finite or infinite. The main disadvantage, however, is the lack of associativity. Indeed, $\langle , \rangle$ being a non-associative binary operation in the universe of all sets, we have to identify $\langle \langle x_1, x_2 \rangle, x_3 \rangle$ and $\langle x_1, \langle x_2, x_3 \rangle \rangle$ etc. by natural mappings (Robinson and Zakon [12]: *groupings*) g: $(X_1 \times X_2) \times X_3 \to X_1 \times (X_2 \times X_3)$ etc. There are ways out of this situation (cf. e.g., [13], [14]). They all amount to replacing $\langle x_1, \ldots, x_k \rangle$ by a "word" of exact "length" k in a (suitably constructed) free semigroup. This would fit perfectly into the following sections. However, all we would so gain here would be the vanishing of groupings (which would become true identities), at some price (in §§5, 6) which, though it is not too bad, we are not willing to pay here.

Besides the *point mapping* i: $A \to E$, one now introduces, for each $k \geqslant 1$, the *set mapping*

$$*_k\colon \boldsymbol{P}(A^k) \to \boldsymbol{P}(E^k)$$

by defining, for each k-ary relation $R \subset A^k$,

(3.1) $$_{*k}R = \{\langle b_1, \ldots, b_k\rangle \in E^k;\ \langle p_t(b_1), \ldots, p_t(b_k)\rangle \in R \text{ a.e.}\}.$$

It follows from this definition and the properties of an ultrafilter that $_{*k}$ is a Boolean homomorphism from $P(A^k)$ into $P(E^k)$. In particular,

(3.2) $$_{*k}A^k = E^k, \quad _{*k}\emptyset = \emptyset.$$

The following statement is a consequence of, even equivalent to, UP2:

(3.3) $$_{*2}\mathrm{id}_A = \mathrm{id}_E.$$

This is the lower analogue of Robinson's and Zakon's normality condition [12], 2.5′.

As a consequence of UP3, one has the following interrelation between the induced point mapping i^k: $A^k \to E^k$ and the set mapping $_{*k}$:

(3.4) $$(i^k)^{-1}\, _{*k}R = R,$$

for each k-ary relation $R \subset A^k$. I.e., i: $\langle A, R\rangle \to \langle E, {}_{*k}R\rangle$ is an embedding. Consequently,

(3.5) $$i^k[R] \subset {}_{*k}R.$$

Note that (3.5) implies UP3: Take $k = 1$ and $R = \{a\}$. Hence UP3, (3.4) and (3.5) are equivalent. As another consequence of (3.5), $_{*k}R = \emptyset$ implies $i^k[R] = \emptyset$ and $R = \emptyset$ making $_{*k}$: $P(A^k) \to P(E^k)$ a Boolean embedding (monomorphism). If R is finite, then equality takes place in (3.5):

(3.6) $$i^k[R] = {}_{*k}R \quad \text{if } R \text{ is finite.}$$

For $R = \{\langle a_1, \ldots, a_k\rangle\}$, this follows from UP3 and UP2. For $k = 1$ and $R = \{a\}$, this is, in fact, the lower order version of Robinson's and Zakon's axiom [12], 2.1.

Another of their axioms, [12], 2.3, concerns permutations of variables. We will even consider arbitrary *transformations* (*of variables*)

$$\sigma\colon \{1, \ldots, k\} \to \{1, \ldots, l\} \quad (k, l \geqslant 1).$$

For an arbitrary set X, one gets an *induced transformation* σ_X: $X^l \to X^k$, defined by

(3.7) $$\sigma_X(\langle x_1, \ldots, x_l\rangle) = \langle x_{\sigma(1)}, \ldots, x_{\sigma(k)}\rangle,$$

for each $x_1, \ldots, x_l \in X$. For every k-ary relation $R \subset X^k$, one gets an l-ary relation $\sigma^{-1}[R] \subset X^l$, defined by

(3.8) $$\sigma^{-1}[R] = \sigma_X^{-1}[R] = \{\langle x_1, \ldots, x_l\rangle;\ \sigma_X(\langle x_1, \ldots, x_l\rangle) \in R\}.$$

With that one has the *covariant transformation functor over* X. Similarly, one obtains the *contravariant transformation functor over* X: For every l-ary relation $S \subset X^l$, one defines a k-ary relation $\sigma[S] \subset X^k$ by

(3.9) $$\sigma[S] = \sigma_X[S] = \{\sigma_X(\langle x_1, \dots, x_l\rangle);\ \langle x_1, \dots, x_l\rangle \in S\}.$$

The second author [14] has shown that (the lower order version of) the axioms 2.1–2.4, 2.5′ of Robinson and Zakon [12] are equivalent to the following two conditions:

(3.10) $${}_{*l}\sigma_A^{-1}[R] = \sigma_E^{-1}[{}_{*k}R]$$

for each $R \subset A^k$, each transformation σ: $\{1, \dots, k\} \to \{1, \dots, l\}$, and

(3.11) $${}_{*k}\sigma_A[S] = \sigma_E[{}_{*l}S]$$

for each $S \subset A^l$, each transformation σ: $\{1, \dots, k\} \to \{1, \dots, l\}$. I.e., the mappings ${}_{*k}$ form a natural transformation between the covariant transformation functors and between the contravariant transformation functors over A and E, respectively. For a permutation σ of $\{1, \dots, k\}$ both (3.10) and (3.11) represent, in fact, the essential part of Robinson's and Zakon's axiom [12], 2.3 (the other part, concerning groupings, being similar to, but not covered by, (3.11)).

For the proof of (3.10), neither of the "axioms" UP1–UP3 is needed, and the same applies to one half (3.11): Its right-hand side is always contained in its left-hand side. The equation (3.11) holds at least for all onto-transformations σ: $\{1, \dots, k\} \to \{1, \dots, l\}$ $(k, l \geqslant 1)$ iff (3.3) holds (cf. [14]), which in turn was equivalent to UP2. In order to prove (3.11) for one-to-one transformations, it finally takes UP1.

(3.10) permits to generalize (3.3) to all *partial diagonals*

(3.12) $$D_{rs}^k(A) = \{\langle x_1, \dots, x_k\rangle;\ x_r = x_s\}$$

$(1 \leqslant r < s \leqslant k,\ k \geqslant 2)$. I.e.,

(3.13) $${}_{*k}D_{rs}^k(A) = D_{rs}^k(E).$$

The *cylindrifications*

(3.14) $$\exists_r^k(R) = \{\langle x_1, \dots, x_k\rangle \in A^k;\ \langle x_1, \dots, x_{r-1}, y, x_{r+1}, \dots, x_k\rangle \in R \text{ for some } y \in A\}$$

$(R \subset A^k,\ 1 \leqslant r \leqslant k,\ k \geqslant 1)$ are preserved,

(3.15) $${}_{*k}\exists_r^k(R) = \exists_r^k({}_{*k}R),$$

iff (3.11) holds for each one-to-one transformation σ: $\{1, \dots, k\} \to \{1, \dots, l\}$ $(k, l \geqslant 1)$. (3.13) and (3.15) make the Boolean embedding ${}_{*k}$ an embedding of the cylindric algebra $\boldsymbol{P}(A^k)$ into the cylindric algebra $\boldsymbol{P}(E^k)$. With that reinterpretation of (3.11), one arrives at a simple algebraic proof (cf. [14]) of Łoś's ultrapower theorem: i: $A \to E$ is, in fact, an elementary embedding of A endowed with its full relational structure, i.e., with all its relations $R \subset A^k$, for every $k \geqslant 1$, into E with the corresponding relations ${}_{*k}R$. This is the lower order version of Robinson's and Zakon's Meta-theorem [12], 3.2 (for which we hope to give a simpler algebraic proof elsewhere, as we wish to keep the formal language out of this paper).

It is well known that a suitable choice of the index-domain T and of the index-filter U, together with UP1, makes the elementary extension E an *enlargement* in the sense of Robinson (cf. Robinson and Zakon [12], § 4). I.e., for every proper filter $\boldsymbol{F} \subset \boldsymbol{P}(A^k)$ $(k \geqslant 1)$, its *monad*

$$\mu(\boldsymbol{F}) = \bigcap \{{}_{*k}F;\ F \in \boldsymbol{F}\} \tag{3.16}$$

is a nonempty subset of E^k (cf. Luxemburg [7], also [8]). This applies, in particular, to the *concurrent* binary relations $R \subset A^m \times A^n$ whose sections $R[\{\langle a_1, \ldots, a_m\rangle\}] \subset A^k$ $(\langle a_1, \ldots, a_m\rangle \in \operatorname{dom} R)$ have the finite intersection property (generate a proper filter $\boldsymbol{F} \subset \boldsymbol{P}(A^k)$).

4. Higher order ultrapowers. We now consider the superstructure $\hat{A}$ over the set A. $\hat{A}$ being as honest a set as any, we can consider some lower order ultrapower of $\hat{A}$, i: $\hat{A} \to E$ with the projections p_t: $E \to \hat{A}$ $(t \in T)$. We get the lower order set mappings

$${}_{*k}\colon \boldsymbol{P}(\hat{A}^k) \to \boldsymbol{P}(E^k)$$

defined, for every $k \geqslant 1$, by (3.1). Recall that ${}_{*k}\hat{A}^k = E^k$ and ${}_{*k}\emptyset = \emptyset$. Instead of ${}_{*1}A_\omega$, one considers

$$Z = \bigcup_{n=0}^{\infty} {}_{*k}A_n. \tag{4.1}$$

By (3.4),

$$i^{-1}[Z] = \bigcup_{n=0}^{\infty} A_n = \hat{A}, \tag{4.2}$$

whence i maps actually into Z,

$$i[\hat{A}] \subset Z. \tag{4.3}$$

We also consider

$${}_{*2}{\in} = \{\langle Y, X\rangle \in E^2;\ p_t(Y) \in p_t(X) \text{ a.e.}\}. \tag{4.4}$$

The four subsequent key lemmas, dealing with ${}_{*2}{\in}$, are using the mappings p_t and i.

LEMMA 4.1. *If* $X \in {}_{*1}A_{n+1}$ *and* $Y_{*2}{\in} X$, *then* $Y \in {}_{*1}A \cup {}_{*1}A_n$.

For $p_t(Y) \in p_t(X) \in A_{n+1}$ a.e., whence $p_t(Y) \in A \cup A_n$ a.e. and so $Y \in_{*1} A \cup {}_{*1} A_n$.

COROLLARY 4.2. *If* $X \in Z \setminus {}_{*1}A$ *and* $Y_{*2}{\in} X$, *then* $Y \in Z$.

Using $\in_A$ defined in (1.15), we can restate this as follows: $X \in Z$ and $Y_{*2}{\in}_A X$ implies $Y \in Z$.

LEMMA 4.3. *Let* $X \in \hat{A}$, $X \subset \hat{A}$, *and* $Y \in E$. *Then*

$$Y_{*2} \in i(X) \quad \text{iff} \quad Y \in_{*1} X. \tag{4.5}$$

For $Y_{*2} \in i(X)$ iff $p_t(Y) \in p_t(i(X))$ a.e. But $p_t(i(x)) = X$ a.e., so $Y_{*2} \in i(X)$ iff $p_t(Y) \in X$ a.e., which by (3.1) is equivalent to $Y \in_{*1} X$.

Note that

$$A_\omega \subset A_{\omega+1} = \boldsymbol{P}(\hat{A}), \tag{4.6}$$

because of (1.12) and (2.1). Hence Lemma 4.3 applies, in particular, to each $X \in A_\omega$.

Since $\emptyset \in A_1 \subset A_\omega$ and ${}_{*1}\emptyset = \emptyset$, we get

COROLLARY 4.4. *There is no* $Y \in E$ *such that* $Y_{*2} \in i(\emptyset)$.

LEMMA 4.5. *Let* $X, X' \in Z \setminus {}_{*1}A$. *Suppose that, for each* $Y \in Z$, $Y_{*2} \in X$ *iff* $Y_{*2} \in X'$. *Then* $X = X'$.

For suppose $X \neq X'$. So $p_t(X) \neq p_t(X')$ a.e. Hence $p_t(X) \not\subset p_t(X')$ a.e. or $p_t(X') \not\subset p_t(X)$ a.e. Assuming the former, we find, for almost all $t \in T$, sets $Y_t \in p_t(X) \setminus p_t(X')$. But $X \in_{*1} A_{n+1}$ for some n, so $p_t(X) \in A_{n+1}$ a.e. and $Y_t \in A \cup A_n$ a.e., whence $Y_t \in \hat{A}$ a.e. So there is $Y \in E$ such that $p_t(Y) = Y_t$ a.e. So $Y_{*2} \in X$ and $Y \in Z$ by Corollary 4.2. But $Y_{*2} \in X'$ does not hold.

Note that we used here the full power of the axiom of choice (as did Davis [4], 1.4, Lemma 5). This makes the discussion of its rôle in this context (cf. Davis [4], p. 8 f., p. 82) appear somewhat unsubstantiated.

LEMMA 4.6. *For any* $X \in Z \setminus {}_{*1}A$, $X = i(\emptyset)$ *iff there is no* $Y \in Z$ *such that* $Y_{*2} \in X$.

For one direction follows from Corollary 4.4. Let now $X \neq i(\emptyset)$. Then $p_t(X) \neq p_t(i(\emptyset))$ a.e. But $p_t(i(\emptyset)) = \emptyset$ a.e. So $p_t(X) \neq \emptyset$ a.e. As in the proof of Lemma 4.5, we find $Y \in Z$ such that $Y_{*2} \in X$. (If we would know $\emptyset \notin A = i^{-1}[{}_{*1}A]$, i.e., $i(\emptyset) \notin {}_{*1}A$, we could have applied Lemma 4.5 itself.)

We introduce the relation

$$\varrho = {}_{*2}\in_A \cap Z^2 = \{\langle Y, X \rangle \in Z^2;\ p_t(Y) \in p_t(X) \notin A \text{ a.e.}\}. \tag{4.7}$$

Recall – (3.5), (4.3) – that i: $\langle \hat{A}, \in_A \rangle \to \langle Z, \varrho \rangle$ is an embedding:

$$Y \in X \notin A \quad \text{iff} \quad i(Y) \varrho i(X) \tag{4.8}$$

for each $X, Y \in \hat{A}$. Recall also – Proposition 1.5 – that $\hat{A}$ is well-founded by $\in_A$, with bottom $A \cup \{\emptyset\}$.

THEOREM 4.7. $\langle Z, \varrho \rangle$ *is well-founded and extensional, with bottom* $Z_0 = {}_{*1}A \cup \{i(\emptyset)\}$.

Proof. Let $M \subset Z$, $M \neq \emptyset$. There is a minimal n such that $M \cap {}_{*1}A_n \neq \emptyset$. So there is $X \in M \cap {}_{*1}A_n$. Let $Y \varrho X$. Then $X \notin_{*1} A$, by the definition of ϱ, whence $n \geqslant 1$. By Lemma 4.1, $Y \in_{*1} A \cup {}_{*1}A_{n-1}$. By the minimal property of n, $Y \notin M$, making X a ϱ-minimal element of M and

$\langle Z, \varrho\rangle$ well-founded. By the definition of ϱ, each $X \in_{*1} A$ is in the bottom of $\langle Z, \varrho\rangle$, and so is $X = i(\emptyset)$, by Corollary 4.4. Conversely, if $X \in Z \setminus_{*1} A$ and $X \neq i(\emptyset)$, then X is not in the bottom by Corollary 4.6. Let now $X, X' \in Z \setminus_{*1} A$ and $X, X' \neq i(\emptyset)$. Let $X \equiv X'$, i.e. $Y \varrho X$ iff $Y \varrho X'$ for each $Y \in Z$. So $Y_{*2} \in X$ iff $Y_{*2} \in X'$ for each $Y \in Z$, whence $X = X'$ by Lemma 4.5, making $\langle Z, \varrho\rangle$ extensional.

THEOREM 4.8. *Let B be a set such that $B \cap B_\omega = \emptyset$. Let f_0 be a one-to-one mapping from $Z_0 = {}_{*1}A \cup \{i(\emptyset)\}$ into $B \cup \{\emptyset\}$ such that*

$$f_0(i(\emptyset)) = \emptyset. \tag{4.9}$$

Let f be its Zermelo–Mostowski extension to all of $\langle Z, \varrho\rangle$. Then f is an embedding of $\langle Z, \varrho\rangle$ into $\langle \hat{B}, \in_B\rangle$. I.e., $f\colon Z \to B$ is one-to-one, and

$$Y_{*2} \in X \in_{*1} A \quad \textit{iff} \quad f(Y) \in f(X) \notin B, \quad \textit{for each } X, Y \in Z. \tag{4.10}$$

Proof. Note that because of (4.9) and Corollary 4.4, the original definition – (1.1) – of f can be conveniently replaced by

$$f(X) = \begin{cases} f_0(X) & \text{if } X \in_{*1} A, \\ \{f(Y);\ Y_{*2} \in X\} & \text{if } X \in Z \setminus_{*1} A. \end{cases} \tag{4.11}$$

If $X \notin_{*1} A$ and $X \neq i(\emptyset)$, then $Y_{*2} \in X$ for some $Y \in Z$ by Lemma 4.6, making $f(X) \neq \emptyset$. Hence

$$f(X) = \emptyset \quad \text{implies} \quad X \in_{*1} A \cup \{i(\emptyset)\}. \tag{4.12}$$

By Lemma 4.1, one gets $f[{}_{*1}A_n \setminus_{*1} A] \subset B_{n+1}$ by induction on $n \geqslant 0$ (for the induction step, note that we might have $\emptyset \in A$, whence we can only claim $f[{}_{*1}A] \subset B \cup B_1$). Hence $f[Z \setminus_{*1} A] \subset B_\omega$ and $f[Z] \subset B \cup B_1 \cup B_\omega = \hat{B}$. Since $B \cap B_\omega = \emptyset$, we get $f[Z \setminus_{*1} A] \cap B = \emptyset$, whence

$$f(X) \notin B \quad \text{implies} \quad X \notin_{*1} A. \tag{4.13}$$

In view of Theorem 4.7, (4.12) and (4.13) represent the nontrivial direction of condition (v), Theorem 1.2. Conditions (iii) and (iv) being already satisfied, Theorem 4.8 becomes a special case of the general Zermelo–Mostowski Theorem 1.2.

It is somewhat surprising that so far A could be any set. This is in some contrast to usual presentations of higher-order enlargements, where much is required of A and not much said (except for some reassuring remarks what could be done) about B (often identified with ${}_{*1}A$). It is about B, namely for (4.13), that we needed the assumption $B \cap B_\omega = \emptyset$, satisfied, for instance, if B is a strong antichain such that $B \cap \emptyset_\omega = \emptyset$, cf. Theorem 2.3. In this context, it is quite remarkable that Robinson and Zakon [12], p. 111 (cf. also Keisler [5], p. 39), came up with conditions "such as" $\emptyset \notin A$ and $\hat{A} \cap \bigcup A = \emptyset$ (the latter making A a weak antichain such that $A \cap A_{\omega+1} = \emptyset$). They find these "small adjustments" necessary to "formalize in any existing set theory" all they did in "the language of naive set theory" (the latter admitting urelements). If we were not "naive" here, then only in succeeding to ban urelements.

5. Higher order ultrapowers, continued. Still assuming $B \cap B_\omega = \emptyset$, we now combine the embeddings i: $\langle \hat{A}, \emptyset_A \rangle \to \langle Z, \varrho \rangle$ and f: $\langle Z, \varrho \rangle \to \langle \hat{B}, \in_B \rangle$ to the embedding

(5.1) $$j = f \circ i \colon \langle \hat{A}, \in_A \rangle \to \langle \hat{B}, \in_B \rangle .$$

So we have, for each X, $Y \in A$,

(5.2) $$Y \in X \notin A \quad \text{iff} \quad j(Y) \in j(X) \notin B .$$

(Stroyan and Luxemburg [16], (3.4.1), state only the trivial direction corresponding to (1.3).)

Actually, the injection j maps into $f[Z]$. Since f is one-to-one, we can also introduce the "projection" mappings

(5.3) $$q_t = p_t \circ f^{-1} \colon f[Z] \to \hat{A} .$$

It is clear that j: $\hat{A} \to f[Z]$ and the mappings q_t: $f[Z] \to \hat{A}$ satisfy UP2 and UP3 of Section 3. So almost all of Section 3 applies here. We again introduce, for each $k \geqslant 1$, the set mapping

$$*^k \colon \boldsymbol{P}(\hat{A}^k) \to \boldsymbol{P}(f[Z]^k)$$

by defining, for each k-ary relation $R \subset \hat{A}^k$:

(5.4) $$*^k R = \{ \langle X_1, \ldots, X_k \rangle \in f[Z]^k ;\ \langle q_t(X_1), \ldots, q_t(X_k) \rangle \in R \text{ a.e.} \} .$$

Again, these set mappings are Boolean embeddings (UP3); again we have, in particular,

(5.5) $$*^k \hat{A}^k = f[Z]^k \quad \text{and} \quad *^k \emptyset = \emptyset$$

(for $k = 1$, cf. Davis [4], p. 29). Again we have (UP2)

(5.6) $$*^2 \mathrm{id}_{\hat{A}} = \mathrm{id}_{f[Z]} ,$$

which is the unrestricted (unbounded, cf. below) version of Robinson's and Zakon's axiom [12], 2.5′. We again have the old interrelations (UP3) between the induced point mapping j^k: $\hat{A}^k \to f[Z]^k$ and the set mapping $*^k$:

(5.7) $$(j^k)^{-1} [*^k R] = R \quad \text{and} \quad j^k [R] \subset *^k R$$

for each $R \subset \hat{A}^k$, making again j: $\langle \hat{A}, R \rangle \to \langle f[Z], *^k R \rangle$ an embedding. Again,

(5.8) $$j^k [R] = *^k R \quad \text{if } R \text{ is finite.}$$

This is a strong version of Robinson's and Zakon's axiom [12], 2.1.

In addition to (5.6), we now have

$$(5.9)\qquad {}^{*2}\in_A = \in_B \cap f[Z]^2,$$

as a consequence of (5.4), (5.3), (4.7) and (4.10). This is the unrestricted version of Robinson's and Zakon's only truly higher order axiom [12], 2.5. Note that (5.2) follows from (5.9) and (5.7). (Stroyan and Luxemburg [16], (3.4.1), list both the (bounded version of) (5.9) and the trivial part of (5.2) as axioms.)

The price we pay for (5.9) is the fact that UP1 does no longer hold without restrictions. A sequence of elements $Y_t \in \hat{A}$ $(t \in T)$, i.e., a function from T into $\hat{A}$, is called *bounded* iff there is an $n \geq 0$ such that $Y_t \in A_n$ a.e. One has the following bounded substitute for UP1:

PROPOSITION 5.1. *Consider a sequence of elements $Y_t \in \hat{A}$ $(t \in T)$. Then there exists $Y \in f[Z]$ such that $q_t(Y) = Y_t$ a.e. iff the sequence is bounded.*

It is (5.9), together with Proposition 5.1, that makes the setting $j\colon \hat{A} \to f[Z]$, $q_t\colon f[Z] \to \hat{A}$ a *higher* order ultrapower of $\hat{A}$, as opposed to the *lower* order ultrapower $i\colon \hat{A} \to E$, $p_t \to \hat{A}$ with its *exotic element relation* ${}_{*2}\in_A \subset E^2$. (Some authors, e.g., Luxemburg [7], do not shrink back from working with ${}_{*2}\in_A$; maybe they should then not call higher order ultrapower what is truly lower order.)

Proposition 5.1 certainly has some damaging effects. Gone is, e.g., the (non-trivial part of) (3.11): While it is still valid for transformations σ from $\{1, \ldots, k\}$ *onto* $\{1, \ldots, l\}$, it is no longer true, without obvious boundedness assumptions (cf. below), for one-to-one mappings. Or to put it this way, ^{*k} does no longer preserve the unrestricted cylindrifications $\exists_r^k$. We will not go into that here. Note only that (3.10), still following just from the definitions, is still valid without any restrictions whatsoever.

We now have a beautiful interrelation between the point mapping and the (unrestricted) set mappings above for which there is no lower order counterpart. We restrict ourselves, in this section, to the case $k = 1$. The case $k \geq 2$ will be treated in the next section. We first observe:

PROPOSITION 5.2. *Let $B \cap B_\omega = \emptyset$. Then*

$$(5.10)\qquad {}^{*1}S = f[{}_{*1}S] = f[Z \cap {}_{*1}S]$$

for each subset $S \subset \hat{A}$.

For any $Y \in f[Z]$, we have $Y \in {}^{*1}S$ iff $q_t(Y) \in S$ a.e., which is equivalent to $f^{-1}(Y) \in {}_{*1}S$, i.e., to $Y \in f[{}_{*1}S]$.

Up to this point, nothing has been assumed about the set A. Conditions on A will surface now.

THEOREM 5.3. *Let $B \cap B_\omega = \emptyset$. Then $j[A] \subset B$ iff $\emptyset \notin A$. Moreover, $j[A_n] \subset B_n$ for each $n \geq 0$ iff $j[A] \subset B$ and $j[A_\omega] \subset B_\omega$ iff $A \cap A_\omega = \emptyset$. In* *that case,*

$$j|A_\omega = {*^1}|A_\omega. \tag{5.11}$$

Finally, $\operatorname{dom} j \cap \operatorname{dom} {*^1} = A_\omega$, *i.e.,* $\hat{A} \cap P(\hat{A}) = A_\omega$, *if* $A \cap A_{\omega+1} = \emptyset$. *In the presence of* $A \cap A_\omega = \emptyset$, *this is the case if and only if* $A \cap A_{\omega+1} = \emptyset$.

Proof. (i) follows from $j[A \cup \{\emptyset\}] \subset B \cup \{\emptyset\}$ and $j(\emptyset) = \emptyset \notin B$. For (ii), let $A \cap A_\omega = \emptyset$. Using Lemma 4.1 once more, one shows that $j[A_n] \subset B_n$ by induction on $n \geqslant 0$, the induction beginning $j[A] \subset B$ being true since $\emptyset \notin A$. With that, $j[A_\omega] \subset B_\omega$. If $j[A] \subset B$ and $j[A_\omega] \subset B_\omega$, then $j[A] \cap j[A_\omega] = \emptyset$ and $A \cap A_\omega = \emptyset$. Assuming $A \cap A_\omega = \emptyset$, we have, for $X \in A_\omega = \hat{A} \setminus A$, that $i(X) \in Z \setminus {_{*1}A}$, whence

$$j(X) = f[{_{*1}X}] = {*^1}X. \tag{5.12}$$

This is, in fact, a consequence of (4.11) (the redefinition of the Zermelo–Mostowski function f), the star-shifting formula (4.5), and of (5.10), of course.

For (iii), assume $A \cap A_{\omega+1} = \emptyset$. One gets $A \cap A_\omega = \emptyset$ and $\hat{A} \cap P(\hat{A}) = (A \cup A_\omega) \cap A_{\omega+1} = A_\omega \cap A_{\omega+1} = A_\omega$. Conversely, let $A \cap A_\omega = \emptyset$ and $\hat{A} \cap P(\hat{A}) = A_\omega$. One gets $(A \cap A_{\omega+1}) \cup A_\omega = (A \cup A_\omega) \cap A_{\omega+1} = A_\omega$, whence $A \cap A_{\omega+1} \subset A \cap A_\omega$ and $A \cap A_{\omega+1} = \emptyset$.

The reader should be duly reminded that the condition $A \cap A_{\omega+1} = \emptyset$ follows from the conditions of Robinson and Zakon [12] and Keisler [5] mentioned at the end of Section 4. At any rate, if $A \cap A_{\omega+1} = \emptyset$ (and there are arbitrarily large sets with this property, cf. Theorem 2.6), then the *pure point part* of j, $j_0 = j|_A$, maps A into B, while the set part, $j_\omega = j|_{A_\omega}$, maps A_ω into B_ω; moreover, the equation (5.12) holds whenever both sides are meaningful, $X \in \hat{A} \cap P(\hat{A}) = A_\omega$. Note that $X \in A_\omega$ iff $Y \subset A \cup A_n$ for some $n \geqslant 0$; these sets $X \subset \hat{A}$ are known as the *bounded sets*. Following Robinson's pioneer work [11], most authors define the mapping ${*^1}$ only for bounded sets, so Robinson and Zakon [12], Luxemburg [8], Machover and Hirschfeld [9], Zakon [18], Stroyan and Luxemburg [16]. In all these approaches, the construction is closely linked together if not directly based upon an appropriate formal language. The first one to have extended the mapping ${*^1}$ to unbounded sets seems to bc Davis [4], p. 29. However, restricting ${*^1}$ to definable bounded or unbounded sets (and every bounded set is definable), he again "constructs" ${*^1}$ via the formal language ((5.12) above corresponds to his Corollary 7.5).

For any bounded set S, we now have, as a consequence of (5.7) and (5.12),

$$j^{-1}[j(S)] = S \quad \text{and} \quad j[S] \subset j(S). \tag{5.13}$$

Any finite set is bounded. Hence, as a consequence of (5.8) and (5.12), we get

$$j[S] = j(S) \quad \text{if } S \text{ is finite.} \tag{5.14}$$

This is another version of the axiom [12], 2.1, more in the spirit of that paper which, after all, considers only one mapping $\Phi: \hat{A} \to \hat{B}$.

6. Relations in higher order ultrapowers. An ordered k-tuple $\langle Y_1, \ldots, Y_k \rangle \in \hat{A}^k$, where $k \geqslant 2$, is a finite subset of $\hat{A}$, hence bounded:

$$\hat{A}^k \subset A_\omega \quad (k \geqslant 2). \tag{6.1}$$

Hence, if $A \cap A_\omega = \emptyset$, we have

$$A \cap \hat{A}^k = \emptyset \quad (k \geqslant 2). \tag{6.2}$$

We need a statement somewhat more subtle than (6.1).

LEMMA 6.1. *If $Y_1, \ldots, Y_k \in A \cup A_n$, $k \geqslant 2$, then $\langle Y_1, \ldots, Y_k \rangle \in A_{n+2k-2}$. Conversely, if $A \cap A_\omega = \emptyset$ and $\langle Y_1, \ldots, Y_k \rangle \in A_m$, then $Y_1, \ldots, Y_k \in A \cup A_m$.*

For the first statement, cf. e.g. Davis [4], p. 14, Lemma 14. The second statement, voidly true for $m < 2$, is proven for $m \geqslant 2$ by induction on $k \geqslant 2$. For the induction beginning, assume $\langle Y_1, Y_2 \rangle \in A_m$. One has $Y_1, Y_2 \in \{Y_1, Y_2\} \in \langle Y_1, Y_2 \rangle \subset A \cup A_{m-1}$. Since $\{Y_1, Y_2\} \in A_\omega$ and $A \cap A_\omega = \emptyset$, we have $\{Y_1, Y_2\} \notin A$. Consequently, $Y_1, Y_2 \in A \cup A_{m-2} \subset A \cup A_m$.

Lemma 6.1 can be reformulated:

$$A_n \cap \hat{A}^k \subset (A \cup A_n)^k \subset A_{n+2k-2} \quad (k \geqslant 2). \tag{6.3}$$

For $k = 2$, we even have

$$A_{n+2} \cap \hat{A}^2 = (A \cup A_n)^2. \tag{6.4}$$

COROLLARY 6.2. *Let $A \cap A_\omega = \emptyset$. Then for any set R and $k \geqslant 2$, $R \subset (A \cup A_n)^k$ for some $n \geqslant 0$ iff $R \subset \hat{A}^k$ and $R \in A_\omega$.*

One direction follows already from (6.1), the other one (which takes $A \cap A_\omega = \emptyset$) from (6.3). Corollary 6.2 has been stated by Robinson and Zakon [12], footnote 1. They call such a relation a *bounded k-ary relation*. By Corollary 6.2, R is bounded as a k-ary relation iff it is bounded as a set (a unary relation).

These were simple technical observations, related to ordered k-tuples (à la Kuratowski), about a single superstructure $\hat{A}$. Returning to our various mappings, we first have

PROPOSITION 6.3. *Let $A \cap A_\omega = \emptyset$. Then for each $k \geqslant 1$:*

$$j|_{\hat{A}^k} = j^k. \tag{6.5}$$

That is,

$$j(\langle Y_1, \ldots, Y_k \rangle) = \langle j(Y_1), \ldots, j(Y_k) \rangle \tag{6.6}$$

for each $Y_1, \ldots, Y_k \in \hat{A}$.

Note that by (6.1) $\hat{A}^k \subset \hat{A}$ for each $k \geqslant 1$. I.e., $\operatorname{dom} j^k \subset \operatorname{dom} j$. For $k = 1$, (6.6) is trivial. For $k = 2$, one uses (5.14), The rest is induction.

For a related result we introduce in the lower order ultrapower E an *exotic ordered k-tuple* by the definition

(6.7) $$p_t([Y_1, \ldots, Y_k]) = \langle p_t(Y_1), \ldots, p_t(Y_k)\rangle \text{ a.e.},$$

for each $k \geqslant 1$, $Y_1, \ldots, Y_k \in E$, existence and uniqueness of $[Y_1, \ldots, Y_k] \in E$ being based upon UP1 and UP2. (If we consider the k-ary operations $\langle, \ldots, \rangle$ and $[, \ldots,]$ as $(k+1)$-ary relations in A and E respectively, the latter is really the $(k+1)$-$_*$ (3.1) of the former.)

LEMMA 6.4. *Let $B \cap B_\omega = \emptyset$. Then $Y_1, \ldots, Y_k \in Z$ implies $[Y_1, \ldots, Y_k] \in Z$. One then has*

(6.8) $$f([Y_1, \ldots, Y_k]) = \langle f(Y_1), \ldots, f(Y_k)\rangle.$$

If also $A \cap A_\omega = \emptyset$, then $Y_1, \ldots, Y_k \in E$ and $[Y_1, \ldots, Y_k] \in Z$ imply $Y_1, \ldots, Y_k \in Z$.

The first and the last statements are immediate consequences of (6.7) and Lemma 6.1. For the equation (6.8), cf. Davis [4], p. 19, Lemma 9.

COROLLARY 6.5. *Let $B \cap B_\omega = \emptyset$. Then $X_1, \ldots, X_k \in f[Z]$ implies $\langle X_1, \ldots, X_k\rangle \in f[Z]$ and one has*

(6.9) $$q_t(\langle X_1, \ldots, X_k\rangle) = \langle q_t(X_1), \ldots, q_t(X_k)\rangle \text{ a.e.}$$

Conversely, if $\langle X_1, \ldots, X_k\rangle \in f[Z]$, then $X_1, \ldots, X_k \in f[Z]$.

The first two statements follow from Lemma 6.4. The last statement, trivial for $k = 1$, holds for $k = 2$ since $B \cap B_\omega = \emptyset$ and $f[Z]$ is transitive over B, cf. (1.2). The rest is induction.

With that, we are ready for our last result:

THEOREM 6.6. *Let $A \cap A_\omega = B \cap B_\omega = \emptyset$. Then for each $k \geqslant 1$:*

(6.10) $$*^k = *^1|_{P(\hat{A}^k)}.$$

(6.11) $$*^k R = *^1 R$$

for each k-ary relation $R \subset \hat{A}^k$.

Proof. Note that $P(\hat{A}^k) \subset P(\hat{A})$, i.e., $\operatorname{dom} *^k \subset \operatorname{dom} *^1$. Let now $\langle X_1, \ldots, X_k\rangle \in *^k R$. I.e., $X_1, \ldots, X_k \in f[Z]$ and $\langle q_t(X_1), \ldots, q_t(X_k)\rangle \in R$ a.e. By Corollary 6.5, $\langle X_1, \ldots, X_k\rangle \in f[Z]$ and $q_t(\langle X_1, \ldots, X_k\rangle) \in R$ a.e., whence $\langle X_1, \ldots, X_k\rangle \in *^1 R$, proving $*^k R \subset *^1 R$ without the assumption $A \cap A_\omega = \emptyset$. Conversely, let $X \in *^1 R$. So $X = f(Y)$, for some $Y \in Z$, and $q_t(X) = p_t(Y) \in R$ a.e. Since $R \subset \hat{A}^k$, we find, for almost all $t \in T$, $Y_{t1}, \ldots, Y_{tk} \in \hat{A}$ (uniquely determined) such that $p_t(Y) = \langle Y_{t1}, \ldots, Y_{tk}\rangle$. By UP1, we find $Y_1, \ldots, Y_k \in E$ (uniquely determined by UP2) such that $p_t(Y_1) = Y_{t1}, \ldots, p_t(Y_k) = Y_{tk}$ a.e., whence $p_t([Y_1, \ldots, Y_k]) = p_t(Y)$ a.e. By UP2, $[Y_1, \ldots, Y_k] = Y \in Z$. Since $A \cap A_\omega = \emptyset$, we get $Y_1, \ldots, Y_k \in Z$, whence $X_1 = f(Y_1), \ldots, X_k = f(Y_k) \in f[Z]$. But $\langle q_t(X_1), \ldots, q_t(X_k)\rangle = \langle p_t(Y_1), \ldots, p_t(Y_k)\rangle = p_t(Y) \in R$ a.e. Hence $X = f(Y) = f([Y_1, \ldots, Y_k]) = \langle X_1, \ldots, X_k\rangle \in *^k R$, completing the proof.

In summary, the higher order case has the following features (in striking contrast to the lower order case, §3): Under the assumption $A \cap A_\omega = B \cap B_\omega = \emptyset$, the mappings j and j^k coincide on $\operatorname{dom} j^k \subset \operatorname{dom} j$. Likewise, the mappings $*^1$ and $*^k$ coincide on $\operatorname{dom} *^k \subset \operatorname{dom} *^1$; we may henceforth just write $*$. Finally, the mappings j and $*$ coincide on A_ω, which is the exact intersection of their domains provided that we have even $A \cap A_{\omega+1} = \emptyset$. In that case, we arrive at the function

$$\Phi = j \cup *: \hat{A} \cup \boldsymbol{P}(\hat{A}) \to \hat{B} \cup \boldsymbol{P}(\hat{B}), \tag{6.12}$$

unambiguously defined by

$$\Phi(S) = \begin{cases} f(i(s)) & \text{if} \quad S \in \hat{A}, \\ f[{}_{*1}S] = X \in f[Z]; \quad q_t(X) \in S \text{ a.e.} & \text{if} \quad S \in \boldsymbol{P}(\hat{A}). \end{cases} \tag{6.13}$$

A final word on higher order enlargements: A proper filter $\boldsymbol{F} \subset \boldsymbol{P}(\hat{A})$ is a *bounded filter* provided that $A \cup A_n \in \boldsymbol{F}$ for some $n \geq 0$ (which is, e.g., the case if even $A_n \in \boldsymbol{F}$ for some $n \geq 0$). Suppose now the index-filter $\boldsymbol{U}$ is adequate (cf. Luxemburg [8], also Bruns and Schmidt [3]) at least for each bounded filter $\boldsymbol{F}$ over $\hat{A}$: There is a sequence of elements $Y_t \in \hat{A}$ $(t \in T)$ such that, for each $F \in \boldsymbol{F}$, $Y_t \in F$ a.e. Then this sequence is bounded (§5), hence – Proposition 5.1 – there exists $X \in f[Z]$ such that $q_t(X) = Y_t$ a.e., making again the monad $\mu(\boldsymbol{F})$ – cf. (3.16) – non-empty. This applies again to such concurrent binary relations $R \subset \hat{A}^2$ whose sections $R[\{Y\}]$ $(Y \in \operatorname{dom} R)$ generate a bounded proper filter $\boldsymbol{F} \subset \boldsymbol{P}(\hat{A})$. Such relations are not necessarily bounded. Nevertheless, there is $X \in f[Z]$ such that $\langle Y, X \rangle \in_* R$ for every $Y \in \operatorname{dom} R$.

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