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WEAK CONVERGENCE AND WEIGHTED AVERAGES FOR GROUPS OF OPERATORS

BY

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1. Introduction. Let $\tau: X \to X$ be an invertible, bimeasurable, measure-preserving transformation of a probability space (X, Σ, μ). In [3] Blum and Hanson proved the following mean ergodic theorem:

The transformation τ is a strong mixing, i.e.,

$$\lim_n \left(\tau^n(A) \cap B\right) = \mu(A)\mu(B) \quad \text{for any } A, B \in \Sigma,$$

if and only if for every real number p with $1 \leqslant p < \infty$, every strictly increasing sequence (k_i) of integers, and every $f \in L^p(\mu)$ we have

$$\lim_n \int \left| \frac{1}{n} \sum_{i=1}^{n} f\left(\tau^{k_i}(x)\right) - \int f\, d\mu \right|^p d\mu(x) = 0.$$

This theorem has subsequently been generalized by many authors, most of interesting results being obtained for cyclic semigroups of contractions (and of power bounded operators) on L^p-spaces (see [11], [13], [1], [2], [7], [18], and [20]). In this context Nagel [15] considered order contractions acting on more general Banach lattices with order continuous norm (see also Schaefer [21], V, §8). Sato [19] and Fong [6] treated of one-parameter continuous semigroups of operators.

In the present note we focus on groups. More specifically, we consider measurable representations of a locally compact group G on relevant Banach spaces. Thus, on the geometric level, the discussion here is confined to groups of measure-preserving transformations of the underlying measure space (Section 4).

Throughout, we make strong use of ideas and methods developed for cyclic and one-parameter semigroups mainly by Akcoglu, Sucheston, Fong, and Nagel.

2. Terminology and notation. Let G be a locally compact non-compact topological (Hausdorff) group. By $G \cup \{\infty\}$ we denote the one-

-point compactification of G. If f is a mapping from G into a topological space S, then we write

$$(*) \qquad \lim_{t\to\infty} f(t) = s$$

if f converges to $s \in S$ on $G \cup \{\infty\}$ along the filter of neighborhoods of ∞. In other words, $(*)$ means that for every neighborhood U of s in S there exists a compact subset $K \subset G$ such that $f(t) \in U$ whenever $t \in G \setminus K$.

A representation $t \to T_t$ of G on a (real or complex) Banach space E with Banach dual E' will be called *measurable* if the function

$$t \to \langle T_t x, y\rangle$$

is Borel measurable for any $x \in E$ and $y \in E'$. Clearly, every weakly continuous representation is measurable. Conversely, every measurable isometric representation of G on a separable Banach space E is automatically continuous (for the case of Hilbert space see [8], Chapter 5, 22.20b; in general case the same argument works).

Let $t \to T_t$ be a measurable representation of G on a Banach space E and suppose that for some $x \in E$ the *orbit*

$$O_x = \{T_t x:\ t \in G\}$$

is bounded in E. Then for every finite signed Radon measure ν on G the mapping

$$y \to \int \langle T_t x, y\rangle d\nu(t)$$

defines a bounded linear functional $\int T_t x d\nu(t)$ on E'. Moreover, if ν is a probability measure, then $\int T_t x d\nu(t)$ belongs to the weak* closed convex hull of O_x in E''. If, in addition, the set O_x is relatively weakly compact, then, by the Krein-Šmulian theorem ([5], V, 6.4), the convex hull of O_x is relatively weakly compact in E, which clearly implies

$$\int T_t x\, d\nu(t) \in E.$$

Now let the family $\{T_t: t \in G\}$ be equicontinuous and suppose that O_x is relatively weakly compact for every x from a dense subset $E_0 \subset E$. Then the mapping

$$x \to \int T_t x\, d\nu(t)$$

defines a bounded linear operator on E_0, extending uniquely to a bounded linear operator $\int T_t d\nu(t)$ on E. If this is the case, we say that *the integral* $\int T_t\, d\nu(t)$ *exists.*

Clearly, if ν has finite support, then for every $x \in E$ the functional $\int T_t x\, d\nu(t)$ is in E and the integral $\int T_t d\nu(t)$ exists. Also, if E is a reflexive Banach space and $\{T_t: t \in G\}$ is equicontinuous, then by the Banach-

-Alaoglu theorem all sets O_x are relatively weakly compact, so that $\int T_t x\, d\nu(t)$ is always in E and $\int T_t d\nu(t)$ exists.

Following Fong's definition for measures on the half-line [6] (cf. also [7]), we denote by $\mathfrak{U}$ the family of all sequences (μ_n) of signed Radon measures on G satisfying

(1) $\sup\limits_n \|\mu_n\| < \infty,$

(2) $\lim\limits_n \mu_n(G) = 1,$

(3) $\lim\limits_n \sup\limits_{t \in G} |\mu_n|(tK) = 0$ for every compact subset $K \subset G$.

Roughly speaking, every $(\mu_n) \in \mathfrak{U}$ corresponds to certain Cesàro type weighted averaging procedure on G. As was kindly pointed out to us by Dr. T. Byczkowski, certain sequences of probability measures satisfying (3) were considered by Csiszár in [4]. In particular, Theorem 3.1 of [4] asserts that if G is a second countable locally compact group, (ν_n) is a sequence of probability measures on G, and $\mu_n = \nu_n * \ldots * \nu_1$ is the convolution product, then either $(\mu_n) \in \mathfrak{U}$ or else there exists a sequence (a_n) in G such that the shifted distributions $\delta_{a_n} * \mu_n$ converge in law as $n \to \infty$.

In the sequel, given a Banach space E, $\mathscr{L}(E)$ will denote the Banach space of all continuous linear operators from E into E. By $\mathscr{L}_s(E)$ and $\mathscr{L}_\sigma(E)$ we denote the space $\mathscr{L}(E)$ endowed with the strong and weak operator topology, respectively.

For Banach lattices, we use the terminology and notation of Schaefer's book [21].

3. Convergence of unitary representations. In the proof of the forthcoming theorem we will need the following group theoretic lemma:

LEMMA 1. *Let G be a locally compact non-compact group and suppose that ∞ is a cluster point of a subset $S \subset G$. Then there exists a sequence (s_n) in S with the following property:*

For any compact subset $K \subset G$ there exists a natural number $n(K)$ such that each translate of K contains at most $n(K)$ elements of (s_n).

Proof. First observe that considering only left translates causes no loss of generality. For every relatively compact and symmetric neighborhood U of identity in G there exists a sequence (s_n) in S such that s_{n+1} does not belong to any of $s_i U^2$ for $i = 1, \ldots, n$. In particular, the sets $s_n U$ are pairwise disjoint. Now, if a compact set K has a covering by k left translates of U, then, clearly, no left translate of K can contain more than k elements of (s_n).

THEOREM 1. *Let G be a locally compact non-compact group and let $t \to T_t$ be a measurable unitary representation of G on a complex Hilbert space H. Then for each $x \in H$ the following conditions are equivalent:*

(i) $\text{weak-}\lim_{t\to\infty} T_t x = x_0$;

(ii) $\lim_n \int T_t x \, d\mu_n(t) = x_0$ *for every sequence* $(\mu_n) \in \mathfrak{U}$;

(iii) $\text{weak-}\lim_n \int T_t x \, d\mu_n(t) = x_0$ *for every sequence* $(\mu_n) \in \mathfrak{U}$ *for which the* μ_n *are probability measures with finite supports.*

Proof. (i) ⇒ (ii). By a standard argument, x can be represented as a sum $x = x_0 + x_1$ such that $T_t x_1$ weakly converges to 0 as $t \to \infty$. Since x_0 is fixed under $(T_t)_{t\in G}$, the integrals

$$\int T_t x_0 \, d\mu_n(t) = \mu_n(G) x_0$$

norm converge to x_0 by (2). Therefore, we may assume that $x = x_1$ or, equivalently, $x_0 = 0$.

We have

$$\begin{aligned} \left\| \int T_t x \, d\mu_n(t) \right\|^2 &= \left(\int T_s x \, d\mu_n(s) \Big| \int T_t x \, d\mu_n(t) \right) \\ &= \int \left(T_s x \Big| \int T_t x \, d\mu_n(t) \right) d\mu_n(s) \\ &= \int d\mu_n(s) \int (T_s x \mid T_t x) \, d\mu_n(t) \\ &= \int d\mu_n(s) \int (x \mid T_{s^{-1}t} x) \, d\mu_n(t). \end{aligned}$$

Let $\varepsilon > 0$ be given. By our assumption, there exists a compact subset $K \subset G$ such that $|(x \mid T_u x)| < \varepsilon$ whenever $u \in G \setminus K$. By (3), there exists a positive integer N such that

$$\sup_{t\in G} |\mu_n|(tK) < \varepsilon \quad \text{for } n \geqslant N.$$

Also, by (1), $\|\mu_n\| \leqslant M < \infty$ for $n \geqslant 1$. Therefore, by splitting the second integral into two parts, we obtain

$$\left| \int (x \mid T_{s^{-1}t} x) \, d\mu_n(t) \right| \leqslant \int_{sK} |(x \mid T_{s^{-1}t} x)| \, d\,|\mu_n|(t) + \int \varepsilon \, d|\mu_n|(t) \leqslant \|x\|^2 \varepsilon + M\varepsilon$$

whenever $n \geqslant N$. This implies

$$\left\| \int T_t x \, d\mu_n(t) \right\|^2 \leqslant (M\|x\|^2 + M^2)\varepsilon \quad \text{for } n \geqslant N,$$

whence

$$\lim_n \int T_t x \, d\mu_n(t) = 0.$$

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i). Our argument is based on Fong [6]. If (i) fails, then necessarily $\text{weak-}\lim_{t\to\infty} T_t x$ does not exist, whence for some $y \in H$ the complex-valued function $h(t) = (T_t x \mid y)$ diverges as $t \to \infty$. Since $h(t)$ is bounded, there exist two disjoint discs A and B in the complex plane, such that ∞ is a cluster point of both $h^{-1}(A)$ and $h^{-1}(B)$. By Lemma 1, we can find two sequences (s_n) and (t_n) in $h^{-1}(A)$ and $h^{-1}(B)$, respectively, such that for every compact subset $K \subset G$ there exists a natural number $n(K)$ with

$$|\{n\colon\ s_n \in tK\}| \leqslant n(K) \quad \text{and} \quad |\{n\colon\ t_n \in tK\}| \leqslant n(K)$$

for all $t \in G$. Now we form two sequences (λ_n) and (ν_n) of Radon probability measures with finite supports on G by putting

$$\lambda_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{s_i} \quad \text{and} \quad \nu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{t_i}.$$

For K and $n(K)$ as above we have

$$\lambda_n(tK) \leqslant \frac{n(K)}{n} \quad \text{and} \quad \nu_n(tK) \leqslant \frac{n(K)}{n}$$

for every $t \in G$. Therefore, (λ_n) and (ν_n) satisfy (3) and are in $\mathfrak{U}$. Thus the sequence $(\mu_n) = (\lambda_1, \nu_1, \lambda_2, \nu_2, \ldots)$ is also in $\mathfrak{U}$. Since A and B are convex and closed, we have $\int h\,d\lambda_n \in A$ and $\int h\,d\nu_n \in B$, whence $\left(\int T_t x\,d\mu_n(t)\,|\,y\right)$ diverges as $n \to \infty$. This contradiction concludes the proof of the theorem.

Now we give a few examples of unitary representations which either satisfy or fail to satisfy (i) of Theorem 1. Example 1 is the classical case of strong mixing.

Example 1. Let G be the group of integers, (X, Σ, μ) any probability space, and τ an invertible, bimeasurable, measure-preserving, and strongly mixing transformation of X. By the definition of strong mixing, the unitary representation

$$T_t f(x) = f\big(\tau^{-t}(x)\big)$$

of G on $L^2(\mu)$ satisfies

$$\lim_{t\to\infty}(T_t f\,|\,g) = \int f\,d\mu \int \bar{g}\,d\mu$$

for any characteristic functions $f, g \in L^2(\mu)$. By equicontinuity,

$$\operatorname*{weak-lim}_{t\to\infty} T_t f = \int f\,d\mu \quad \text{for every } f \in L^2(\mu).$$

Example 2. Let G be a locally compact non-compact group, (X, Σ, μ) a probability space, and A an index set. For every $a \in A$ put $X_a = X$ and denote by $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu})$ the product probability space $\prod_{a\in A} X_a$. Further, we assume that to each $t \in G$ there correspond a permutation ϱ_t of A and an invertible, bimeasurable, measure-preserving transformation σ_t of X such that $\varrho_s\varrho_t = \varrho_{st}$ and $\sigma_s\sigma_t = \sigma_{st}$ hold for all $s, t \in G$.

Assume, in addition, that for every $\alpha, \beta \in A$ the set $\{t \in G\colon \varrho_t(\alpha) = \beta\}$ is relatively compact. (It is easy to see that under this assumption the set A is infinite; this is, e.g., the case where $A = G/G_0$ for a compact subgroup $G_0 \subset G$, and $t \to \varrho_t$ is the canonical action of G on the quotient space.) The formula

$$\big(\tau_t(x)\big)(\alpha) = \sigma_t\big(x\,(\varrho_t^{-1}(\alpha))\big)$$

defines an action $t \to \tau_t$ of G on $\tilde{X}$. Clearly, all transformations τ_t are measure-preserving and invertible, and $\tau_s\tau_t = \tau_{st}$ holds for any $s, t \in G$. Therefore,

$$T_t f(x) = f\big(\tau_{t^{-1}}(x)\big)$$

is a unitary representation of G on $L^2(\tilde{\mu})$. Let $\tilde{B}$ and $\tilde{C}$ be two cylinders in $\tilde{\Sigma}$ with bases in finite-dimensional product spaces $\prod_{\alpha\in B} X_\alpha$ and $\prod_{\alpha\in C} X_\alpha$, respectively. By assumption, there exists a compact subset $K \subset G$ such that $\varrho_t(B) \cap C = \emptyset$ for every $t \in G\setminus K$. Therefore, the cylinders $\tau_t(\tilde{B})$ and $\tilde{C}$ are stochastically independent and we have

$$(T_t\chi_{\tilde{B}}\,|\,\chi_{\tilde{C}}) = \tilde{\mu}\big(\tau_t(\tilde{B})\cap\tilde{C}\big) = \tilde{\mu}(\tilde{B})\tilde{\mu}(\tilde{C})$$

whenever $t \in G\setminus K$. Since the characteristic functions of cylinders form a linearly dense subset of $L^2(\tilde{\mu})$, we have

$$\lim_{t\to\infty}(T_t f\,|\,g) = \int f\,d\tilde{\mu}\int \bar{g}\,d\tilde{\mu} \quad \text{for all } f, g \in L^2(\tilde{\mu})$$

or, equivalently,

$$\text{weak-}\lim_{t\to\infty} T_t f = \int f\,d\tilde{\mu} \quad \text{for every } f \in L^2(\tilde{\mu}).$$

Let us specify two simple special cases of Example 2:

(a) If G is the group of integers, $A = G$, and $\varrho_t(\alpha) = \alpha+t$, $\sigma_t(x) = x$ ($t \in G$, $\alpha \in A$, $x \in X$), then we obtain the classical bilateral shift.

(b) X is an infinite compact group with normalized Haar measure μ, G is the group X endowed with discrete topology, $A = G$, and $\varrho_t(\alpha) = t\alpha$, $\sigma_t(x) = tx$ ($t \in G$, $\alpha \in A$, $x \in X$).

Example 3. Let G be a locally compact non-compact group with left Haar measure dt and let $t \to T_t$ be the left regular representation of G on $L^2(G)$, i.e.,

$$T_t f(s) = f(t^{-1}s) \quad (s, t \in G).$$

For two compact sets K and L in G we have $tK \cap L = \emptyset$, whence $(T_t\chi_K\,|\,\chi_L) = 0$ whenever $t \in G\setminus(LK^{-1})$. Therefore, by equicontinuity,

$$\text{weak-}\lim_{t\to\infty} T_t f = 0 \quad \text{for every } f \in L^2(G).$$

Example 4. Let A be a Hermitian operator on a complex Hilbert space H and let G denote the group of reals. Then A induces a continuous unitary representation $t \to T_t = e^{itA}$ of G on H. If A has an eigenvector x pertaining to a non-zero eigenvalue λ (e.g., if A is non-zero and compact), then $T_t x = e^{it\lambda}x$ and, clearly, weak-$\lim_{t\to\infty} T_t x$ does not exist.

4. Groups of point transformations. In terms of ergodic theory, Examples 1 and 2 of the preceding section can be viewed as strongly mixing flows on probability spaces. Let us examine this case closer.

We say that a locally compact group G *acts measurably* on the probability space (X, Σ, μ) if to each $t \in G$ there corresponds an invertible, bimeasurable, measure-preserving transformation τ_t of X such that

(a) $\tau_s \tau_t = \tau_{st}$ holds throughout,

(b) the real function $t \to \mu(\tau_t(A) \cap B)$ is Borel measurable on G for all $A, B \in \Sigma$.

It is easy to see that (b) is implied by the joint measurability of the mapping $(t, x) \to \tau_t(x)$.

For every $1 \leqslant p < \infty$ the measurable action $t \to \tau_t$ of G on X induces an isometric representation $t \to T_t$ on (complex or real) $L^p(\mu)$ by $T_t f(x) = f(\tau_{t^{-1}}(x))$. By (b), $t \to T_t$ is a measurable representation. Clearly, the operators T_t are positive (in the Banach lattice sense) and satisfy $T_t 1 = 1$, where 1 denotes the constant-one function on X. Therefore, for every bounded function $f \in L^p(\mu)$, the orbit O_f is order bounded in the Banach lattice $L^p(\mu)$, hence relatively weakly compact ([21], II, 5.10 f). Thus, the integral $\int T_t d\nu(t)$ exists in $\mathscr{L}(L^p(\mu))$ for any finite signed Radon measure ν on G (Section 2).

As an application of Theorem 1 we obtain the following extension of the Blum-Hanson theorem. (By $\mu \otimes 1$ we denote the one-dimensional projection $f \to \int f d\mu$ of $L^p(\mu)$ onto the constants.)

COROLLARY. *Let a locally compact non-compact group G act measurably on a probability space (X, Σ, μ). With τ_t and T_t as above, the following conditions are equivalent for every $1 \leqslant p < \infty$:*

(o) $\lim_{t\to\infty} \mu(\tau_t(A) \cap B) = \mu(A)\mu(B)$ *for all* $A, B \in \Sigma$;

(i) $\lim_{t\to\infty} T_t = \mu \otimes 1$ *in* $\mathscr{L}_\sigma(L^p(\mu))$;

(ii) $\lim_n \int T_t d\mu_n(t) = \mu \otimes 1$ *in* $\mathscr{L}_s(L^p(\mu))$ *for every* $(\mu_n) \in \mathfrak{U}$;

(iii) $\lim_n \int T_t d\mu_n(t) = \mu \otimes 1$ *in* $\mathscr{L}_\sigma(L^p(\mu))$ *for every* $(\mu_n) \in \mathfrak{U}$ *for which all* μ_n *are probability measures with finite supports.*

Proof. Considering the linearly dense subset of characteristic functions in $L^p(\mu)$, we infer that (i) $\Rightarrow$ (o) is trivial and (o) $\Rightarrow$ (i) follows from equicontinuity. (iii) $\Rightarrow$ (i) can be proved in exactly the same way as in Theorem 1. The implication (ii) $\Rightarrow$ (iii) is trivial.

We prove (i) $\Rightarrow$ (ii). In the case of $p = 2$, the implication follows directly from Theorem 1. In the general case, by equicontinuity, it suffices to prove

$$\lim_n \int T_t f d\mu_n(t) = \int f d\mu$$

in $L^p(\mu)$ for every $0 \leqslant f \leqslant 1$. If $\|\mu_n\| \leqslant M$, then for every such function f we have

$$\left|\int T_t f d\mu_n(t)\right| \leqslant M,$$

whence the set $\{\int T_t f d\mu_n(t)\colon n \geqslant 1\}$ is order bounded in $L^\infty(\mu)$. Since on order bounded sets the L^p-topologies for $1 \leqslant p < \infty$ coincide ([21], V, 8.3), we have the required L^p-convergence by (ii) in Theorem 1.

Remark 1. By the proof of the Corollary, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) holds for any measurable isometric representation $t \to T_t$ of G on $L^p(\mu)$ $(1 \leqslant p < \infty)$ satisfying $T_t \geqslant 0$ and $T_t 1 = 1$ for $t \in G$. On the other hand, if a measurable isometric representation satisfies the two conditions above and if, in addition, (X, Σ, μ) is a standard Borel space, then, by Theorem 2 of [9], for every $t \in G$ there exists a measure-preserving transformation $\tau_{t^{-1}}$ of X such that $T_t f(x) = f(\tau_{t^{-1}}(x))$ for every $f \in L^p(\mu)$ (cf. also [12], Theorem 3.1). By the essential uniqueness of $\tau_{t^{-1}}$, we have $\tau_s(\tau_t(x)) = \tau_{st}(x)$ almost everywhere on X for all $s, t \in G$ (the exceptional set of measure zero depends on s and t). If, moreover, G is second countable, then, by a slight modification [1] of Mackey's Theorem 1 in [14], the transformations τ_s can be chosen so that $\tau_s \tau_t = \tau_{st}$ holds throughout and the mapping $(t, x) \to \tau_t(x)$ is jointly measurable. Thus, under the assumptions above on X and G, the representation $t \to T_t$ is induced by a measurable action of G on X.

5. Order contractive groups in Banach lattices. We can generalize the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) in the Corollary by considering arbitrary Banach lattices with order continuous norm. The method is due to Nagel [15] who also introduced the concept of order contraction (cf. [21], V, §8).

Let E be a (complex or real) Banach lattice with order continuous norm. According to [21], V, 8.1, an equicontinuous (semi-) group $(T_t)_{t \in G}$ of operators on E, each having a modulus, is called *order contractive* if

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(1) By using either Satz 1 in [16] or Theorem 6.1 in [22], and the fact that every uncountable standard Borel probability space contains a null set of cardinality continuum (see footnote 15 in [16]), we can carry the action of G from a standard Borel space S into X.

there exist a quasi-interior element $u \in E_+$ and a strictly positive linear form μ on E such that $|T_t|u \leqslant u$ and $|T_t|'\mu \leqslant \mu$ for all $t \in G$. In the like manner, we say that a representation $t \to T_t$ of the group G on E is *order contractive* if each T_t has a modulus and $(T_t)_{t \in G}$ is an order contractive group of operators. Every order interval in E is relatively weakly compact ([21], II, 5.10 and 5.12), therefore, if $t \to T_t$ is a measurable order contractive representation and ν is a finite signed Radon measure on G, then $\int T_t x \, d\nu(t)$ is in E for every x from the norm dense principal ideal $E_u \subset E$ generated by u. Thus, by Section 2, the integral $\int T_t d\nu(t)$ exists in $\mathscr{L}(E)$.

The following result corresponds to Nagel's Theorem 5.2 in [15]:

THEOREM 2. *Let $t \to T_t$ be a measurable order contractive representation of a locally compact non-compact group G on a (complex or real) Banach lattice E with order continuous norm. Then the following conditions are equivalent:*

(i) $\lim_{t \to \infty} T_t = P$ *in* $\mathscr{L}_\sigma(E)$;

(ii) $\lim_n \int T_t d\mu_n(t) = P$ *in* $\mathscr{L}_s(E)$ *for every* $(\mu_n) \in \mathfrak{U}$;

(iii) $\lim_n \int T_t d\mu_n(t) = P$ *in* $\mathscr{L}_\sigma(E)$ *for every* $(\mu_n) \in \mathfrak{U}$ *for which all* μ_n *are probability measures with finite supports.*

Proof. For (iii) $\Rightarrow$ (i) see Theorem 1, (ii) $\Rightarrow$ (iii) is trivial.

We prove (i) $\Rightarrow$ (ii). Our argument is modelled on [21], V, 8.4. There exists a compact space K (the structure space of E) such that μ can be viewed as an order continuous finite Radon measure on K, u as the constant-one function, and the principal ideal E_u generated by u can be identified with the Banach lattice $C(K)$. Moreover, $C(K) \subset E \subset L^1(\mu)$, both inclusions of dense ideals (see [21], V, 8.4, for details). Now $t \to T_t | C(K)$ is an isometric representation of G on $C(K)$. Dually, each T_t extends to an isometry $\tilde{T}_t$ on $L^1(\mu)$, giving rise to a measurable isometric representation $t \to \tilde{T}_t$ of G on $L^1(\mu)$. Now, by the M. Riesz convexity theorem (see, e.g., [21], V, 8.2), the restrictions $\tilde{T}_t | L^2(\mu)$ yield a measurable unitary representation of G on the complex Hilbert space $L^2(\mu)$. Since, by [21], V, 8.3, both the weak and the norm topologies of E and of $L^2(\mu)$ coincide on order intervals in $C(K)$, we have

$$\lim_n \int T_t x \, d\mu_n(t) = Px$$

in E for every $x \in C(K)$ (Theorem 1). Since $C(K)$ is dense in E, (ii) follows by equicontinuity.

Remark 2. In the definition of the order contractive representation we assumed the equicontinuity of $(T_t)_{t \in G}$. If, however, the representation is weakly continuous (i.e., the function $t \to \langle T_t x, y\rangle$ is continuous for all $x \in E$, $y \in E'$), then the condition

$$\lim_{t\to\infty} T_t = P \quad \text{in } \mathscr{L}_\sigma(E)$$

alone implies that the set $\{T_t\colon t \in G\}$ is relatively compact in $\mathscr{L}_\sigma(E)$, hence equicontinuous. Indeed, for every neighborhood U of P in $\mathscr{L}_\sigma(E)$ there exists a compact subset $L \subset G$ such that $T_t \in U$ whenever $t \in G \setminus L$. The set $\{T_t\colon t \in L\}$ is compact in $\mathscr{L}_\sigma(E)$ as a continuous image of the compact set L. Therefore, by the arbitrariness of U, the set $\{P\} \cup \{T_t\colon t \in G\}$ is compact in $\mathscr{L}_\sigma(E)$.

6. Groups of isometries on Lebesgue spaces. Let E be a (complex or real) AL-space, i.e., a Banach lattice whose norm is additive on the positive cone. The dual Banach lattice E' can be identified with the Banach lattice $C(K)$ of all continuous (complex or real) functions on a compact space K (see [21], II, §9). If $T \in \mathscr{L}(E)$ is an isometry onto E, then the adjoint T' is an isometry from E' onto E'. Therefore, by the classical Banach-Stone theorem ([5], V, 8.8), there exist a homeomorphism φ_T of K and a function $r_T \in C(K)$ with $|r_T| = 1$ such that

$$T'f(x) = r_T(x) f\big(\varphi_T(x)\big)$$

for every $f \in C(K)$ (cf. [12], Theorem 3.1). We have clearly

$$|T'|f(x) = f\big(\varphi_T(x)\big),$$

so $|T'|$ is a Banach lattice automorphism of $C(K)$ and

$$|T'| = \bar{r}_T T'.$$

Since $|T'| = |T|'$ by Lemma 3 in [10] (and its obvious modification in the complex case), we obtain

$$|ST|' = |(ST)'| = |T'S'| = \big|r_T(r_S \circ \varphi_r)|T|'|S|'\big| = |T|'|S|' = (|S||T|)',$$

whence $|ST| = |S||T|$ for any two isometries $S, T \in \mathscr{L}(E)$. Therefore, for any group $(T_t)_{t\in G}$ of isometries of E, $(|T_t|)_{t\in G}$ is a group of Banach lattice automorphisms.

LEMMA 2. *Let $(T_t)_{t\in G}$ be a group of isometries of an AL-space E. Suppose that there exists an element $0 \neq x \in E$ whose orbit O_x is relatively weakly compact. Then there exists a non-zero element $u \in E_+$ with $|T_t|u = u$ for all $t \in G$.*

Proof (cf. [21], V, 8.6). Let B denote the set of all weak cluster points of the relatively weakly compact (by [21], II, 8.8, Corollary) subset $\{|T_t x|\colon t \in G\} \subset E$. Clearly, all elements of B have the same norm $\|x\|$. It is a routine to check that $|T_t|B \subset B$ for every $t \in G$. Consequently, $|T_t|C \subset C$, where C denotes the weak closed (i.e., norm closed) convex hull of B. Since B is relatively weakly compact, C is weakly compact by the Krein-Šmulian theorem. From the preceding discussion it follows that $(|T_t|)_{t\in G}$ acts as a group of isometries on the weakly compact convex set C. Since all elements of C have non-zero norm ($\|x\|$), the assertion follows from the Ryll-Nardzewski fixed point theorem [17].

PROPOSITION. *Let (X, Σ, μ) be a σ-finite measure space, $E = L^1(\mu)$, and let G be a locally compact non-compact group. Then every weakly continuous isometric representation $t \to T_t$ of G on E such that $\lim\limits_{t\to\infty} T_t$ exists in $\mathcal{L}_\sigma(E)$ is order contractive.*

Proof (cf. [21], V, 8.7). Let $v \in E$ be a fixed, everywhere positive function. A set $A \in \Sigma$ will be called *G-invariant* if for every $f \in E_+$ the inclusion

$$\{x\colon f(x) > 0\} \subset A$$

implies

$$\mu\big(\{x\colon |T_t|f(x) > 0\}\setminus A\big) = 0$$

for all $t \in G$. Let $\boldsymbol{J}$ denote the family of all G-invariant sets $J \in \Sigma$ for which there exists a non-zero function u_J, $0 \leqslant u_J \in E$, satisfying both $\{x\colon u_J(x) > 0\} = J$ and $|T_t|u_J = u_J$ for all $t \in G$. Since for every pair $I, J \in \boldsymbol{J}$ we have

$$|T_t|(u_I+u_J) = u_I+u_J \quad \text{and} \quad \{x\colon u_I(x)+u_J(x) > 0\} = I\cup J,$$

the family $\boldsymbol{J}$ is upwards directed. Therefore, the order bounded family $\{v\chi_J\colon J \in \boldsymbol{J}\} \subset E$ is also upwards directed. Since every AL-space is super Dedekind complete, there exists a function v_0, $0 \leqslant v_0 \in E$, such that

$$v_0 = \sup_{J\in\boldsymbol{J}} v\chi_J = \sup_n v\chi_{J_n},$$

where (J_n) is a sequence in $\boldsymbol{J}$ (the suprema being taken in the Banach lattice sense). By letting

$$u = \sum_n 2^{-n} u_{J_n}/\|u_{J_n}\|$$

we obtain $u \in E_+$ and $|T_t|u = u$ for every $t \in G$.

Now the set X decomposes into two G-invariant subsets

$$Y = \{x\colon u(x) > 0\} \quad \text{and} \quad Z = X\setminus Y.$$

Clearly, $\mu(J\cap Z) = 0$ for every $J \in \boldsymbol{J}$. We claim that $\mu(Z) = 0$. To this end denote by ν the restriction of μ to Z and consider the representation $t \to T_t|L^1(\nu)$ of G on $L^1(\nu)$. By Remark 2 (Section 5), the orbits of elements in $L^1(\nu)$ are relatively weakly compact. Thus, by Lemma 2, there are no non-zero elements in $L^1(\nu)$, implying $\mu(Z) = 0$.

We have proved so far that $|T_t|u = u$ $(t \in G)$ for a positive quasi-interior element $u \in E$. Since, clearly, $|T_t|'1 = 1$, the representation $t \to T_t$ is order contractive.

Next we get rid of the assumption of σ-finiteness of μ. The proof of the forthcoming lemma is included here for the sake of completeness.

LEMMA 3 ([5], IV, Exercise 13.54, II). *Let (X, Σ, μ) be a (not necessarily σ-finite) measure space and let F be a relatively weakly compact subset of $L^1(\mu)$. Then there exists a set $A \in \Sigma$ of σ-finite μ-measure such that all functions from F are supported by A.*

Proof. The space $L^1(\mu)$ can be viewed as a closed subspace of $\mathrm{ca}(\Sigma)$, the Banach space of all countably additive measures on Σ with bounded variation. Therefore, each relatively weakly compact subset of $L^1(\mu)$ is relatively weakly compact in $\mathrm{ca}(\Sigma)$. By the Bartle-Dunford-Schwartz theorem ([5], IV, 9.2), there exist a sequence (ν_n) in F and a sequence (a_n) of positive real numbers with $\sum a_n < \infty$ such that every $\nu \in F$ is absolutely continuous with respect to the positive measure

$$\lambda = \sum a_n |\nu_n| \in L^1(\mu).$$

Therefore, every element of F is supported by the σ-finite μ-support of λ.

Our last result corresponds to Theorem 2.1 in [1]:

THEOREM 3. *Let E be a (complex or real) AL-space and let $t \to T_t$ be a weakly continuous isometric representation on E of a locally compact non-compact group G. Then the following conditions are equivalent:*

(i) $\lim\limits_{t\to\infty} T_t = P$ *in* $\mathscr{L}_\sigma(E)$;

(ii) *the integrals* $\int T_t d\mu_n(t)$ *exist and converge in* $\mathscr{L}_s(E)$ *to* P *(as* $n \to \infty$*) for every* $(\mu_n) \in \mathfrak{U}$;

(iii) $\lim \int T_t d\mu_n(t) = P$ *in* $\mathscr{L}_\sigma(E)$ *for every* $(\mu_n) \in \mathfrak{U}$ *for which all* μ_n *are probability measures with finite supports.*

Proof. Since (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are standard, we prove only (i) $\Rightarrow$ (ii). By the Kakutani representation theorem (see, e.g., [21], II, 8.5), E can be identified with $L^1(\mu)$ for some positive Radon measure μ on a locally compact (Hausdorff) space X. By (i) and Remark 2, all orbits O_f $(f \in E)$ are relatively weakly compact, whence, by Lemma 3, each O_f is supported by a set $A(f)$ of σ-finite μ-measure. Let $E(f)$ denote the closed Banach lattice ideal of E generated by O_f, i.e.,

$$E(f) = L^1(\nu_f),$$

where ν_f is the restriction of μ to $A(f)$. The restricted representation $t \to T_t | E(f)$ of G on $L^1(\nu_f)$ is, by the Proposition, order contractive. Thus, for every finite signed Radon measure λ on G, the integral $\int T_t g\, d\lambda(t)$ is in $E(f) \subset E$ for every g from a dense subset of $E(f)$ (see the discussion preceding Theorem 2). Since f is arbitrary, the integral is in E for every g from a dense subset $E_0 \subset E$. Therefore, the integrals $\int T_t d\mu_n(t)$ exist in $\mathscr{L}(E)$. By equicontinuity, it suffices to prove the convergence in (ii) for individual relatively compact orbits O_g, $g \in E_0$. Thus the result follows from Lemma 3, the Proposition, and Theorem 2.

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