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ON PRODUCTS OF NEAT STRUCTURES

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In this note we give some extensions of Theorems 1 and 2 of [2]. It is a result of a discussion on the seminar on model theory in Wrocław.

0. Let $C = (C, T)$ denote the Cantor discontinuum. If A is a non-empty set, then A^C denotes the set of all continuous functions from C into A regarded as a discrete space. If $\mathfrak{A}$ is a relational structure, then $\mathfrak{A}^C$ is the substructure of $\mathfrak{A}^C$ with the universe A^C. It is obvious that if A is infinite, then $|A^C| = |A|$.

An *autonomous system* (see [1]) is a triple $\{S, \pi, \varrho\}$ such that S is a set of formulas which form a partition of logical unit and π is a commutative and associative function from $S \times S$ into S with the property that for every relational structures $\mathfrak{A}$ and $\mathfrak{B}$ and every $\alpha, \beta \in S$, if $\mathfrak{A} \models \alpha$ and $\mathfrak{B} \models \beta$, then $\mathfrak{A} \times \mathfrak{B} \models \pi(\alpha, \beta)$. We do not quote the properties of ϱ, since we are not going to use it.

It was proved by Galvin [1] that for every formula φ there is an autonomous system $\{S, \pi, \varrho\}$ such that $\vdash \varphi \leftrightarrow \bigvee S_1$, where S_1 is a subset of S.

A relational structure $\mathfrak{A}$ is *neat* if for every formula φ of the language of $\mathfrak{A}$ there is in this language a predicate R_φ such that $\mathfrak{A} \models \varphi \leftrightarrow R_\varphi$.

1. THEOREM 1. *If $\mathfrak{A}_0, \ldots, \mathfrak{A}_{n-1}$ are relational structures which are atomic and neat, then the direct product $\mathfrak{A} = \mathfrak{A}_0 \times \ldots \times \mathfrak{A}_{n-1}$ is atomic.*

Proof. Let $a \in A^k$. We are going to prove that there is an atom α of $F_k(\mathrm{Th}(\mathfrak{A}))$ such that $\mathfrak{A} \models \alpha[a]$. To simplify the notation we assume $k = 1$ (it is obvious how to generalize this proof to arbitrary $k < \omega$).

Let $a = (a_0, \ldots, a_{n-1})$. Since $\mathfrak{A}_i$ is atomic, for every $i < n$ there is an atom α_i of $F_1(\mathrm{Th}(\mathfrak{A}_i))$ such that $\mathfrak{A}_i \models \alpha_i[a_i]$. Moreover, we can assume that, for every $i, j < n$,

$$(1) \qquad \mathfrak{A}_i \models \exists x \alpha_j(x) \quad \text{iff } \mathfrak{A}_j \equiv \mathfrak{A}_i.$$

In fact, if $\mathfrak{A}_i \equiv \mathfrak{A}_j$ does not hold, then there is a sentence φ_{ij} such that $\mathfrak{A}_i \models \varphi_{ij}$ and $\mathfrak{A}_j \models \neg\varphi_{ij}$. If we put

$$a_i' = a_i \wedge \bigwedge \{\varphi_{ij}: \mathfrak{A}_i \not\equiv \mathfrak{A}_j, i \neq j < n\} \quad \text{for } i < n,$$

then a_i' is an atom and (1) is satisfied for every $i, j < n$.

Let us consider the formula

$$a = R_{\vee\{a_i: i<n\}} \wedge \bigwedge \{\neg R_{\neg a_i}: i < n\}.$$

Let n^n denote the set of all functions of n into n and S_n be the set of all permutations of n. Then we have

$$R^{\mathfrak{A}}_{\vee\{a_i: i<n\}} = \bigcup_{f\in n^n} (R^{\mathfrak{A}_0}_{a_{f(0)}} \times \ldots \times R^{\mathfrak{A}_{n-1}}_{a_{f(n-1)}}) \tag{2}$$

and

$$a^{\mathfrak{A}} = \bigcup_{f\in S_n} (R^{\mathfrak{A}_0}_{a_{f(0)}} \times \ldots \times R^{\mathfrak{A}_{n-1}}_{a_{f(n-1)}}). \tag{3}$$

It is clear that $\mathfrak{A} \models a[(a_0, \ldots, a_{n-1})]$. We shall prove that a is an atom of $F_1(\mathrm{Th}(\mathfrak{A}))$.

Let γ be a formula of $F_1(\mathrm{Th}(\mathfrak{A}))$ such that

$$\mathfrak{A} \models (\gamma \to a). \tag{4}$$

In particular, we have

$$\mathfrak{A} \models \gamma \to R_{\vee\{a_i: i<n\}}. \tag{5}$$

We shall check that $\gamma = a$.

Let S_γ be an autonomous system for γ. We can assume that $\gamma \in S_\gamma$ and $\mathfrak{A} \models \gamma[(a_0, \ldots, a_{n-1})]$. It follows from the definition of an autonomous system that there are formulas $\gamma_0, \ldots, \gamma_{n-1} \in S_\gamma$ such that

$$\mathfrak{A}_i \models \gamma_i[a_i] \tag{6}$$

and

$$\pi(\gamma_0, \ldots, \gamma_{n-1}) = \gamma. \tag{7}$$

By (5) and (6) we have $\mathfrak{A}_i \models \gamma_i \to (a_0 \vee \ldots \vee a_{n-1})$ for every $i < n$ and, since a_i is an atom of $F_1(\mathrm{Th}(\mathfrak{A}_i))$, we get

$$\mathfrak{A}_i \models (a_i \to \gamma_i) \quad \text{for } i < n. \tag{8}$$

Consequently,

$$\mathfrak{A}_i \models \gamma_i \leftrightarrow (a_{i_0} \vee \ldots \vee a_{i_{k-1}}), \tag{9}$$

where $k \leqslant n$, $i_j < n$ for $j < k$ and, for some $j < k$, $i_j = i$. We claim that, for every $i < n$,

$$\mathfrak{A}_i \models (\gamma_i \leftrightarrow a_i). \tag{10}$$

Suppose to the contrary that this does not hold. Then by (8) there is $j < n$, $j \neq i$, such that a_j is an atom of $F_1(\mathrm{Th}(\mathfrak{A}_i))$ and $\mathfrak{A}_i \models (a_i \vee a_j) \to \gamma_i$.

Consequently, if

$$B = R_{a_0}^{\mathfrak{A}_0} \times \ldots \times R_{a_i}^{\mathfrak{A}_i} \times \ldots \times R_{a_j}^{\mathfrak{A}_j} \times \ldots \times R_{a_{n-1}}^{\mathfrak{A}_{n-1}},$$

then $B \subseteq \gamma^{\mathfrak{A}}$ and, by (4), $B \subseteq a^{\mathfrak{A}}$. This contradicts (3) and completes the proof of (10).

By the commutativity of π, for every $f \in S_n$ we have

$$\gamma = \pi(a_{f(0)}, \ldots, a_{f(n-1)}).$$

Consequently, by (3), $a^{\mathfrak{A}} \subseteq \gamma^{\mathfrak{A}}$ and, by (4), $\mathfrak{A} \models (\gamma \leftrightarrow a)$. The proof is completed.

We say that a relational structure $\mathfrak{A}$ is *neat with respect to atoms* if for every $n < \omega$ and every formula a, which is an atom of $F_n(\mathrm{Th}(\mathfrak{A}))$, there is a predicate R_a in $L(\mathfrak{A})$ such that $\mathfrak{A} \models (a \leftrightarrow R_a)$.

By an easy modification of the proof of Theorem 1 we can get the following

COROLLARY. *If $\mathfrak{A}_0, \ldots, \mathfrak{A}_{n-1}$ is a sequence of relational structures which are atomic and neat with respect to atoms and which are mutually non-elementarily equivalent, then the direct product $\mathfrak{A} = \mathfrak{A}_0 \times \ldots \times \mathfrak{A}_{n-1}$ is atomic and neat with respect to atoms.*

2. In [2] (see Theorem 1) Pacholski proved that if $2_{\mathfrak{F}}^I$ is atomless and $\mathfrak{A}$ is prime and neat, then $\mathrm{Th}(\mathfrak{A}_{\mathfrak{F}}^I)$ has a prime model. We shall prove that $\mathfrak{A}^C$ is such a model.

It is well known that $\mathrm{Th}(\mathfrak{A}^C) = \mathrm{Th}(\mathfrak{A}_{\mathfrak{F}}^I)$ (see, e.g., [5]).

THEOREM 2. *If $\mathfrak{A}$ is neat and prime, then $\mathfrak{A}^C$ is a prime model.*

Proof. Let $\mathfrak{A}$ be a relational structure which is neat and prime. Then, by Theorem 3.5 of Vaught [4], $\mathfrak{A}$ is atomic and countable. By Vaught [4], p. 303-321, it suffices to check that $\mathfrak{A}^C$ is atomic.

As in the proof of Theorem 1 we restrict ourselves to the case $k = 1$.

Let $f_0, \ldots, f_{n-1} \in A^C$. We are going to prove that there is an atom a in $F_n(\mathrm{Th}(\mathfrak{A}^C))$ such that

$$\mathfrak{A} \models a[f_0, \ldots, f_{n-1}].$$

Since for $i < n$ the function f_i has only finitely many values, there is a decomposition $\{U_0; \ldots; U_{k-1}\}$ of C into a finite number of closed and open sets such that functions $f_0, \ldots, f_{n-1}$ are constant on every U_i $(i < k)$.

Let us assume, for $i < k$, that a_i is an atom of $F_n(\mathrm{Th}(\mathfrak{A}))$ such hat $\mathfrak{A} \models a_i[f_0(x), \ldots, f_{n-1}(x)]$ for $x \in U_i$. We put

$$a = R_{\vee\{a_i:\, i<k\}} \wedge \bigwedge \{\neg R_{\neg a_i}:\ i < n\}.$$

The proof that a is an atom of $F_n(\mathrm{Th}(\mathfrak{A}^C))$ is exactly the same as the proof of case 1 in the proof of Theorem 1 in [2].

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