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COMPACTIFYING THE SPACE OF HOMEOMORPHISMS

BY

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If X is a compact metric space, it is well known that $H(X)$, the space of homeomorphisms of X onto itself, is a complete separable metric topological group. [In nearly all the interesting cases, these spaces of homeomorphisms turn out to be very badly *not* locally compact and very badly *not* abelian.] This is unfortunate in at least one respect: Although topological groups and topological transformation groups have been widely studied, nearly all that work has been done in a locally compact or an abelian setting.

Nevertheless, spaces of homeomorphisms, and the homogeneity properties that go along with them, are of very deep interest to this author. In this paper, we will consider a very natural compactification of the space of homeomorphisms to see what information we can gain about our base space X and our space of homeomorphisms $H(X)$. Of course, as we gain compactness, we generally lose our topological group structure. As much as anything this paper represents an attempt to learn more about homogeneity properties by looking at the "bad" places in the group of homeomorphisms. Ironically, when the "bad stuff" is put in, the resulting space is in some senses a nicer space.

1. Background, notation, definitions. In this paper a *continuum* will be a compact, connected metric space. We will use N to denote the positive integers. If A is a collection of sets, A^* will denote the union of the members of A.

If X is a compact metric space, $H(X)$ will denote the set of all homeomorphisms from X onto itself. As has been noted, $H(X)$ is a complete metric separable topological group [2]. The topology on $H(X)$ is the compact-open topology. There are several natural ways to define a metric on $H(X)$ compatible with this topology. We will look more closely at some of these in the next section. Composition is, of course, the group operation on $H(X)$. We will use 1 to denote the identity in $H(X)$.

Now $H(X)$ acts on X, as does any subgroup G of $H(X)$. Also, $H(X)$ and its subgroups act in a very natural way on many other spaces: For example, $H(X)$ acts on itself and on X^n, where $n \in N$. (In this last case we define the action of $H(X)$ on X^n as follows: If $h \in H(X)$ and $x = (x_1, \ldots, x_n) \in X^n$, $h(x) = (h(x_1), h(x_2), \ldots, h(x_n)) \in X$.)

Suppose that the topological group G acts on the metric space X. If $x \in X$, $Gx = \{h(x) | \ h \in G\}$ is called the *orbit* of x under the action of G. The action of G on X is said to be *transitive* if $Gx = X$ for each x in X. If $A \subseteq G$,

$$Ax = \{h(x) \in X | \ h \in A\}.$$

The action of G on X is *micro-transitive* if for every x in X and every open neighborhood u of 1 in G, ux is open in X.

Next we state *Ancel's version of the Effros Theorem* [1]: Suppose that a separable complete metric topological group acts transitively on a metric space X. Then G acts micro-transitively on X if and only if X has a complete metric.

For more details and background about these ideas, the reader is referred to [13].

2. $H(X)$ as a subspace of $2^{X \times X}$. Suppose X is a compact metric space and d is a metric on X compatible with its topology. Then ϱ will denote the "sup" metric on $H(X)$ with respect to d: i.e., if $g, h \in H(X)$,

$$\varrho(g, h) = \text{lub}\,\{d(g(x), h(x)) | \ x \in X\},$$

and if $\varepsilon > 0$,

$$N_\varepsilon(h) = \{f \in H(X) | \ \varrho(f, h) < \varepsilon\}.$$

The sup metric is the one most commonly used by topologists when X is compact metric, but $H(X)$ is not generally complete with respect to this metric, although this metric is certainly compatible with the compact-open topology on $H(X)$.

We will use ϱ' to denote the usual complete metric on $H(X)$ with respect to d: if $g, h \in H(X)$,

$$\varrho'(g, h) = \max\{\varrho(g, h), \varrho(g^{-1}, h^{-1})\}.$$

Further, if $\varepsilon > 0$, let

$$N'_\varepsilon(h) = \{f \in H(X) | \ \varrho'(f, h) < \varepsilon\}.$$

If $x \in X$, $\varepsilon > 0$, $D_\varepsilon(x)$ will denote $\{y \in X | \ d(x, y) < \varepsilon\}$. Then d' will denote the "taxicab" metric on X^2 with respect to d; i.e., if $x = (x_1, x_2)$, $y = (y_1, y_2) \in X^2$,

$$d'(x, y) = d(x_1, y_1) + d(x_2, y_2).$$

If $x \in X^2$, $\varepsilon > 0$,

$$D'_\varepsilon(x) = \{y \in X^2 | \ d'(x, y) < \varepsilon\}.$$

We will use the Vietoris topology and Hausdorff metric on $2^{X \times X}$: If $C, D \in 2^{X \times X}$,

$$v(C, D) = \text{glb}\{\varepsilon > 0 \mid C \subset D'_\varepsilon(D) \text{ and } D \subset D'_\varepsilon(C)\}.$$

If $C \in 2^{X \times X}$, $\varepsilon > 0$,

$$S_\varepsilon(C) = \{D \in 2^{X \times X} \mid v(C, D) < \varepsilon\}.$$

Now, if $f \in H(X)$, f can be thought of as a closed collection of ordered pairs in $X \times X$ or a point in $2^{X \times X}$. Thus, we will be associating f with $\text{gr} f$ (and, when no confusion arises, be somewhat sloppy about it by calling both f and $\text{gr} f$ just f).

OBSERVATION 1. *Suppose (X, d) is a compact metric space. Then the space $H(X)$ can be embedded in $2^{X \times X}$. Further, if f, $h \in H(X)$, then*

$$\varrho'(f, h) \geqslant \varrho(f, h) \geqslant v(f, h).$$

Proof. Surely, this is essentially already known. However, since the author does not know where to find such a result, and it is crucial to what follows, we prove it here. Define the function

$$\phi\colon H(X) \to Z, \quad \text{where } Z = \{\text{gr} f \mid f \in H(X)\} \subseteq 2^{X \times X},$$

as follows:

$$\phi(f) = \text{gr} f.$$

Thus ϕ is a one-to-one function from $H(X)$ onto Z. We show that it is a homeomorphism:

Suppose $\varepsilon > 0$, $h \in H(X)$. Then $\phi(N_\varepsilon(h)) \subseteq S_\varepsilon(\text{gr}\, h)$: If $f \in N_\varepsilon(h)$, then, for each x in X,

$$d(f(x), h(x)) < \varepsilon, \quad (x, f(x)) \in D'_\varepsilon(x, h(x)) \subseteq D'_\varepsilon(\text{gr}\, h),$$

and

$$(x, h(x)) \in D'_\varepsilon(x, f(x)) \subset D'_\varepsilon(\text{gr} f).$$

Thus

$$\text{gr} f \subset D'_\varepsilon(\text{gr}\, h), \quad \text{gr}\, h \subset D'_\varepsilon(\text{gr} f), \quad \text{gr} f \in S_\varepsilon(\text{gr}\, h),$$

and ϕ is continuous.

There is some $\delta > 0$ such that $\phi^{-1}(S_\delta(\text{gr}\, h)) \subseteq N_\varepsilon(h)$: Suppose not. Then for each i in N,

$$\phi^{-1}(S_{2^{-i}}(\text{gr}\, h)) \nsubseteq N_\varepsilon(h)$$

and

$$\phi^{-1}(S_{2^{-i}}(\text{gr}\, h)) \circ h^{-1} \nsubseteq N_\varepsilon(h) \circ h^{-1} = N_\varepsilon(1).$$

For each i, there is some $f_i \in H(X)$ such that

$$\text{gr} f_i \in S_{2^{-i}}(\text{gr}\, h) \quad \text{and} \quad f_i h^{-1} \notin N_\varepsilon(1).$$

For each i, there is x_i in X such that

$$d(f_i h^{-1}(x_i), x_i) \geqslant \varepsilon.$$

Let $h^{-1}(x_i) = y_i$. Then

$$\begin{aligned}\varepsilon \leqslant d(f_i h^{-1}(x_i), x_i) &= d(f_i h^{-1}(x_i), hh^{-1}(x_i)) \\ &= d'\big((y_i, f_i(y_i)), (y_i, h(y_i))\big).\end{aligned}$$

Then we can find, for each i, z_i in X that

$$d'\big((y_i, f_i(y_i)), (z_i, h(z_i))\big) < 2^{-i}.$$

There is some $\delta > 0$, $\delta < \varepsilon/4$, such that if $r, t \in X$, $d(r, t) < \delta$, then $d(h(r), h(t)) < \varepsilon/4$; and so we can find $M \in N$ such that if $m > M$, then $2^{-m} < \delta/2$. Choose $m > M$. Then

$$\begin{aligned}\varepsilon &\leqslant d'\big((y_m, f_m(y_m)), (y_m, h(y_m))\big) \\ &\leqslant d'\big((y_m, f_m(y_m)), (z_m, h(z_m))\big) + d'\big((z_m, h(z_m)), (y_m, h(y_m))\big) \\ &< 2^{-m} + 2^{-m} + \varepsilon/4 < \varepsilon.\end{aligned}$$

But this is a contradiction. It follows that ϕ^{-1} is a homeomorphism.

Then $H(X)$ can be embedded quite naturally in $2^{X \times X}$. Hereafter, we will dispense with distinguishing between f and $\mathrm{gr} f$ and $H(X)$ and Z. We have also shown that v is a metric on $H(X)$ compatible with its usual topology, and that, for $f, g \in H(X)$, $\varrho(f, g) \geqslant v(f, g)$. It is clear that $\varrho'(f, g) \geqslant \varrho(f, g)$.

OBSERVATION 2. *The group $H(X)$ acts on $2^{X \times X}$. If $f \in H(X)$, then $H(X)(f) = H(X)$, and thus*

(1) *is a G_δ-set in $2^{X \times X}$;*

(2) *is a dense G_δ-set in $\overline{H(X)} = Y \subseteq 2^{X \times X}$;*

(3) *is a topological group lying in $2^{X \times X}$.*

Proof. Again we include a proof only to ensure clearness.

Now $H(X)$ acts on $2^{X \times X}$: Define ϕ: $H(X) \times 2^{X \times X} \to 2^{X \times X}$ as follows:

$$\phi(h, C) = \{(x, h(y)) \mid (x, y) \in C\} \quad \text{for } (h, C) \in H(X) \times 2^{X \times X}.$$

It is easy to check that $\phi(h, C) \in 2^{X \times X}$, and ϕ is an action of $H(X)$ on $2^{X \times X}$, since ϕ is continuous, $\phi(1, C) = C$ for $C \in 2^{X \times X}$ and, for $f, h \in H(X)$, $\phi(fh, C) = \phi(f, \phi(h, C))$. We denote $\phi(h, C)$ by hC.

Since for $f \in H(X)$ we have (in our abused notation)

$$H(X)(\mathrm{gr} f) = H(X)(f) = \{hf \mid h \in H(X)\} = H(X),$$

the rest of the observation is true. (Recall that the completeness of $H(X)$ implies that $H(X)$ is G_δ in Y and $2^{X \times X}$.)

COROLLARY 1. *If X is a compact metric space, then $H(X)$ acts on a compact metric space Y which admits under the action of $H(X)$ a dense G_δ-orbit.*

Proof. Of course, our $Y = \overline{H(X)} \subseteq 2^{X\times X}$. (Note that $h \in H(X)$, $C \in Y$ imply $hC \in Y$.)

OBSERVATION 3. *Suppose X is a compact metric space.*

(1) *Define $\hat{\phi}$: $H(X) \times 2^{X\times X} \to 2^{X\times X}$ as follows:*

$$\hat{\phi}(h, C) = \{(x, z) | \text{ there is some } y \text{ in } X \text{ such that } (x, y) \in h, (y, z) \in C\}$$

for (h, C) in $H(X) \times 2^{X\times X}$. Then $\hat{\phi}$ also gives an action of $H(X)$ on $2^{X\times X}$.

(2) *Define β: $2^{X\times X} \to 2^{X\times X}$ by*

$$\beta(C) = \{(y, x) | \ (x, y) \in C\} \quad \text{for } C \in 2^{X\times X}.$$

Then $\beta \in H(2^{X\times X})$.

Proof. Note that, for $f, h \in H(X)$,

$$\hat{\phi}(hf, C) = C(hf) = \hat{\phi}(f, \hat{\phi}(h, C)) = (Ch)f, \quad \text{where } Ch = \hat{\phi}(h, C).$$

The proofs of these statements are straightforward, so we omit them.

OBSERVATION 4. *Suppose X is a compact metric space. Then $C \in \overline{H(X)} = Y$ implies that if $x \in X$, then there are y and z in X such that $(x, y) \in C$ and $(z, x) \in C$. If C and D are in $\overline{H(X)} = Y$, and $f_1, f_2, \ldots$ is a sequence of $H(X)$ which converges to C, $h_1, h_2, \ldots$ is a sequence of $H(X)$ which converges to D, then each convergent subsequence of $f_1 h_1, f_2 h_2, \ldots$ converges to some subset of $CD = \{(x, y) | \text{ there is some } z \text{ in } X \text{ such that } (x, z) \in D \text{ and } (z, y) \in C\}$.*

This is straightforward, so the proof is omitted.

Note that the operation defined in Observation 4 [If $C, D \in 2^{X\times X}$, $CD = \{(x, y) \in X^2 |$ there is some z in X such that $(x, z) \in D$ and $(z, y) \in D\}$.] does *not* make $2^{X\times X}$ or Y into a topological group: this operation is not continuous, generally.

However, it is sometimes true that $H(X)$ is locally compact or even compact. In these cases though, X does not admit under the action of $H(X)$ an orbit which contains a nondegenerate continuum and is G_δ in the space [7]. If $H(X)$ is compact, $Y = \overline{H(X)} = H(X)$ and Y is a topological group. Otherwise, the composition operation on $H(X)$ cannot be extended to Y to make it into a topological group: $H(X)$ is G_δ in Y, and, if this happened, $H(X)$ would be a dense G_δ-subgroup of the group Y and has to be closed in Y. Otherwise, there is an h in Y such that

$$hH(X) \cap H(X) = \emptyset,$$

where $hH(X)$ and $H(X)$ are dense G_δ-sets of Y, which is impossible. In fact, Y may actually have a fixed point under the action of $H(X)$. As we shall see later, X^2 is often an element of Y.

THEOREM 1. *Suppose X is a compact metric space. Then X admits a dense orbit if and only if, for each (x, y) in X^2, there is some $C_{(x,y)}$ in Y such that $(x, y) \in C_{(x,y)}$.*

Proof. Suppose that X admits a dense orbit $H(X)(z)$ and $(x, y) \in X^2$. For each $i \in N$ one can find $h_i \in H(X)$, $r_i \in D_{2^{-i}}(x)$, and $s_i \in D_{2^{-i}}(y)$, such that $r_i \in H(X)(z)$ and $h_i(r_i) = s_i$. Now $h_1, h_2, \ldots$ has a limit point C in Y, and some subsequence of $h_1, h_2, \ldots$ converges to C. Then $(x, y) \in C$.

Suppose that for each $(x, y) \in X^2$ there is some $C_{(x,y)}$ in Y such that $(x, y) \in C_{(x,y)}$. We use here an argument very similar to one given by Lewis [9]. Suppose $b_0, b_1, b_2, \ldots$ is a basis for X. If $(x_0, y_0) \in b_0 \times b_1$, there is some C_1 in Y such that $(x_0, y_0) \in C_1$, and there is h_0 in $H(X)$ such that

$$h_0(y_1) = x_1 \quad \text{for some } (x_1, y_1) \in b_0 \times b_1.$$

There is some open $o_0 \subseteq b_1$ containing y_1 such that

$$\operatorname{diam} o_0 < 1/2, \quad \operatorname{diam} h_0(o_0) < 1/2, \quad h_0\overline{(o_0)} \subseteq b_0.$$

Consider b_2. If $(x_1, y_1') \in h_0(o_0) \times b_2$, there is some C_2 in Y such that $(x_1, y_1') \in C_2$, and there is some h_1 in $H(X)$ such that

$$h_1(y_2) = x_2 \quad \text{for some } (x_2, y_2) \in h_0(o_0) \times b_2.$$

Then there is some open $o_1 \subseteq b_2$ containing y_2 such that

$$\operatorname{diam} o_1 < 1/4, \quad \operatorname{diam} h_1(o_1) < 1/4, \quad h_1(\bar{o}_1) \subseteq h_0(o_0).$$

Consider b_3 and likewise obtain h_2 in $H(X)$ such that

$$h_2(y_3) = x_3 \quad \text{for some } (x_3, y_3) \in h_1(o_1) \times b_3.$$

Continue this process.

Now

$$\bigcap_{i=0}^{\infty} h_i(o_i) = \bigcap_{i=0}^{\infty} \overline{h_i(o_i)} = \{z\} \quad \text{for some } z \in X.$$

Then $H(X)(z)$ is dense in X: If u is open, there is some $i \geqslant 2$ such that $b_i \subseteq u$. Then

$$h_{i-1}(y_i) = x_i \quad \text{for some } (x_i, y_i) \in h_{i-2}(o_{i-2}) \times b_i;$$

$$\overline{h_{i-1}(o_{i-1})} \subseteq h_{i-2}(o_{i-2}), \quad \operatorname{diam} h_{i-1}\overline{(o_{i-1})} < 2^{-i}, \quad \text{and} \quad o_{i-1} \subseteq b_i.$$

Since $z \in h_{i-1}(o_{i-1})$, we have $h_{i-1}^{-1}(z) \in o_{i-1} \subseteq b_i$.

THEOREM 2. *Suppose X is a compact metric space such that every point of X is a limit point of X. Then the following are equivalent:*

(1) $X^2 \in Y = \overline{H(X)}$.

(2) *If $u_1, u_2, \ldots, u_n$ and $o_1, o_2, \ldots, o_n$ are each n-element subcollections*

of nonempty open sets in X, then there is some h in $H(X)$ such that, for each $i \leqslant n$, there is x_i in u_i such that $h(x_i) \in o_i$.

(3) *There is a countable dense subset $D = p_1, p_2, \ldots$ of X such that if $o_1, o_2, \ldots, o_n$ is an n-element subcollection of nonempty open subsets of X, then there is some h in $H(X)$ such that, for each $i \leqslant n$, $h(p_i) \in o_i$.*

Proof. (1)$\Rightarrow$(2). Suppose $X^2 \in Y$. There is a sequence $h_1, h_2, \ldots$ in $H(X)$ such that $h_1, h_2, \ldots$ converges to X^2. Suppose that $u_1, u_2, \ldots, u_n$ and $o_1, o_2, \ldots, o_n$ are n-element sequences of nonempty open sets in X. Without loss of generality assume $u_1, \ldots, u_n$ and $o_1, \ldots, o_n$ are both disjoint collections. Then for each $i \leqslant n$ choose open $u_i' \neq \emptyset$ such that $\overline{u_i'} \subseteq u_i$ and choose for each $i \leqslant n$ open o_i' such that $\overline{o_i'} \subseteq o_i$.

Now $X - \{\overline{u_i'}\}^* = u_{n+1}$ is open, as is $X - \{\overline{o_i'}\}^* = o_{n+1}$. Let

$$B = \{u_i \times o_j |\ i \leqslant n+1,\ j \leqslant n+1\}.$$

Let $\langle B \rangle$ denote the basic open subset of $2^{X \times X}$ consisting of all points C of $2^{X \times X}$ which have the property that $C \cap (u_i \times o_j) \neq \emptyset$ for any $i, j \leqslant n+1$, and $C \subseteq B^*$. Then $X^2 \in \langle B \rangle$. There is $M \in N$ such that, for each $m > M$, $h_m \in \langle B \rangle$. Then

$$h_m \cap (u_i \times o_i) \neq \emptyset \qquad \text{for } i \leqslant n,$$

and there is $(x_i, y_i) \in h_m \cap (u_i \times o_i)$. Thus $h_m(x_i) = y_i$ for each $i \leqslant n$.

(2)$\Rightarrow$(3). Suppose $B = b_1, b_2, \ldots$ is a basis for X. Let

$$\hat{B} = \{d(1) \times \ldots \times d(n) |\ n \in N \text{ and } (d(1), \ldots, d(n)) \in B^n\}.$$

List the elements of $\hat{B}$: $\hat{b}_1, \hat{b}_2, \ldots$ Now there are $e(1) \in N$ and a collection of open sets $\{d(1)^1, \ldots, d(e(1))^1\}$ such that

$$\hat{b}_1 = d(1)^1 \times \ldots \times d(e(1))^1.$$

Let $t_1 = X^{e(1)-1} \times b_1$. There are h_1 in $H(X)$ and $x_1 = (x_{11}, \ldots, x_{1e(1)}) \in \hat{b}_1$ such that $h_1(x_1) \in t_1$. (Recall that $h_1(x_1) = (h_1(x_{11}), \ldots, h_1(x_{1e(1)}))$.) There is some basic open subset

$$o_1 = o(1, 1) \times \ldots \times o(1, e(1))$$

of $X^{e(1)}$ such that

$$x_1 \in o_1 \subseteq \hat{b}_1, \qquad h_1(x_1) \in h_1(\bar{o}_1) \subseteq t_1,$$
$$\operatorname{diam} \bar{o}_1, \operatorname{diam} h(\bar{o}_1) < 1/2.$$

Let $a'(1) = e(1)$.

Now there are $e(2) \in N$ and a collection of open sets $\{d(1)^2, \ldots, d(e(2))^2\}$ such that

$$\hat{b}_2 = d(1)^2 \times \ldots \times d(e(2))^2.$$

Let

$$a'(2) = e(2)+a'(1)+1, \qquad b_2^{\#} = \hat{b}_2 \times X^{a(1)},$$

where $a(1) = a'(1)+1$, and

$$t_2 = h_1(o_1) \times X^{e(2)} \times b_2.$$

There are h_2 in $H(X)$ and $x_2 \in b_2^{\#}$ such that $h_2(x_2) \in t_2$. Then there is some basic open set o_2 such that

$$x_2 \in o_2 \subseteq b_2^{\#}, \qquad h_2(x_2) \in h_2(\bar{o}_2) \subseteq t_2,$$
$$\operatorname{diam} \bar{o}_2, \operatorname{diam} h_2(\bar{o}_2) < 1/4.$$

Continue this process, obtaining at the m-th step the following: There are $e(m) \in N$ and a collection of open sets $\{d(1)^m, \ldots, d(e(m))^m\}$ such that

$$\hat{b}_m = d(1)^m \times \ldots \times d(e(m))^m.$$

Let

$$a'(m) = e(m)+a'(m-1)+1, \qquad b_m^{\#} = \hat{b}_m \times X^{a(m-1)},$$

where $a(m-1) = a'(m-1)+1$, and

$$t_m = h_{m-1}(o_{m-1}) \times X^{e(m)} \times b_m.$$

There are some h_m in $H(X)$ and $x_m \in b_m^{\#}$ such that $h_m(x_m) \in t_m$. Then there is some basic open subset o_m of $X^{a'(m)}$ such that

$$x_m \in o_m \subseteq b_m^{\#}, \qquad h_m(x_m) \in h_m(\bar{o}_m) \subseteq t_m,$$
$$\operatorname{diam} \bar{o}_m, \operatorname{diam} h_m(\bar{o}_m) < 2^{-m}.$$

Now for each $i \in N$ consider

$$\bigcap_{j=i}^{\infty} h_j\big(o(j, a'(i))\big),$$

where $o_j = o(j, 1) \times \ldots \times o(j, a'(j))$, $a'(j) = M_j$ for each $j \in N$. Note that

$$b_i \supseteq h_i\big(\overline{o(i, a'(i))}\big) \supseteq h_i\big(o(i, a'(i))\big) \supseteq h_{i+1}\big(\overline{o(i+1, a'(i))}\big) \supseteq \ldots$$

and since the diameters of the sets in this monotone sequence go to 0, $\bigcap_{j=i}^{\infty} h_j\big(\overline{o(j, a'(i))}\big)$ is degenerate, but not empty. For each i, let

$$\{p_i\} = \bigcap_{j=i}^{\infty} h_j\big(\overline{o(j, a'(i))}\big)$$

and let $D = \{p_1, p_2, \ldots\}$. Then D is a countable dense subset of X.

Suppose $\hat{o}(1), \hat{o}(2), \ldots, \hat{o}(n)$ is an n-element sequence of nonempty open subsets of X. For each $i \leqslant n$, there is some $d(i) \in B$ such that $d(i) \subseteq \hat{o}_i$, and $d(1), d(2), \ldots, d(n)$ is a finite subsequence of $b_1, b_2, \ldots$ If $m \in N$,

$$h_m(o_m) \subseteq t_m \subseteq X^{e(1)-1} \times b_1 \times X^{e(2)} \times b_2 \times \ldots \times X^{e(m)} \times b_m.$$

Now

$$d(1) \times \ldots \times d(n) = \{w \in \hat{b}_l \mid w_i \in \hat{b}(l, i) \text{ for } i \in \{a'(1), \ldots, a'(n)\}\},$$

where $\hat{b}_l = \hat{b}(l, 1) \times \ldots \times \hat{b}(l, m) \in \hat{B}$ for infinitely many l. Pick one such l. Then $x_l \in b_l^{\#}$ and $h_l(x_l) \in t_l$. Thus

$$h_l(x_{la'(i)}) \in b_i \quad \text{and} \quad x_{la'(i)} \in d(i) \quad \text{for each } i \leqslant n.$$

Also, $h_l(\bar{o}_l) \subseteq t_l$, and

$$p_i \in h_l(o(l, a'(i))) \subseteq b_i.$$

Then $h_l^{-1}(p_i) \in (o(l, a'(i))) \subseteq d(i) \subseteq \hat{o}_i$, and h_l^{-1} is the desired homeomorphism.

(3) $\Rightarrow$ (1). For each $i \in N$, suppose

$$G_i = \{g(i, 1), \ldots, g(i, n_i)\}$$

is a finite open cover of X, has mesh less than 2^{-i}, and is a refinement of G_{i-1} if $i \geqslant 2$.

Consider

$$G_1 \times G_1 = \{g(1, i) \times g(1, j) \mid 1 \leqslant i, j \leqslant n_1\} = \{\hat{g}(1, i) \mid i \leqslant n_1^2\}.$$

There is a finite subsequence $d(1, 1), \ldots, d(1, n_1^2)$ of D such that, for each $j \leqslant n_1^2$,

$$d(1, j) \in u(1, j) \in G_1$$

(where $\hat{g}(1, j') = u(1, j') \times v(1, j')$ for each $j' \leqslant n_1^2$). Let

$$d^1 = (d(1, 1), \ldots, d(1, n_1^2)).$$

There is some h_1 in $H(X)$ such that

$$h_1(d^1) \in v(1, 1) \times \ldots \times v(1, n_1^2).$$

Consider

$$G_2 \times G_2 = \{g(2, i) \times g(2, j) \mid 1 \leqslant i, j \leqslant n_2\} = \{\hat{g}(2, i) \mid i \leqslant n_2^2\}.$$

There is a finite subsequence $d(2, 1), \ldots, d(2, n_2^2)$ of D such that, for each $j \leqslant n_2^2$,

$$d(2, j) \in u(2, j)$$

(again $\hat{g}(2, j') = u(2, j') \times v(2, j')$ for each $j' \leqslant n_2^2$). Let

$$d = \big(d(2, 1), \ldots, d(2, n_2^2)\big).$$

There is some h_2 in $H(X)$ such that

$$h_2(d^2) \in v(2, 1) \times \ldots \times v(2, n_2^2).$$

Continue this process. The resulting sequence $h_1, h_2, \ldots$ of $H(X)$ has a limit point C in Y. Thus, some subsequence $f_1, f_2, \ldots$ of $h_1, h_2, \ldots$ converges to C. Suppose that, for each i, f_i is the $q(i)$-th member of the sequence $h_1, h_2, \ldots$

Suppose $(x, y) \in X^2$ and $\varepsilon > 0$. There is $l \in N$ such that $2^{-l} < \varepsilon/16$. Suppose $i \geqslant l$. Then there is $j < n_{q(i)}^2$ such that

$$(x, y) \in u\big(q(i), j\big) \times v\big(q(i), j\big) \subseteq D_{\varepsilon/2}(x) \times D_{\varepsilon/2}(y),$$

and

$$f_i\big(d(q(i), j)\big) \in v\big(q(i), j\big) \quad \text{with } d\big(q(i), j\big) \in u\big(q(i), j\big).$$

Thus $(x, y) \in C$, and $C = X^2$.

A separable space X is *countable dense homogeneous* if whenever A and B are countable dense subsets of X there is some $h \in H(X)$ such that $h(A) = B$. Ungar [14] has shown that if X is a countable dense homogeneous continuum *other than* the simple closed curve, then X is strongly n-homogeneous for each n, i.e., if $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two n-element sets in X, then there is some h in $H(X)$ such that $h(a_i) = b_i$ for each $i \leqslant n$. Thus, if X is a countable dense homogeneous continuum other than the simple closed curve or if X is the pseudo-arc (see [8]), then $X^2 \in Y = \overline{H(X)}$. Now, if S is a simple closed curve, then $S^2 \notin \overline{H(S)}$. ($S$ cannot satisfy condition (2) of the preceding theorem.) However, S is a homogeneous continuum with rather unique properties, and thus we ask the following:

QUESTION (**P 1351**). If X is a homogeneous continuum other than a simple closed curve, is $X^2 \in Y = \overline{H(X)}$?

3. Connectivity relations. Keesling [6] has shown that if X is a compact metric space and $H(X)$ is locally compact, then $H(X)$ is 0-dimensional. However, if X is a compact metric space and X admits under the action of $H(X)$ a complete orbit which contains a nondegenerate continuum, then $H(X)$ is not locally compact [7].

Now, it has been conjectured that if X is a homogeneous continuum (nondegenerate), then $H(X)$ is not 0-dimensional. However, it is known that $H(X)$ can be totally disconnected [4]. We give here some relations between the connectedness of X and the connectedness of Y.

THEOREM 3. *If the compact metric space X contains a nondegenerate continuum P such that, for some x in X, $P \subseteq H(X)(x)$ and $H(X)(x)$ is complete, then $\overline{H(X)}$ is not totally disconnected.*

(In fact, one needs only to assume that G is a complete subgroup of $H(X)$ such that $Gx \supseteq P$ and Gx is complete to conclude that $\bar{G}$ (closure taken in $2^{X \times X}$) is not totally disconnected.)

Proof. Suppose that G is a complete subgroup of $H(X)$ and $x \in X$ such that Gx is complete and there is some nondegenerate continuum P such that $P \subseteq Gx$. If $\varepsilon > 0$, there is some $\delta > 0$ such that

$$Gx \cap D_\delta(z) \subseteq N_\varepsilon(1)(z) \quad \text{for each } z \text{ in } P.$$

[We take some metric d on X, and the discs $D_\alpha(y)$ for $\alpha > 0$, $y \in X$, with respect to this d. Then we take ϱ on $H(X)$ and G with respect to d, and here $N_\alpha(h) = \{f \in G \mid \varrho(h, f) < \alpha\}$.]

Without loss of generality, assume $x \in P$. Choose $y \in P$, $x \neq y$. There is, for each i in N, $\delta_i > 0$ such that

$$Gx \cap \overline{D_{\delta_i}(z)} \subseteq N_{2^{-i}}(1)(z) \quad \text{for each } z \text{ in } P.$$

Then there is a finite sequence

$$x = x(i, 0),\ x(i, 1), \ldots, x(i, n_i) = y$$

in P such that $d(x(i, j-1), x(i, j)) < \delta_i$ for each j such that $1 \leqslant j \leqslant n_i$. And then one can find $f(i, j)$ in $N_{2^{-i}}(1)$ such that

$$f(i, j)\big(x(i, j-1)\big) = x(i, j)$$

for $i \in N$, $1 \leqslant j \leqslant n_i$. In order to simplify notation, let

$$f(i, j) \circ f(i, j-1) \circ \ldots \circ f(i, 2) \circ f(i, 1) = \hat{f}(i, j) \quad \text{for } i \in N,\ 1 \leqslant j \leqslant n_i.$$

Thus $A_i = \{1 = \hat{f}(i, 0), \hat{f}(i, 1), \hat{f}(i, 2), \ldots, \hat{f}(i, n_i)\}$ has the following properties:

(1) $\varrho(\hat{f}(i, j-1), \hat{f}(i, j)) < 2^{-i}$ for each j such that $1 \leqslant j \leqslant n_i$;
(2) $\hat{f}(i, 0)(x) = 1(x) = x$;
(3) $\hat{f}(i, n_i)(x) = y$.

Since, for each i in N, A_i is a closed subset of $Y = \overline{H(X)}$, there are some closed subset A of Y and a subsequence $B_1, B_2, \ldots$ of $A_1, A_2, \ldots$ such that $B_1, B_2, \ldots$ converges to A (in $2^Y \subseteq 2^{2^{X \times X}}$). Now $1 \in B_i$ for each i means $1 \in A$. Also, some subsequence $g_1, g_2, \ldots$ of $f(1, n_1), f(2, n_2), \ldots$ such that, for each i, $g_i \in B_j$ for some j converges to some C in Y. Since $(x, y) \in \hat{f}(i, n_i)$ for each i, $(x, y) \in C$ and $C \neq 1$, A is not degenerate, but A is a continuum. Then $A \subseteq \bar{G} \subseteq Y$.

Remark. If X is a compact metric space, and $\overline{H(X)} = Y$ contains a nondegenerate continuum, then X contains a nondegenerate continuum. But this follows from an elementary fact: If Z is a compact metric space and 2^Z contains a nondegenerate continuum, then Z contains a nondegenerate continuum.

A space X is *representable* if for each $x \in X$ and u open in X such that $x \in u$ there is some open set v in X such that $x \in v \subseteq u$, and if $y \in v$, there is some homeomorphism from the space onto itself such that $h(x) = y$ and $h(z) = z$ for each $z \notin v$. [The preceding definition was actually first given as the definition of strongly locally homogeneous. Bales [3] proved that these 2 terms are equivalent, however.] Each representable continuum is countable dense homogeneous and locally connected. Observe the following: If X is a representable continuum, and u is a connected open set in X, then if $x, y \in u$, there is some h in $H(X)$ such that $h(x) = y$ and $h(z) = z$ for each $z \notin u$. (For more background, references for these terms, we refer the reader to [13].) Examples of representable continua include manifolds, including Hilbert cube manifolds, as well as the Menger universal curve.

THEOREM 4. *Suppose X is a representable continuum which is not a simple closed curve. Then if C is a subcontinuum of X,*

$$1 \cup (C \times C) \in \overline{H(X)} = Y.$$

Proof. Since X is not a simple closed curve, and X is countable dense homogeneous, X is strongly n-homogeneous for each n. Also, no finite subset of X separates X (see [14] and [15]).

Further, X has no local cut-points (see [5]), so X has a basis B of connected open sets such that no finite set separates any b in B. For each j in N, there is an open cover

$$U_j = \{u(j, 1), u(j, 2), \ldots, u(j, n_j)\} \subseteq B$$

of C such that for $i \leqslant n_j$ the diameter of $u(j, i)$ is less than 2^{-j}. Note that U_j^* is connected and that no finite set separates U_j^*. Next choose a finite subset of $(U_j^*)^2$ as follows: For each $(k, l) \in \{1, \ldots, n_j\}^2$,

$$x(j, k, l) \in u(j, k) \quad \text{and} \quad y(j, k, l) \in u(j, l)$$

and for $(k, l) \neq (k', l')$,

$$x(j, k, l) \neq x(j, k', l'), \quad y(j, k, l) \neq y(j, k', l');$$

for (k, l), (k', l') (equal or not),

$$x(j, k, l) \neq y(j, k', l').$$

Then find f_j in $H(X)$ such that

$$f_j(x(j, k, l)) = y(j, k, l) \quad \text{for each } (k, l) \text{ in } \{1, \ldots, n_j\}^2$$

and

$$f_j(z) = z \quad \text{for each } z \notin U_j^*.$$

To see how to do this – first find h_1 in $H(X)$ such that

$$h_1(x(j, k, l)) = y(j, k, l) \quad \text{for some } (k, l) \text{ in } \{1, \ldots, n_j\}^2$$

and

$$h_1(z) = z \quad \text{for } z \notin U_j^*.$$

Now $U_j^* - \{y(j, k, l)\}$ is connected, and there is h_2 in $H(X)$ such that

$$h_2 h_1(x(j, k', l')) = y(j, k', l') \quad \text{for } (k', l') \in \{1, \ldots, n_j\}^2 - \{(k, l)\}$$

and

$$h_2(z) = z \quad \text{for } z \notin U_j^* - \{y(j, k, l)\}.$$

Continue this process.

Then one can check that $f_1, f_2, \ldots$ converges to $1 \cup (C \times C)$.

Remarks. It is a corollary to the above theorem that if X is a representable continuum, then $\overline{H(X)}$ contains arcs (see [16], p. 186).

However, it is not true that, for X compact metric, $Y = \overline{H(X)}$ contains an arc implies X contains an arc. If P denotes the pseudo-arc, then $\overline{H(P)}$ contains arcs, as we see below.

THEOREM 5. *If P denotes the pseudo-arc, then $\overline{H(P)}$ contains arcs, and $H(P)$ is arcwise connected in $\overline{H(P)}$. Also, if P' is a subcontinuum of P, then*

$$1 \cup (P' \times P') \in Y.$$

Proof. Suppose $x \in P$ and Cx denotes the composant of P which contains x. Choose $y \neq x$ from Cx. There is a copy P' of the pseudo-arc in Cx such that $x \in P'$ and $y \in P'$. There is an arc A, which is a monotone family of continua in P, in 2^P such that $\{x\}$, $P' \in A$ (see [16], p. 186). Let α: $[0, 1] \to A$ be a homeomorphism such that $\alpha(0) = \{x\}$ and $\alpha(1) = P'$.

Fix $r \in [0, 1]$. For each $j \in N$ there is an open cover

$$U_j = \{u(j, i), \ldots, u(j, n_j)\}$$

of $\alpha(r)$ such that, for $i \leqslant n_j$, $\operatorname{diam} \overline{u(j, i)} < 2^{-j}$. If $r > 0$, for each $(k, l) \in \{1, \ldots, n_j\}^2$ choose a composant $K(j, k, l)$ of $\alpha(r)$ such that, for $(k, l) \neq (k', l')$,

$$K(j, k, l) \neq K(j, k', l').$$

There is a subcontinuum $\hat{K}(j, k, l)$ of $K(j, k, l)$ such that

$$\hat{K}(j, k, l) \cap u(j, k) \neq \emptyset \quad \text{and} \quad \hat{K}(j, k, l) \cap u(j, l) \neq \emptyset.$$

We now make use of the following theorem of Lewis [11]:

Let u be an open subset of the pseudo-arc P. Let p and q be distinct points of P so that the subcontinuum M irreducible between p and q does not intersect $\bar{u}$. Then there exists a homeomorphism h in $H(P)$ with $h(p) = q$ and $h|u = 1|u$.

For us, this means that we may now choose a homeomorphism k_j such that

$$k_j(\hat{x}) = \hat{x} \quad \text{for } \hat{x} \notin D_{2^{-j}}(\alpha(r)),$$

$$k_j(u(j, k)) \cap u(j, l) \neq \emptyset \quad \text{for } (k, l) \in \{1, \ldots, n_j\}^2 = F.$$

There is a collection $E_j = \{E(j, k, l) | \; (k, l) \in F\}$ of mutually exclusive open subsets of P such that

$$E_j^* \subseteq D_{2^{-j}}(\alpha(r)) \quad \text{and} \quad \hat{K}(j, k, l) \subseteq E(j, k, l) \text{ for each } (k, l) \in F.$$

Obtain a homeomorphism $h(j, k, l)$ for each $E(j, k, l)$ such that

$$h(j, k, l)(u(j, k)) \cap u(j, l) \neq \emptyset$$

and

$$h(j, k, l)(\hat{x}) = \hat{x} \quad \text{for } \hat{x} \notin E(j, k, l).$$

Then "combine" the $h(j, k)$'s to obtain k_j. One can then check that $k_1, k_2, \ldots$ converges to some E_r in $\overline{H(P)}$ such that

$$E_r = 1 \cup (\alpha(r) \times \alpha(r)).$$

Let $E_0 = 1$. Note that $E_0 = 1$; $E_1 = P'^2$; if $r < r'$ in $[0, 1]$, then $E_r \subsetneqq E_{r'}$; and $\{E_r | \; r \in [0, 1]\}$ is a continuum in $\overline{H(P)}$. Thus $\{E_r | \; r \in [0, 1]\}$ is an arc in $\overline{H(P)}$.

We could have taken P' to be P above, and the result would be the same; i.e., there is an arc E in $\overline{H(P)}$ from 1 to P^2. If $h \in H(P)$, then $h(E)$ is an arc in $\overline{H(P)}$ from h to $h(P^2) = P^2$.

Remarks. The pseudo-arc itself, of course, contains no arcs. Note also the following: Lewis has recently shown that $H(P)$ contains no nondegenerate subcontinua [11], but each h in $H(P)$ can be written as a finite composition of ε-homeomorphisms for $\varepsilon > 0$ (see [10]). (It follows that $\overline{H(P)}$ is a continuum.)

QUESTION **(P 1352)**. If X is a homogeneous continuum, does $\overline{H(X)}$ contain arcs?

THEOREM 6. *If X is a countable dense homogeneous continuum other than the circle, $Y = \overline{H(X)}$ is a continuum.*

Proof. Suppose D is a countable dense subset of X. Let

$$E = \{e_1, e_2, \ldots\} = D^2.$$

Ungar [14] showed that $H(X)$ acts transitively and micro-transitively on $F^n(X)$ for $n \in N$ in this case. Note that $F^n(X)$ is locally compact and if $\varepsilon > 0$ and

$$T = \{x = (x_1, \dots, x_n) \in X^n \mid d(x_i, x_j) \geqslant \varepsilon \text{ for } i, j \leqslant n, \ i \neq j\},$$

then T is a compact subset of $F^n(X)$. If $\varepsilon_n > 0$, then there is some $\delta_n > 0$ such that if $\{z_1, \dots, z_n\}$ and $\{y_1, \dots, y_n\}$ are two n-element subsets of X with

$$d(z_k, y_k) < \delta_n \quad \text{for each } k \leqslant n,$$

$$d(z_k, z_l) \geqslant \varepsilon_n \quad \text{and} \quad d(y_k, y_l) \geqslant \varepsilon_n \quad \text{for } l, k > n, \ l \neq k,$$

then there is some h in $N_{\varepsilon_n}(1)$ such that $h(z_k) = y_k$ for each $k \leqslant n$.

For each $n \in N$, $j \leqslant n$, choose a finite sequence

$$A(n, j) = a(n, j, 0), \dots, a(n, j, m(n, j))$$

of points of X as follows:

(1) Consider e_j first, $j \leqslant n$, and denote e_j by $(d(j, 1), d(j, 2))$. Choose $a(n, j, 0)$ such that

$$d(a(n, j, 0), d(j, 1)) < 2^{-n} \quad \text{for each } j$$

and

$$a(n, j, 0) \neq a(n, j', 0) \quad \text{for } j \neq j'.$$

Choose $a(n, j, m(n, j))$ such that

$$d(a(n, j, m(n, j)), d(j, 2)) < 2^{-n} \quad \text{for each } j$$

and

$$a(n, j, m(n, j)) \neq a(n, j', m(n, j')) \quad \text{for } j \neq j', \ j, j' \leqslant n,$$

and

$$a(n, j, m(n, j)) \neq a(n, j', 0) \quad \text{for } j, j' \leqslant n.$$

(2) Now no finite set separates X (see [15]). Thus, if

$$B(n, j) = \{d(1, 1), d(2, 1), \dots, d(n, 1), d(1, 2), \dots, d(n, 2)\} \cup \bigcup_{k \neq j} \{a(n, k, 0), a(n, k, m(n, k))\},$$

then $X - B(n, j)$ is connected, locally connected [15] and arcwise connected for each $j \leqslant n$. Suppose $P(n, j)$ is an arc in $X - B(n, j)$ from $a(n, j, 0)$ to $a(n, j, m(n, j))$. There is $\varepsilon_n > 0$ such that

$$\varepsilon_n < d(B(n, j), P(n, j))$$

in X for $j \leqslant n$ and $\varepsilon_n < 2^{-n}$. There is some $\delta_n > 0$, $\delta_n < \varepsilon_n$, such that if $\{z_1, \ldots, z_n\}$ and $\{y_1, \ldots, y_n\}$ are two n-element subsets of X with

$$d(z_k, y_k) < \delta_n \quad \text{for } k \leqslant n,$$

$$d(z_k, z_l) \geqslant \varepsilon_n \quad \text{and} \quad d(y_k, y_l) \geqslant \varepsilon_n \quad \text{for } k, l \leqslant n, \ k \neq l,$$

then there is $h \in N_{2^{-n}}(1)$ such that $h(z_k) = y_k$ for $k \leqslant n$.

Now for each $j \leqslant n$ choose $a(n, j, 1), \ldots, a(n, j, m(n, j)-1)$ so that

(1) $d(a(n, j, i-1), a(n, j, i)) < \delta_n$ for $1 \leqslant i \leqslant m(n, j)$;

(2) $a(n, j, i) \in P(n, j)$ for $1 \leqslant i \leqslant m(n, j)$;

(3) $A(n, j) \cap A(n, j') = \emptyset$ for $j \neq j'$, $j, j' \leqslant n$.

Let

$$\sum_{j=1}^{n} m(n, j) = m(n).$$

Choose a sequence $f(n, 1), \ldots, f(n, m(n))$ of homeomorphisms from $N_{2^{-n}}(1)$ as follows:

(1) $f(n, 1) \in N_{2^{-n}}(1)$ such that

$$f(n, 1)(a(n, 1, 0)) = a(n, 1, 1),$$

$$f(n, 1)(a(n, j, 0)) = a(n, j, 0) \quad \text{for } j \neq 1.$$

(2) $f(n, 2) \in N_{2^{-n}}(1)$ such that

$$f(n, 2)(a(n, 1, 1)) = a(n, 1, 2)$$

and

$$f(n, 2)(a(n, j, 0)) = a(n, j, 0) \quad \text{for } j \neq 1.$$

. . . .

$(m(n, 1))$ $f(n, m(n, 1)) \in N_{2^{-n}}(1)$ such that

$$f(n, m(n, 1))(a(n, 1, m(n, 1)-1)) = a(n, 1, m(n, 1))$$

and

$$f(n, m(n, 1))(a(n, j, 0)) = a(n, j, 0) \quad \text{for } j \neq 1.$$

$(m(n, 1)+1)$ Let $m(n, 1)+1 = b$. Then $f(n, b) \in N_{2^{-n}}(1)$ such that

$$f(n, b)(a(n, 1, m(n, 1))) = a(n, 1, m(n, 1)),$$

$$f(n, b)(a(n, 2, 0)) = a(n, 2, 1)$$

and

$$f(n, b)(a(n, j, 0)) = a(n, j, 0) \quad \text{for } 2 < j \leqslant n.$$

. . . .

$(m(n, 1)+m(n, 2))$ Let $m(n, 1)+m(n, 2) = b'$. Then $f(n, b') \in N_{2^{-n}}(1)$

such that

$$f(n, b')(a(n, 1, m(n, 1))) = a(n, 1, m(n, 1)),$$
$$f(n, b')(a(n, 2, m(n, 2)-1)) = a(n, 2, m(n, 2)),$$

and

$$f(n, b')(a(n, j, 0)) = a(n, j, 0) \quad \text{for } 2 < j \leqslant n.$$

. . . .

$(\sum_{j=1}^{n-1} m(n, j)+1)$ Let

$$\hat{b} - \sum_{j=1}^{n-1} m(n, j)+1.$$

Then $f(n, \hat{b}) \in N_{2^{-n}}(1)$ such that

$$f(n, \hat{b})(a(n, k, m(n, k))) = a(n, k, m(n, k)) \quad \text{for } k \leqslant n-1$$

and

$$f(n, \hat{b})(a(n, n, 0)) = a(n, n, 1).$$

$(m(n) = \sum_{j=1}^{n} m(n, j))$ $f(n, m(n)) \in N_{2^{-n}}(1)$ such that

$$f(n, m(n))(a(n, k, m(n, k))) = a(n, k, m(n, k)) \quad \text{for } k \leqslant n-1$$

and

$$f(n, m(n))(a(n, n, m(n)-1)) = a(n, n, m(n)).$$

As before, let

$$\hat{f}(n, j) = f(n, j) \circ f(n, j-1) \circ \ldots \circ f(n, 1).$$

For each n, let $\hat{B}_n = \{\hat{f}(n, 1), \ldots, \hat{f}(n, m(n))\}$. Some subsequence of $B_1, B_2, \ldots$ converges to a continuum B in Y. Now, for each $i \in N$, $\hat{f}(i, 1) \in N_{2^{-i}}(1)$, so $1 \in B$.

Since, for $i \in N$,

$$\hat{f}(i, m(i))(a(i, j, 0)) = a(i, j, m(i, j)) \quad \text{for } j \leqslant i,$$

one can check that some subsequence of the appropriate subsequence of this sequence converges to $X^2 \in B$. We are done, since if $h \in H(X)$, then $h(B)$ is a continuum from h to X^2.

QUESTION (**P 1353**). Suppose X is a homogeneous continuum. Is $\overline{H(X)} = Y$ connected? (If X is a simple closed curve, $\overline{H(X)}$ is connected, although the preceding argument would not give this.)

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