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INDECOMPOSABLE $Z_p[G]$-LATTICES FOR A CLASS OF METABELIAN GROUPS

BY

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1. Introduction. Let Q_p be the field of p-adic numbers and Z_p its ring of integers. In this paper we determine all finitely generated and Z_p-torsion-free $Z_p[G]$-modules for a class of metabelian groups G (Theorem 2.4). This generalizes Théorème II.5 of [3]. An application of this result to the study of the Galois structure of unit groups in certain real algebraic number fields will be given elsewhere (see [4]).

In the sequel G will denote a non-abelian group of order pm which is a non-trivial semidirect product of the cyclic group S of order p by an abelian group T of order m such that the order of every element of T divides $p-1$. It is known (see [3], Proposition 1.3) that every such group G is uniquely determined by S, T and a non-trivial p-adic character χ of T. We have then $tst^{-1} = s^{\chi(t)}$ for $s \in S$, $t \in T$.

We shall apply the method of Rosen [7] which utilizes skew group rings. The same approach was earlier used by Pu [6] for classifying integral representations of metacyclic groups of order pq (with prime p, q).

Let R be an integral domain and let Λ be an R-algebra. A left Λ-module which is finitely generated and projective as an R-module will be called a *Λ-lattice*. Observe that in the case of a Dedekind ring R every finitely generated and torsion-free module is projective.

If Γ is a finite group, R is a ring and $\mathrm{Aut}(R)$ denotes the group of all automorphisms of R, then the skew group ring $R * \Gamma$ is defined as a free left R-module with Γ serving as a system of free generators, in which multiplication is defined by putting

$$rx \cdot sy = r\Psi_x(s)xy \tag{1.1}$$

(for $x, y \in \Gamma$, $r, s \in R$), where Ψ is a fixed homomorphism of G into $\mathrm{Aut}(R)$. The image of x under Ψ will be denoted by Ψ_x.

Let M, N be $R[\Gamma]$-modules. We treat $\mathrm{Hom}_R(N, M)$ as an $R[\Gamma]$-module with the action of Γ defined by

$$(xf)(m) = xf(x^{-1}m) \tag{1.2}$$

for $x \in \Gamma$, $m \in N$ and $f \in \mathrm{Hom}_R(N, M)$.

Every cocycle $F \in Z^1(\Gamma, \mathrm{Hom}_R(N, M))$ (i.e. a map $\Gamma \to \mathrm{Hom}_R(N, M)$ satisfying $F_{xy} = xF_y + F_x$ for all $x, y \in \Gamma$) defines an $R[\Gamma]$-extension of M by N (treated both as $R[\Gamma]$-modules), which is defined as the R-direct sum $M \oplus N$ on which Γ acts by

$$x(m, n) = \big(xm + F_x(n), xn\big) \quad (m \in M, n \in N, x \in \Gamma).$$

We shall denote this extension by $(M, N; F)$, and in the cases where the choice of F is obvious we write simply (M, N).

We shall use the following two lemmas:

(1.3) LEMMA ([5], Lemme III.2). *Let Γ be a finite group, Δ its normal subgroup and M an $R[\Gamma]$-module such that $M^\Delta = 0$. If we define the action of Γ/Δ on $H^1(\Delta, M)$ by the formula*

$$(x\Delta)F_g = x^{-1}F_{xgx^{-1}} \quad \textit{for } g \in \Delta,\ x \in \Gamma \textit{ and } F \in Z^1(\Delta, M),$$

then the groups $H^1(\Gamma, M)$ and $H^1(\Delta, M)^{\Gamma/\Delta}$ are isomorphic.

(1.4) LEMMA ([1], Corollary (3.45)). *Let M, N be $R[\Gamma]$-lattices and let F, F' be cocycles in $Z^1(\Gamma, \mathrm{Hom}_R(N, M))$. Then the extensions $(M, N; F)$ and $(M, N; F')$ are $R[\Gamma]$-isomorphic if and only if there exist*

$$a \in \mathrm{Aut}_{R[\Gamma]}(M), \quad b \in \mathrm{Aut}_{R[\Gamma]}(N), \quad c \in \mathrm{Hom}_R(N, M)$$

such that for all $x \in \Gamma$

$$aF_x(xm) - F'_x(xbm) = xc(m) - c(xm).$$

We shall consider skew group-rings of the form $R * T$, where R is either $\boldsymbol{Z}_p[\xi]$ or $\boldsymbol{Q}_p(\xi)$, and the homomorphism Ψ in (1.1) is defined by $\Psi_t(\xi) = \xi^{\chi(t)}$ for any $t \in T$, where ξ is a fixed primitive p-th root of unity.

Some further notation is needed. We put:

$$L = \boldsymbol{Q}_p(\xi), \quad R_L = \boldsymbol{Z}_p[\xi], \quad T_0 = \mathrm{Ker}\,\chi, \quad n_0 = \# T_0, \quad n = m/n_0,$$

t_1 is a fixed representative of the generating coset in T/T_0, and $P = (1-\xi)R_L$.

$\hat{H}$ will denote the character group of an abelian group H. For any $X \in \hat{T}_0$ we choose a character $\bar{X} \in \hat{T}$, whose restriction to T_0 coincides with X, and by χ_1 and $\bar{\chi}_1$ we denote the principal characters of T_0 and T, respectively. Finally, for any $X \in \hat{T}$ we set

$$e_X = \big(\sum_{t \in T} X(t^{-1})\,t\big)/m.$$

2. Indecomposable lattices over the skew group-ring $R_L * T$. Let

$$T = \bigcup_{i=0}^{n-1} T_0 t_1^i$$

be the decomposition of T into disjoint cosets with respect to T_0.

(2.1) PROPOSITION. *The left ideals $Le_{\bar{X}}$ (where $X \in \hat{T}_0$) form a complete set of simple $L * T$-modules.*

Proof. Since $L * T$ is obviously semi-simple, it suffices to show that all minimal ideals of $L * T$ are of the asserted form. First of all observe that

$$L * T = \bigoplus_{X \in \hat{T}} Le_X.$$

Since all ideals Le_X are minimal, it remains to show that for any X, Y in $\hat{T}$ which coincide on T_0 we have $Le_X \simeq Le_Y$. Indeed, under this condition the element $X(t_1)/Y(t_1)$ of $\boldsymbol{Q}_p$ is an n-th root of unity, whence its norm from $\boldsymbol{Q}_p$ to $\boldsymbol{Q}_p(\xi)$ is 1. Using Hilbert's Theorem 90 we can write (with a suitable $c \in \boldsymbol{Q}_p(\xi)$)

$$X(t_1)/Y(t_1) = \Psi_{t_1}(c)/c,$$

and this allows us to construct an $L * T$-isomorphism of Le_X onto Le_Y mapping ae_X onto ace_Y for $a \in L$.

We need two lemmas:

(2.2) LEMMA. *The ring $R_L * T$ is hereditary.*

Proof. Since the extension L/L^T is tamely ramified, the trace map $\mathrm{Tr} = \mathrm{Tr}_{L/L^T}$ on integers is surjective, and hence there exists $s_0 \in \boldsymbol{Z}_p[\xi]$ such that

$$\mathrm{Tr}(s_0) = \sum_{t \in T} \Psi_t(s_0) = n_0.$$

If now M is a left ideal of $R_L * T$, then it is also an R_L-submodule of the R_L-lattice $R_L * T$. Therefore it is R_L-projective. Now let $F\colon N \to M$ be an $R_L * T$-surjection of an $R_L * T$-module N on M. Then there exists $a \in \mathrm{Hom}_{R_L}(M, N)$ which splits F, i.e. $Fa = 1_M$. Put

$$a' = \Big(\sum_{t \in T} t s_0 a t^{-1}\Big)/n_0 \in \mathrm{Hom}_{R_L * T}(M, N).$$

Then

$$Fa' = \Big(\sum_{t \in T} t s_0 F a t^{-1}\Big)/n_0 = \Big(\sum_{t \in T} \Psi_t(s_0)\Big)/n_0 = 1,$$

whence a' splits F, and this shows that M is $R_L * T$-projective.

(2.3) LEMMA ([1], 26.12 (ii)). *Let R be a Dedekind ring and assume that the skew group-ring $R * T$ is hereditary. Then an $R * T$-lattice M is indecomposable if and only if $K \otimes M$ is a simple $K * T$-module, where K denotes the quotient field of R^T (T acting on R by right multiplication, as defined in* (1.1)).

Now we can prove the main result of this section:

(2.4) THEOREM. *Every indecomposable R_L-lattice is isomorphic to $P^j e_{\bar{X}}$ with a suitable $0 \leqslant j < n$ and $X \in \hat{T}_0$.*

Proof. From Proposition 2.1 and Lemmas 2.2 and 2.3 it follows that M is an indecomposable $R_L * T$-lattice if and only if $L \otimes M = Le_{\bar{X}}$ with a suitable $X \in \hat{T}_0$.

The module M has R_L-rank one, hence it is isomorphic to R_L as an R_L-module. Consequently, in view of $M \subset L * T$ we arrive at $M = R_L x e_{\bar{X}}$ with a suitable x in L, and this is of the asserted form.

3. Indecomposable $Z_p[G]$-lattices.

(3.1) PROPOSITION. *For $X \in \hat{T}$, $Y \in \hat{T}_0$ and $j = 0, 1, \ldots, n-1$ let*

$$M = \mathrm{Hom}_{Z_p}(Z_p e_X, P^j e_{\bar{Y}}).$$

Then the group $H^1(G, M)$ is cyclic of p elements if $X = \chi^{j-1}\bar{Y}$, and is zero otherwise.

Proof. Since $M^S = 0$, Lemma 1.3 reduces our task to computing $H^1(S, M)^{G/S}$.

Denote by σ a generator of S and let c be a cocycle in $Z^1(S, M)$. It is determined by $c_\sigma(e_X)$. The fact that the class of c is fixed by G/S is equivalent to the existence of an element v of M such that, for any $t \in T$,

$$t^{-1} c_{t\sigma t^{-1}} - c_\sigma = t^{-1} c_{\sigma^{\chi(t)}} - c_\sigma = (1-\xi) v.$$

Hence, with $y_t = (1-\xi^{\chi(t)})/(1-\xi)$, we have $y_t c_\sigma - t c_\sigma = (1-\xi^{\chi(t)}) t v$ because $c_{\sigma^{\chi(t)}} = y_t c_\sigma$. Since $(tc_\sigma)(x) = tc_\sigma(t^{-1}x)$ for any $x \in Z_p e_X$, and with suitable $a \in R_L$ we have $c_\sigma(e_X) = (1-\xi)^j a e_{\bar{Y}}$, we get

$$y_t(1-\xi)^j a e_{\bar{Y}} - X^{-1}(t)(1-\xi^{\chi(t)})\,\Psi_t(a)\,\bar{Y}(t)\, e_{\bar{Y}} \in P^{j+1} e_{\bar{Y}}.$$

Using $y_t \equiv \chi(t) \pmod P$ and $\Psi_t(a) \equiv a \pmod P$ (for $t \in T$), we infer finally that for all t in T

$$aX(t)\,\bar{Y}^{-1}(t) - a\chi^{j-1}(t) \in P. \tag{3.2}$$

If now $X \neq \bar{Y}\chi^{j-1}$, then a must belong to P, since otherwise we would have $X(t) \equiv \bar{Y}\chi^{j-1} \pmod P$, which gives a contradiction since all n_0-th roots of unity are distinct $(\mathrm{mod}\, P)$. But $a \in P$ implies that c must be the zero cocycle, and hence $H^1(G, M) = 0$ in this case.

If $X = \bar{Y}\chi^{j-1}$, then (3.2) is satisfied by all $a \in R_L$, which implies that $H^1(S, M)^{G/S} = H^1(S, M)$, so it remains to show that the last group has p elements. To do this consider a cocycle f which is determined by the value

$$f_\sigma(e_X) = (1-\xi)^j \alpha e_{\bar{Y}} \quad \text{with } \alpha \in R_L.$$

Write $\alpha = a + h(1-\xi)$ with $0 \leqslant a < p$, $h \in R_L$ and put $g_\sigma(e_X) = (1-\xi)^j a e_{\bar{Y}}$. Then

$$g_\sigma(e_X) - f_\sigma(e_X) = (1-\xi)\, w e_X,$$

where $w \in M$ and $w(e_X) = h(1-\xi)^j$. Thus g and f are equivalent, and so define the same element of $H^1(S, M)$.

If two cocycles f, g are equivalent and

$$f_\sigma(e_X) = (1-\xi)^j a e_{\bar{Y}}, \quad g_\sigma(e_X) = (1-\xi)^j a' e_{\bar{Y}}$$

with $0 \leqslant a$, $a' < p$ and $a \neq a'$, then there exists $b \in M$ such that

$$f_\sigma(e_X) - g_\sigma(e_X) = \sigma b(e_X) - b(\sigma e_X),$$

whence $a \equiv a' \pmod{P}$, thus $a = a'$, a contradiction. This shows that $H^1(S, M)$ has p elements, and so the proposition is established.

(3.3) LEMMA. *For every integer j satisfying $0 \leqslant j < n$ and every Y in $\hat{T}_0$ there exists exactly one (up to $\mathbf{Z}_p[G]$-isomorphism) non-trivial extension of $P^j e_{\bar{Y}}$ by $\mathbf{Z}_p e_{\chi^{j-1}\bar{Y}}$.*

Proof. First we show that all non-trivial extensions are isomorphic. Let M be defined as in Proposition 3.1, with $X = \chi^{j-1}\bar{Y}$. Write

$$f_\sigma(e_X) = (1-\xi)^j a e_{\bar{Y}}, \quad f'_\sigma(e_X) = (1-\xi)^j a' e_{\bar{Y}}$$

with a, $a' \in \mathbf{Z}_p \backslash p\mathbf{Z}_p$. Since a/a' is a p-adic unit, the $\mathbf{Z}_p[G]$-endomorphism $\alpha\colon x \to (a/a')x$ of $P^j e_{\bar{Y}}$ is in fact an automorphism. Since $f_\sigma(e_X) - \alpha f'_\sigma(e_X) = 0$, Lemma 1.4 implies that the extensions defined by f and f' are $\mathbf{Z}_p[G]$-isomorphic.

It remains to prove the existence of a non-trivial extension. Let $f_\sigma(e_X) = (1-\xi)^j e_{\bar{Y}}$. If the extension defined by f were trivial, then there would exist a $\mathbf{Z}_p[G]$-automorphism α of $P_j e_{\bar{Y}}$ and $c \in M$ such that

$$\alpha f_\sigma(\sigma e_X) = \sigma c(e_X) - c(\sigma e_X) = (\xi - 1) c(e_X).$$

Since $c(e_X) \in P^j e_{\bar{Y}}$, this would imply $\alpha((1-\xi)^j e_{\bar{Y}}) \in P^{j+1} e_{\bar{Y}}$, and hence

$$\alpha(P^j e_{\bar{Y}}) \subset P^{j+1} e_{\bar{Y}}.$$

Thus α could not be an automorphism.

Finally we prove our main result, giving a classification of indecomposable $\mathbf{Z}_p[G]$-lattices:

(3.4) THEOREM. *Every indecomposable $\mathbf{Z}_p[G]$-lattice is isomorphic to one of the following modules: $\mathbf{Z}_p e_X$, $P^j e_{\bar{Y}}$ and $(P^j e_{\bar{Y}}, \mathbf{Z}_p e_{\chi^{j-1}\bar{Y}})$, where $X \in \hat{T}$, $Y \in \hat{T}_0$ and $0 \leqslant j < n$.*

Proof. Since $p \nmid [G:S]$, the Lemma in [2] shows that every indecomposable $\mathbf{Z}_p[G]$-lattice is isomorphic to a direct summand of $\mathbf{Z}_p[G] \otimes M$ for a certain indecomposable $\mathbf{Z}_p[S]$-module M. We have three choices for M, namely $\mathbf{Z}_p$, R_L and $\mathbf{Z}_p[S]$.

Observe that

$$\mathbf{Z}_p[G] \simeq \bigoplus_{X \in \hat{T}} \mathbf{Z}_p[S]\, e_X, \tag{3.5}$$

the isomorphism given by the map

$$x \mapsto \sum_{X \in \hat{T}} x e_X \qquad (x \in \boldsymbol{Z}_p[G]).$$

Since the element σ acts trivially on $\boldsymbol{Z}_p$, we have

$$\boldsymbol{Z}_p[S]\, e_X \otimes_{\boldsymbol{Z}_p[S]} \boldsymbol{Z}_p \simeq \boldsymbol{Z}_p e_X;$$

hence

$$\boldsymbol{Z}_p[G] \otimes_{\boldsymbol{Z}_p[S]} \boldsymbol{Z}_p \simeq \bigoplus_{X \in \hat{T}} (\boldsymbol{Z}_p[S]\, e_X \otimes_{\boldsymbol{Z}_p[S]} \boldsymbol{Z}_p) \simeq \bigoplus_{X \in \hat{T}} \boldsymbol{Z}_p e_X.$$

Now consider

$$N = \boldsymbol{Z}_p[G] \otimes_{\boldsymbol{Z}_p[S]} R_L \simeq \bigoplus_{t \in T} (t\boldsymbol{Z}_p[S] \otimes R_L) \simeq \bigoplus_{t \in T} t \otimes R_L.$$

Clearly, the $\boldsymbol{Z}_p$-rank of N equals $(p-1)m$. Since $S = 1+\sigma+\sigma^2+\ldots+\sigma^{p-1}$ annihilates N, it follows that N (as a $\boldsymbol{Z}_p[G]$-module) is an $R_L * T$-lattice, because the rings $\boldsymbol{Z}_p[G]/\tilde{S}\boldsymbol{Z}_p[G]$ and $R_L * T$ are isomorphic. Thus Theorem 2.4 implies

$$N \simeq \bigoplus_{j=0}^{n} \bigoplus_{l=0}^{n_0-1} P^{r_j} e_{\bar{Y}_l}$$

with suitable $0 \leqslant r_j < n$ and $Y_l \in \hat{T}_0$. Since N is $\boldsymbol{Z}_p[G]$-cyclic, the summands here are pairwise non-isomorphic. Hence

$$\boldsymbol{Z}_p[G] \simeq \boldsymbol{Z}_p[G] \otimes_{\boldsymbol{Z}_p[S]} \boldsymbol{Z}_p[S] \simeq \bigoplus_{j=0}^{n-1} \bigoplus_{Y \in \hat{T}_0} P^j e_{\bar{Y}}.$$

In view of (3.5) it remains to decompose the modules $\boldsymbol{Z}_p[S]\, e_X$ for $X \in \hat{T}$. Let $X = \chi^{j-1}\bar{Y}$, where $0 \leqslant j < n$ and $Y \in \hat{T}_0$, and consider the exact sequence of $\boldsymbol{Z}_p[G]$-lattices:

$$0 \to \boldsymbol{Z}_p[S](1-\sigma)e_X \to \boldsymbol{Z}_p[S]\, e_X \to \tilde{S}\boldsymbol{Z}_p e_X \to 0, \tag{3.6}$$

where the last epimorphism is the multiplication by $\tilde{S}$.

Now observe that the first non-zero term of this sequence is isomorphic to $P^j e_{\bar{Y}}$ and the last one to $\boldsymbol{Z}_p e_X$. In fact, the second statement is obvious, and to prove the first one write, for $j = 0, 1, \ldots, n-1$,

$$v_j = \Big(\sum_{t \in T} \chi^j(t^{-1})\, \xi^{\chi(t)}\Big)/m.$$

Using Proposition II.7 of [3] we get $R_L v_j = P^j$ $(j = 0, 1, \ldots, n-1)$. If we put

$$e_\chi(\sigma) = \Big(\sum_{t \in T} \chi(t^{-1})\, \sigma^{\chi(t)}\Big)/m,$$

then by the Corollary (Scolie) to Proposition II.11 in [3] we obtain

$$e_\chi(\sigma) \equiv (1-\sigma) \pmod{(1-\sigma)^2},$$

and thus $\boldsymbol{Z}_p[S]\, e_\chi(\sigma) = \boldsymbol{Z}_p[S](1-\sigma)$.

Now we obtain the asserted isomorphism from

$$\boldsymbol{Z}_p[S]\,(1-\sigma)\, e_{\chi^{j-1}\bar{Y}} = \boldsymbol{Z}_p[S]\, e_\chi(\sigma)\, e_{\chi^{j-1}\bar{Y}}$$

to $\boldsymbol{Z}_p[S]\, v_j e_{\bar{Y}}$ by putting, for $g \in \boldsymbol{Z}_p[X]$,

$$g(\sigma)\, e_\chi(\sigma)\, e_{\chi^{j-1}\bar{Y}} \mapsto g(\xi)\, v_j e_{\bar{Y}}.$$

Now (3.6) implies

$$\boldsymbol{Z}_p[S]\, e_X \simeq (P^j e_{\bar{Y}},\, \boldsymbol{Z}_p e_{\chi^{j-1}\bar{Y}}) \tag{3.7}$$

for any $X \in \hat{T}$ of the form $X = \chi^{j-1}\bar{Y}$.

To complete the proof of the theorem it remains to show that the modules listed in its statement are indeed indecomposable.

The modules $\boldsymbol{Z}_p e_X$ are obviously indecomposable. Every decomposition of $P^j e_{\bar{Y}}$, as a $\boldsymbol{Z}_p[G]$-module, would be also an $R_L * T$-decomposition, which cannot exist by Theorem 2.4.

For any $\boldsymbol{Z}_p[G]$-module M define

$$\tilde{M} = \{m \in M\colon\ \tilde{S}m = 0\}.$$

If we would have $\boldsymbol{Z}_p[S]\, e_X = M_1 \oplus M_2$, then

$$(\boldsymbol{Z}_p[S]\, e_X)^{\sim} = \boldsymbol{Z}_p[S](1-\sigma)\, e_X = \tilde{M}_1 \oplus \tilde{M}_2,$$

and by the indecomposability of $\boldsymbol{Z}_p[S](1-\sigma)\, e_X$ as an $R_L * T$-module we have, say, $\tilde{M}_1 = \boldsymbol{Z}_p[S](1-\sigma)\, e_X$.

Comparing the $\boldsymbol{Z}_p$-ranks we see that if $M_1 \neq \tilde{M}_1$, then $M_1 = \boldsymbol{Z}_p[S]\, e_X$. If, however, $M_1 = \tilde{M}_1$, then $\boldsymbol{Z}_p[S](1-\sigma)\, e_X$ is a direct summand of $\boldsymbol{Z}_p[S]\, e_X$, but this is impossible because $\boldsymbol{Z}_p[S]\, e_X$ is a non-trivial extension of $\boldsymbol{Z}_p[S](1-\sigma)\, e_X$ by $\boldsymbol{Z}_p e_X$. The theorem is thus proved.

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