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RECTILINEARLY AND RECTIFIABLY AMBIGUOUS POINTS OF A FUNCTION HARMONIC INSIDE A SPHERE

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Denote the Cartesian coordinates of a point in three-dimensional Euclidean space by x, y, z, and set

$$S = \{(x, y, z):\ x^2+y^2+z^2 < 1\},$$
$$T = \{(x, y, z):\ x^2+y^2+z^2 = 1\}.$$

Suppose that f is a single-valued, real-valued function defined for every point $P \in S$. A point $Q \in T$ is called an *ambiguous point of* f, if there exist Jordan arcs J_1 and J_2 that lie in S except for their common end point Q, on which the limits

$$\lim_{\substack{P \to Q \\ P \in J_1}} f(P) \quad \text{and} \quad \lim_{\substack{P \to Q \\ P \in J_2}} f(P)$$

exist and are unequal. Such arcs J_1 and J_2 are called *arcs of ambiguity of* f at Q. If f has a pair of rectilinear arcs of ambiguity at Q, then Q is called a *rectilinearly ambiguous point of* f.

THEOREM. *There exists a harmonic function* $h(P)$ $(P \in S)$ *and an everywhere dense subset* D *of* T *with* $|D| = 2^{\aleph_0}$ *such that every point of* D *is a rectilinearly ambiguous point of* h *and every point* $Q \in T \setminus D$ *is an ambiguous point of* h *with arcs of ambiguity* J_1^Q *and* J_2^Q *at* Q *such that* J_1^Q *is a rectilinear segment and* J_2^Q *is a rectifiable Jordan arc.*

Proof. To simplify the description of our construction, it is convenient to work initially with the cube

$$A = \{(x, y, z):\ 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1\}$$

instead of S, and confine our attention to the face

$$F = \{(x, y, 0): 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$$

instead of T. In the final stages of the proof, we shall suppose that the construction has been carried out for S and T, which entails no conceptual difficulty.

We form two sets, G_1 and G_2, in A.

To define G_1, consider the square (interior and boundary) W with vertices $(\frac{1}{8}, \frac{1}{8}, \frac{7}{8})$, $(\frac{7}{8}, \frac{1}{8}, \frac{7}{8})$, $(\frac{7}{8}, \frac{7}{8}, \frac{7}{8})$ and $(\frac{1}{8}, \frac{7}{8}, \frac{7}{8})$. Construct (see, e.g., [4, p. 135]) the familiar perfect nowhere dense subset V of W by first dividing W into nine equal squares, retaining the four at the corners of W, and labeling these four W_1, W_2, W_3, W_4 in the same order as the corresponding vertices of W were listed above. Then divide each W_j ($j = 1, 2, 3, 4$) into nine equal squares, retain the four at the corners of W_j, and label these four W_{j1}, W_{j2}, W_{j3}, W_{j4}, in the same order as before. Continuing in this way, we obtain for every sequence $k_1, \ldots, k_n$, where each k_m ($m = 1, 2, \ldots, n$) is one of the numbers 1, 2, 3, 4, a square $W_{k_1 \ldots k_n}$. Then the set V is the set of all points of the form

$$W_{k_1} \cap W_{k_1 k_2} \cap \ldots \cap W_{k_1 k_2 \ldots k_n} \cap \ldots \tag{1}$$

where $k_1, k_2, \ldots, k_n, \ldots$ is any infinite sequence whose terms belong to the set $\{1, 2, 3, 4\}$. Next we divide the square F into four equal squares F_1, F_2, F_3, F_4, then divide each F_j ($j = 1, 2, 3, 4$) into four equal squares F_{j1}, F_{j2}, F_{j3}, F_{j4}, and so on, each time labeling them in the same order as the corresponding subsquares of W. Join the point (1) of the set V to the point

$$F_{k_1} \cap F_{k_1 k_2} \cap \ldots \cap F_{k_1 k_2 \ldots k_n} \cap \ldots \tag{2}$$

of the square F by a rectilinear segment. This segment shall contain the point (1) but not the point (2). Define G_1 to be the union of all such segments.

The construction of G_2 takes place in the truncated pyramid M whose base is F and whose upper base is W. The interior and boundary of the pyramid, with the exception of the base F, are regarded as belonging to M. Let

$$\tfrac{7}{8} > s_2 > s_3 > \ldots > s_n > s_{n+1} > \ldots > 0, \qquad \lim_{n \to \infty} s_n = 0.$$

For each $n = 1, 2, 3, \ldots$ we define 2^n planes. For $n = 1$, each of the two planes is determined by the center of the square W and one of the two lines dividing the square F into the four equal squares F_j ($j = 1, 2, 3, 4$), where we consider only that part of the plane that belongs to M. For $n > 1$, each of the 2^n planes is determined by the center of a square $W_{k_1 \ldots k_{n-1}}$ and one of the two lines dividing the square $F_{k_1 \ldots k_{n-1}}$ into the four equal squares

$F_{k_1 \ldots k_{n-1} j}$ $(j = 1, 2, 3, 4)$, where we consider only that part of the plane lying in the truncated pyramid

$$M \cap \{(x, y, z): 0 < z \leqslant s_n\}.$$

Define G_2 to be the union of all such parts of planes.

It is evident from the construction of G_1 and G_2 that $G_1 \cap G_2 = \emptyset$. Let D be the union of the segments used above to divide F, each F_{k_1}, each $F_{k_1 k_2}, \ldots$, each $F_{k_1 k_2 \ldots k_{n-1}}, \ldots$ into four equal squares. Then clearly $|D| = 2^{\aleph_0}$ and D is an everywhere dense subset of F. If $Q \in D$, then there is a rectilinear segment at Q belonging to G_1; and there is a plane, and hence a rectilinear segment, at Q belonging to G_2. If $Q \in F \setminus D$, then there is a rectilinear segment at Q belonging to G_1; and there is a simple polygonal rectifiable arc at Q that is a subset of G_2.

Turning now to S and T, we may assume that we have constructed two sets, G_1 and G_2, in S, where G_1 is the union of rectilinear segments and G_2 is the union of parts of planes, and an everywhere dense subset D of T, for which the assertions in the preceding paragraph hold.

Let $f(P)$ $(P \in S)$ be a real-valued continuous function mapping S onto the unit interval in such a way that $f(G_1) = 0$ and $f(G_2) = 1$. Set $G = G_1 \cup G_2$.

Let

$$0 < r_0 < r_1 < \ldots < r_n < r_{n+1} < \ldots < 1, \qquad \lim_{n \to \infty} r_n = 1,$$

and, for $n = 0, 1, 2, \ldots,$ set

$$S_n = \{(x, y, z): x^2 + y^2 + z^2 < r_n^2\},$$
$$T_n = \{(x, y, z): x^2 + y^2 + z^2 = r_n^2\},$$
$$K_n = (S_n \cup T_n \cup G) \cap (S_{n+1} \cup T_{n+1}).$$

The next step is potential-theoretical. It is clear that K_n is a compact set; denote its complement by CK_n. Consider any boundary point of K_n (such a boundary point is also a boundary point of CK_n). A sphere with this point as center and radius ϱ contains on its surface a continuum belonging to CK_n with diameter greater than ϱ, as is evident from the construction of G. It follows [5, p. 294, Theorem 5.4′] that every boundary point of K_n is a regular point of CK_n. Consequently ([5, p. 308, Theorem 5.10]) CK_n is not thin at any boundary point, and so ([2, p. 60]) K_n has no unstable boundary point. Therefore ([3]) any continuous function on K_n that is harmonic at every interior point of K_n can be uniformly approximated on K_n as closely as desired by a harmonic polynomial.

With this in hand, it is now possible to construct, by a method like that employed in [1, pp. 153–154], a harmonic function $h(P)$ $(P \in S)$ such that

$$\lim_{\substack{P \to T \\ P \in G}} [h(P) - f(P)] = 0,$$

which implies our theorem. □

Remark. It would be interesting to know if there exists a harmonic function $h(P)$ $(P \in S)$ such that *every* point of T is a rectilinearly ambiguous point of h.

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