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METRIZABLE APPROXIMATIONS OF SEMIGROUPS

BY

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Introduction. In [6] Numakura proved the now well-known result that a compact totally disconnected semigroup is the inverse limit of finite discrete semigroups. A later work of Bednarik and Wallace [1] generalizes this result to compact totally disconnected acts. By using uniformity techniques, Hofmann and Mostert prove in their book [5] that any compact semigroup is the inverse limit of compact metric semigroups having subinvariant metrics. Most recently, in a paper involving the methods of universal algebra, Taylor [7] proves that a compact act is the inverse limit of compact metric acts (without, however, the inclusion of any subinvariance).

The general theme in all these results is to write a given semigroup or act as the inverse limit of somewhat less complicated objects, but the techniques vary considerably. Our purpose herein* is to develop a singularly uncomplicated method of forming inverse limits which, besides reproducing all of the above results, shows promise of being extremely valuable in the theory of compact semigroups and acts. Our Theorem III, which says that a compact UDC is the inverse limit of compact metric UDC's, is one example of the value of the techniques. We then conclude with a suggestion as to one possible way in which this method might be employed.

Preliminaries. By a *topological semigroup* we mean a Hausdorff space S together with a jointly continuous, associative multiplication on S. If S does not contain a two-sided identity, S^1 will denote the topological semigroup obtained by adjoining to S a two-sided identity as an isolated point; otherwise, $S^1 = S$. If X is a space, $C(X)$ will denote the algebra of continuous real-valued functions defined on S. All spaces in this paper will be presumed Hausdorff.

Given a topological semigroup T and a space X, an act is a continuous function

$$(t, x) \mapsto t\cdot x\colon\ T\times X \to X$$

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such that $(t_1t_2)\cdot x = t_1\cdot(t_2\cdot x)$ for all $t_1, t_2 \in T$ and all $x\in X$. In this case, T is said *to act on* X, and X is called a *T-space.* If both X and Y are T-spaces, a T-morphism is a continuous function $f: X\to Y$ such that $f(t\cdot x) = t\cdot f(x)$ for all $t\in T$ and all $x\in X$. If f is also a homeomorphism, we shall refer to f as a T-isomorphism and say that X and Y are *T-isomorphic.* If X is a T-space, and T has an identity 1, we shall assume $1\cdot x = x$ for all $x\in X$. If T does not have an identity, we make X into a T^1-space by defining $1\cdot x = x$ for all $x\in X$.

Assume we are given a topological semigroup T. For each a in a directed set D, suppose X_a is a compact T-space, and that for $a\leqslant\beta$, $a, \beta\in D$, we are given a T-morphism $\varphi_a^\beta : X_\beta \to X_a$ such that: (1) $\varphi_a^a = 1_a$ $=$ identity function of X_a and (2) $a\leqslant \beta\leqslant\gamma$, $\varphi_a^\beta\circ\varphi_\beta^\gamma = \varphi_a^\gamma$. We infer that $(X_a, \varphi_a^\beta, D)$ is an inverse system. Denote by $\operatorname{invlim}(X_a, \varphi_a^\beta, D)$ the set $\bigcap\{K(a,\beta)\colon a\leqslant\beta\}$, where $K(a,\beta) = \{x\in\prod\{X_a\colon a\in D\}\colon \varphi_a^\beta(x_\beta) = x_a\}$. Then $\operatorname{invlim}(X_a, \varphi_a^\beta, D)$ is a compact T-space, where for $t\in T$ and $x\in\operatorname{invlim}(X_a,\varphi_a^\beta, D)$ we define $(t\cdot x)_a = t\cdot x_a$ for all $a\in D$.

Section I. Throughout this section, we assume that T is a topological semigroup and X is a compact T-space.

PROPOSITION 1.1. *Given $A\subseteq C(X)$, define a relation $T(A)$ on X as follows*: $T(A) = \{(x,y)\in X\times X\colon f(t\cdot x) = f(t\cdot y)$ *for all* $t\in T^1$ *and* $f\in A\}$. *Then*:

(a) *$T(A)$ is a closed equivalence relation on X.*

(b) *If $A\subseteq B\subseteq C(X)$ then $T(B)\subseteq T(A)$.*

(c) *Denote by $X(A)$ the quotient space of X modulo $T(A)$ and by η_A the natural surmorphism of X onto $X(A)$; then $X(A)$ is a compact T-space and η_A is a T-morphism with respect to the action on $X(A)$ by T.*

Proof. (a) and (b) are clear.

Let $(x,y)\in T(A)$ and $t_0\in T$. For each $f\in A$ and $t\in T^1$ we have $f(t\cdot x) = f(t\cdot y)$. Hence, $f\big(t\cdot(t_0\cdot x)\big) = f\big((tt_0)\cdot x\big) = f\big((tt_0)\cdot y\big) = f\big(t\cdot(t_0\cdot y)\big)$ and we have $(t_0\cdot x, t_0\cdot y)\in T(A)$. It now follows that $\big(t,\eta_A(x)\big)\mapsto\eta_A(t\cdot x)\colon T\times X(A) \doteq X(A)$ is an act and (c) is established.

PROPOSITION 1.2. *Let $A\subseteq C(X)$ and $B\subseteq C(X)$ with $A\subseteq B$. There exists a T-morphism φ_A^B: $X(B)\to X(A)$ of $X(B)$ onto $X(A)$ such that $\varphi_A^B\circ\eta_B = \eta_A$.*

Proof. By proposition 1.1 (b), $T(B)\subseteq T(A)$ and the function φ_A^B is the function induced in the diagram

$$\begin{tikzcd} X(B) \arrow[rr, "\varphi_A^B"] & & X(A) \\ & X \arrow[ul, "\eta_B"] \arrow[ur, "\eta_A"'] & \end{tikzcd}$$

PROPOSITION 1.3. *Let $\mathfrak{A}$ be a family of subsets of $C(X)$ which is directed by inclusion. For $A \in \mathfrak{A}$, let $T(A)$, $X(A)$, and η_A be as defined in Proposition* 1.1. *For $A, B \in \mathfrak{A}$ with $A \subseteq B$, let φ_A^B: $X(B) \to X(A)$ be as defined in Proposition* 1.2. *Then:*

(a) $(X(A), \varphi_A^B, \mathfrak{A})$ *is an inverse system.*

(b) *The function $x \mapsto \bar{x}$: $X \to \mathrm{invlim}(X(A), \varphi_A^B, \mathfrak{A})$ defined by $\bar{x}(A) = \eta_A(x)$ for all $A \in \mathfrak{A}$ is a T-morphism and is onto.*

(c) *If $\bigcup \mathfrak{A}$ separates the points of X (i.e., for $x \neq y$, $x, y \in X$ there exists $f \in \bigcup \mathfrak{A}$ with $f(x) \neq f(y)$), the function $x \mapsto \bar{x}$ is a T-isomorphism.*

Proof. (a) is easily verified; that $x \mapsto \bar{x}$ is a T-morphism follows from the fact that each η_A: $X \to X(A)$ is a T-morphism (1.1 (c)). That $x \mapsto \bar{x}$ is a epimorphism is a well-known property of inverse limits (cf. [5], p. 49). Finally, to prove (c) let $x, y \in X$ with $x \neq y$. There exists $A \in \mathfrak{A}$ and $f \in A$ with $f(x) \neq f(y)$. Hence, $f(1 \cdot x) \neq f(1 \cdot y)$ so that $(x, y) \notin T(A)$ and $\bar{x}(A) = \eta_A(x) \neq \eta_A(y) = \bar{y}(A)$; consequently, $\bar{x} \neq \bar{y}$. It now follows that $x \mapsto \bar{x}$ is one to one and, together with part (b), is a T-isomorphism.

PROPOSITION 1.4. *Let T be a compact semigroup and X a compatct T-space. Let A be a finite subset of $C(X)$ such that for $f \in A$, $f(X)$ is finite; then $X(A)$ is finite and discrete.*

Proof. For $f \in A$ and $x \in X$, define $G(f; x) = \{y \in X: f(t \cdot y) = f(t \cdot x)$ for all $t \in T^1\}$. Let $\{y_a\}_{a \in D}$ be a net in $X - G(f; x)$ converging to $y_0 \in X$. For each $a \in D$, there exists $t_a \in T^1$ such that $f(t_a \cdot y_a) \neq f(t_a \cdot x)$. Since T^1 is compact, we may assume the net $\{t_a\}_{a \in D}$ converges to $t_0 \in T^1$ and by continuity of f, $\{f(t_a \cdot y_a)\}_{a \in D}$ and $\{f(t_a \cdot x)\}_{a \in D}$ converge to $f(t_0 \cdot y_0)$ and $f(t_0 \cdot x)$ respectively. But $f(X)$ is finite, so eventually $f(t_a \cdot y_a) = f(t_0 \cdot y_0)$ and $f(t_a \cdot x) = f(t_0 \cdot x)$ and thus, $f(t_0 \cdot y) \neq f(t_0 \cdot x)$. It follows that $y_0 \in X - G(f; x)$ and that $G(f; x)$ is open. Now $\eta_A^{-1}(\eta_A(x))' = \{y \in X: f(t \cdot y) = f(t \cdot x)$ for all $t \in T^1$ and $f \in A\} = \bigcap \{G(f; x): f \in A\}$ is open in X since A is finite; consequently, $\eta_A(x)$ is open in $X(A)$ for each $x \in X$ so that $X(A)$ is finite and discrete.

THEOREM I. *Let T be a compact semigroup and X a compact totally disconnected T-space. Let $\mathfrak{A}$ be the family of all finite subsets $A \subseteq C(X)$ such that $f(X)$ is finite for each $f \in A$. Then each $X(A)$, $A \in \mathfrak{A}$, is a finite discrete T-space and X is T-isomorphic to $\mathrm{invlim}(X(A), \varphi_A^B, \mathfrak{A})$ under the T-isomorphism $x \mapsto \bar{x}$ (of Proposition* 1.3 (*b*)).

Proof. In view of Propositions 1.3 (c) and 1.4, we need only show that $\bigcup \mathfrak{A}$ separates the points of X. Given $x, y \in X$, $x \neq y$, there exists a compact open set $E \subseteq X$ such that $x \in E$ and $y \notin E$. If f_E denotes the characteristic function of E, then $f_E \in C(X)$, $f_E(X) = \{0, 1\}$ and $f_E(x) = 1$, $f_E(y) = 0$. The set $A = \{f_E\}$ belongs to $\mathfrak{A}$ and we are done.

Let T be a topological semigroup and X a T-space. A metric d on X is T-subinvariant if $d(t \cdot x, t \cdot y) \leqslant d(x, y)$ for all $t \in T$ and $x, y \in X$.

PROPOSITION 1.5. *Let T be a compact semigroup and X a compact T-space. Let $A \subseteq C(X)$, A countable. Suppose that, for $f \in A$, $\sup\{|f(x)|: x \in X\} \leqslant 1$. The T-space $X(A)$ is metrizable with a T-subinvariant metric.*

Proof. Let $A = \{f_1, f_2, \ldots\}$ be an enumeration of A. For $x, y \in X$ define:

$$d(\eta_A(x), \eta_A(y)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup\{|f_n(t\cdot x) - f_n(t\cdot y)|: t \in T^1\}.$$

It is a simple matter to verify that d is a well-defined metric on $X(A)$. Now let $x \in X$ and pick $\varepsilon > 0$. There exists $N > 0$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2n} < \frac{\varepsilon}{4}.$$

For each n, $1 \leqslant n \leqslant N$, there is an open set V_n, $x \in V_n$, such that for $y \in V_n$ and $t \in T^1$,

$$|f_n(t\cdot y) - f_n(t\cdot x)| < \frac{\varepsilon}{4}.$$

Setting

$$V = \bigcap_{n=1}^{N} V_n, \qquad x \in V,$$

V is open, and for $y \in V$,

$$\begin{aligned} d(\eta_A(x), \eta_A(y)) &= \sum_{n=1}^{N} \frac{1}{2^n} \sup\{|f_n(t\cdot y) - f_n(t\cdot x): t \in T^1\} + \\ &\quad + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \sup\{|f_n(t\cdot y) - f_n(t\cdot x)|: t \in T^1\} \\ &\leqslant \frac{\varepsilon}{4} \sum_{n=1}^{N} \frac{1}{2^n} + 2 \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Hence, $\eta_A: X \to X(A)$ is continuous in the metric topology on $X(A)$ and since X is compact, the metric topology equals the quotient topology. Further,

$$\begin{aligned} d(t_0\cdot\eta_A(x), t_0\cdot\eta_A(y)) &= d(\eta_A(t_0\cdot x), \eta_A(t_0\cdot y)) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \sup\{|f_n(t\cdot(t_0\cdot x)) - f_n(t\cdot(t_0\cdot y)): t \in T^1\} \\ &\leqslant d(\eta_A(x), \eta_A(y)), \end{aligned}$$

so that d is subinvariant and we are finished.

THEOREM II. *Let T be a compact semigroup and X a compact T-space. Let $\mathfrak{A}$ be the family of all countable subsets A of $C(X)$ such that for $f \in A$,*

$$\sup\{|f(x)|: x \in X\} \leqslant 1.$$

Then each $X(A)$, $A \in \mathfrak{A}$, is a metrizable T-space with a T-subinvariant metric and X is T-isomorphic to $\operatorname{invlim}(X(A), \varphi_A^B, \mathfrak{A})$ *under the T-isomorphism $x \mapsto \bar{x}$ (of Proposition* 1.3 (b)).

Proof. Again, in view of Proposition 1.3 (c) and 1.5, we need only show that $\bigcup \mathfrak{A}$ separates the points of X. However, this follows from Urysohn's Lemma.

Section II (Applications to semigroups). Let S be a compact topological semigroup and let $T = S^1 \times S^1$. We define multiplication in T by $(a, b)(c, d) = (ac, db)$ (Note the reversal in the second coordinate). With respect to this multiplication, T is a compact topological semigroup and $T^1 = T$. The function $(a, b)\cdot s \mapsto asb: T\times S \to S$ is an act and, hence, S is a compact T-space.

PROPOSITION 2.1. *Given $A \subseteq C(S)$.*

(a) $T(A) = \{(x, y) \in S\times S \mid f(axb) = f(ayb)$ *for all* $a, b \in S^1$ *and* $f \in A\}$.

(b) *$T(A)$ is a closed congruence on S.*

(c) *$S(A)$ is a compact topological semigroup and $\eta_A: S \to S(A)$ is a continuous homomorphism as well as a T-morphism.*

Proof. Given $A \subseteq C(X)$, by definition

$$\begin{aligned} T(A) &= \{(x, y) \in S\times S \mid f(t\cdot x) = f(t\cdot y) \text{ for all } t \in T^1 \text{ and } f \in A\} \\ &= \{(x, y) \in S\times S \mid f\big((a, b)\cdot x\big) = f\big((a, b)\cdot y\big) \text{ for all } (a, b) \in S^1\times S^1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } f \in A\} \\ &= \{(x, y) \in S\times S \mid f(axb) = f(ayb) \text{ for all } a, b \in S^1 \text{ and } f \in A\} \end{aligned}$$

and part (a) is established. That $T(A)$ is a closed equivalence is Proposition 1.1 (a). If $(x, y) \in T(A)$ and $s \in S$ then for each $f \in A$ and $(a, b) \in S^1\times S^1$ we have $f\big(a(sx)b\big) = f\big((as)xb\big) = f\big((as)yb\big) = f\big(a(sy)b\big)$ so that $(sx, sy) \in T(A)$ and similarly $(xs, ys) \in T(A)$. We have established part (b) and part (c) follows.

PROPOSITION 2.2. *If $A \subseteq B \subseteq C(S)$ φ_A^B: $S(B) \to S(A)$ is a continuous homomorphism.*

Proof. The function φ_A^B: $S(B) \to S(A)$ is induced and satisfies $\varphi_A^B \circ \eta_B = \eta_A$. Further, φ_A^B is a T-morphism by Proposition 1.2. Thus, for $a, b \in S$, let $t = (a, 1)$; then we have $\varphi_A^B\big(\eta_B(a)\eta_B(b)\big) = \varphi_A^B\big(\eta_B(ab)\big) = \varphi_A^B\big(\eta_B(t\cdot b)\big) = t\cdot\varphi_A^B\big(\eta_B(b)\big) = t\cdot\eta_A(b) = \eta_A(t\cdot b) = \eta_A(ab) = \eta_A(a)\eta_A(b) = \varphi_A^B\big(\eta_B(a)\big)\cdot\varphi_A^B\big(\eta_B(b)\big)$.

As a result of the above and Proposition 1.3 (c) we have:

PROPOSITION 2.3. *If $\mathfrak{A}$ is a family of subsets of $C(S)$ and $\bigcup\mathfrak{A}$ separates the points of S, then the function $x \mapsto \bar{x}$ (of Proposition* 1.3 (b)) *is an isomorphism of S onto* $\operatorname{invlim}\left(S(A), \varphi_A^B, \mathfrak{A}\right)$.

COROLLARY 2.1 (Numakura [6]). *If S is a compact totally-disconnected semigroup, then S is (isomorphic to) an inverse limit of finite discrete semigroups.*

Proof. If $\mathfrak{A}$ denotes the family of all finite subsets $A \subseteq C(S)$ such that $f \in A$, $f(S)$ is finite, then $\bigcup\mathfrak{A}$ separates the points of S and by Proposition 1.4, $S(A)$ is finite and discrete. The result now follows from Proposition 2.3.

Given a semigroup S, a metric d on S is subinvariant if $d(sa, sb) \leqslant d(a, b)$ and $d(as, bs) \leqslant d(a, b)$ for all a, b and $s \in S$.

COROLLARY 2.2 (Hofmann and Mostert [5], p. 49). *If S is a compact semigroup, then S is (isomorphic to) an inverse limit of compact metric semigroups each of which has a subinvariant metric.*

Proof. Let $\mathfrak{A}$ be the family of countable subsets $A \subseteq C(X)$ such that for $f \in A$, $\sup\{|f(s)|: s \in S\} \leqslant 1$. Again, $\bigcup\mathfrak{A}$ separates the points of S. For each $A \in \mathfrak{A}$, $S(A)$ is a T-space and has a T-subinvariant metric d by Proposition 1.5. By Proposition 2.1 (c), $S(A)$ is a compact semigroup and $\eta_A: S \to S(A)$ is a homomorphism and a T-morphism. Thus, for a, b, and $s \in S$, let $t = (s, 1)$; then

$$\begin{aligned} d\left(\eta_A(s)\eta_A(a), \eta_A(s)\eta_A(b)\right) &= d\left(\eta_A(sa), \eta_A(sb)\right) = d\left(\eta_A(t\cdot a), \eta_A(t\cdot b)\right) \\ &= d\left(t\cdot\eta_A(a), t\cdot\eta_A(b)\right) \leqslant d\left(\eta_A(a), \eta_A(b)\right). \end{aligned}$$

Similarly,

$$d\left(\eta_A(a)\eta_A(s), \eta_A(b)\eta_A(s)\right) \leqslant d\left(\eta_A(a), \eta_A(b)\right)$$

and d is subinvariant. The Proposition now follows.

A semigroup S is (uniquely) divisible if for each $x \in S$ and each integer $n \geqslant 1$, there is a (unique) $y \in S$ satisfying $y^n = x$. If S is uniquely divisible, one can define positive rational powers of elements of S; further, if S is commutative, the usual laws of exponentiation hold. If S is a compact uniquely divisible commutative (UDC) semigroup and r is a positive rational, the function $x \mapsto x^r$ is an automorphism of S. The reader is referred to [2], [3], and [4] for a discussion of divisible semigroups.

THEOREM III. *A compact UDC semigroup S is (isomorphic to) an inverse limit of compact metric UDC semigroups, each having a subinvariant metric.*

Proof. For $f \in C(X)$ and r a positive rational, define $f^r \in C(X)$ by $f^r(x) = f(x^r)$. A subset $A \subseteq C(S)$ is rationally closed if whenever $f \in A$, $f^r \in A$ for all positive rationals r. Let $\mathfrak{A}$ denote the family of all countable rationally closed subsets $A \subseteq C(S)$ such that for $f \in A$, $\sup\{|f(t)|: t \in S\} \leqslant 1$. As in Corollary 2.2, for each $A \in \mathfrak{A}$, $S(A)$ is a compact metric semigroup with a subinvariant metric. Since $\eta_A: S \to S(A)$ is a surmorphism, $S(A)$ is divisible. Now let $x, y \in S$, $n \geqslant 1$ and suppose $\eta_A(x)^n = \eta_A(y)^n$. Then $\eta_A(x^n) = \eta_A(y^n)$ and it follows that $f(ax^n b) = f(ay^n b)$ for all $a, b \in S^1$ and $f \in A$. Then for $f \in A$, $a, b \in S^1$ we have $f^{1/n} \in A$ and $f(axb) = f^{1/n}(a^n x^n b^n)$ $= f^{1/n}(a^n y^n b^n) = f(ayb)$ so that $\eta_A(x) = \eta_A(y)$. Hence, $S(A)$ is uniquely divisible. If $x, y \in S$ and $x \neq y$, there exists $f \in C(S)$, $\sup\{|f(s)|: s \in S\} \leqslant 1$ and $f(x) \neq f(y)$. Let $A = \{f^r : r \text{ positive rational}\}$; then $A \in \mathfrak{A}$ and $f \in A$. Hence, $\bigcup \mathfrak{A}$ separates the points of S and, consequently, S is isomorphic to $\operatorname{invlim}(S(A), \varphi_A^B, \mathfrak{A})$ by Proposition 2.3.

QUESTION. Using this method, can one find a reasonable category of compact semigroups such that each semigroup in the category is the inverse limit of finite-dimensional compact semigroups. It would seem that if A is a "nice" finite subset of $C(S)$, $S(A)$ might have ε-maps into an n-cube. (**P 785**)

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