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ON CONFORMALLY RELATED CONFORMALLY RECURRENT METRICS I. SOME GENERAL RESULTS

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1. Introduction. Let (M, g) be a Riemannian manifold with a (possibly indefinite) metric g.

A tensor field $T^{i_1\ldots i_p}{}_{j_1\ldots j_q}$ of type (p, q) on M is called *recurrent* if

$$T^{h_1\ldots h_p}{}_{l_1\ldots l_q} T^{i_1\ldots i_p}{}_{j_1\ldots j_q,k} = T^{h_1\ldots h_p}{}_{l_1\ldots l_q,k} T^{i_1\ldots i_p}{}_{j_1\ldots j_q}, \tag{1}$$

where the comma denotes covariant differentiation with respect to g.

Relation (1) states that at any point $x \in M$ such that $T(x) \neq 0$ there exists a (unique) covariant vector u (called the *recurrence vector* of T) which satisfies the condition

$$T^{i_1\ldots i_p}{}_{j_1\ldots j_q,k}(x) = u_k T^{i_1\ldots i_p}{}_{j_1\ldots j_q}(x). \tag{2}$$

A Riemannian manifold (M, g) is called *recurrent* (*Ricci-recurrent*) if its curvature tensor (Ricci tensor) is recurrent.

According to Adati and Miyazawa [1], an n-dimensional $(n \geqslant 4)$ Riemannian manifold (M, g) is called *conformally recurrent* if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{ik}g_{hj}) \tag{3}$$

is recurrent.

If $C_{hijk,l} = 0$ everywhere on M and $\dim M \geqslant 4$, then (M, g) is said to be *conformally symmetric* [3]. Such a manifold is called *essentially conformally symmetric* [5] if it is neither conformally flat $(C_{hijk} = 0)$ nor locally symmetric $(R_{hijk,l} = 0)$.

Clearly, the class of conformally recurrent manifolds contains all conformally symmetric as well as all recurrent manifolds of dimension $n \geqslant 4$.

The existence of *essentially conformally recurrent manifolds*, i.e., of conformally recurrent manifolds which lie beyond the two classes mentioned above, has been established in [9]. Namely, using the expressions for components of the Weyl conformal curvature tensor and its covariant derivative (see [9], Lemmas 2-3), one can easily verify the following theorem:

THEOREM A. *Let M denote the Euclidean n-space ($n \geqslant 4$) endowed with the metric g_{ij} given by*

$$g_{ij}dx^i dx^j = Q(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n,\quad Q = [(\exp(x^1))c_{\lambda\mu} + k_{\lambda\mu}]x^\lambda x^\mu, \tag{4}$$

where $i, j = 1, 2, \ldots, n$, $\lambda, \mu = 2, 3, \ldots, n-1$, $(k_{\lambda\mu})$ is a symmetric and non-singular matrix, and $(c_{\lambda\mu})$ is a symmetric matrix satisfying

$$(c_{\lambda\mu}) \neq \frac{1}{n-2}(k_{\lambda\mu}) \quad \textit{and} \quad k^{\lambda\mu}c_{\lambda\mu} = 1 \textit{ with } (k^{\lambda\mu}) = (k_{\lambda\mu})^{-1}.$$

Then (M, g) is an essentially conformally recurrent Ricci-recurrent manifold satisfying

$$C_{hijk,lm} - C_{hijk,ml} = 0 \tag{5}$$

and its recurrence vector is non-zero everywhere on M.

Let (M, g) be an n-dimensional Riemannian manifold whose metric g need not be definite. If $\bar{g}$ is another metric on M and there exists a function p on M such that $\bar{g} = (\exp(2p))g$, then g and $\bar{g}$ are said to be *conformally related* or *conformal to each other*, and such a change of metric $g \to \bar{g}$ is called a *conformal change*. If $p = \text{const}$, then the conformal change of metric is called *trivial* or a *homothety*.

Conformally related conformally symmetric metrics have been studied by Adati and Miyazawa. Their main result ([2], Theorem 4.1) can be formulated as follows:

THEOREM B. *Let M be a conformally symmetric manifold with positive definite metric g. If $\bar{g}$ is a conformally symmetric metric on M such that g and $\bar{g}$ are conformally related, then both g and $\bar{g}$ are conformally flat or the conformal change of g is a homothety.*

Similar problems were studied by Miyazawa for positive definite conformally recurrent metrics (see [7], Theorem 1.1) as well as for conformally recurrent metrics with the same recurrence vectors (see [8], Theorem 3).

The purpose of the present paper is to investigate (without additional assumptions) conformally related conformally recurrent metrics. More precisely, we shall prove the following theorems:

THEOREM 1. *Suppose that M admits two conformally recurrent metrics g and $\bar{g}$ conformally related by $\bar{g} = (\exp(2p))g$. Let a_j and $\bar{a}_j$ be the recurrence vectors of C and $\bar{C}$, respectively.*

Then:

(a) $p_l C^h{}_{ikj} + p_k C^h{}_{ijl} + p_j C^h{}_{ilk} = 0$ *everywhere on* M.

(b) *At each given point of M we have $C^h{}_{ijk} = 0 = \bar{C}^h{}_{ijk}$ or $\bar{a}_j = a_j - 4p_j$ and $p^r p_r = 0$, where $p_j = \partial_j p$.*

THEOREM 2. *Suppose that (M, g) is conformally recurrent. If p is a function on M satisfying condition* (a), *then $\bar{g} = (\exp(2p))g$ is conformally recurrent.*

THEOREM 3. *Let (M, g) be conformally symmetric. If $\bar{g}$ is a conformally symmetric metric on M such that g and $\bar{g}$ are conformally related, then both g and $\bar{g}$ are conformally flat or the conformal change of g is a homothety.*

We shall also show that there exist conformally related and non-homothetic essentially conformally recurrent metrics.

All manifolds under consideration are assumed to be connected and of class C^∞. The Riemannian metrics are not assumed to be definite.

2. Preliminary results. In the sequel we shall need the following lemmas:

LEMMA 1. *The Weyl conformal curvature tensor satisfies the following well-known relations:*

$$C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \tag{6}$$

$$C_{hijk} + C_{hjki} + C_{hkij} = 0, \quad C^r{}_{ijr} = C^r{}_{irk} = C^r{}_{rjk} = 0. \tag{7}$$

LEMMA 2. *If c_j and T_{ij} are numbers satisfying*

$$c_l T_{ij} + c_i T_{lj} + c_j T_{li} = 0, \tag{8}$$

then each c_j is zero or each T_{ij} is zero.

The proof is trivial.

LEMMA 3. *If c_j, p_j, and B_{hijk} are numbers satisfying*

$$c_l B_{hijk} + p_h B_{lijk} + p_i B_{hljk} + p_j B_{hilk} + p_k B_{hijl} = 0, \tag{9}$$

$$B_{hijk} = B_{jkhi} = -B_{hikj}, \quad B_{hijk} + B_{hjki} + B_{hkij} = 0, \tag{10}$$

then each $b_j = c_j + 2p_j$ is zero or each B_{hijk} is zero.

Proof. Suppose that one of the b's, say b_q, is not zero. Then (9) with $l = h = k = q$ gives $b_q B_{qijq} = 0$, since $B_{qqjq} = 0 = B_{qjqq}$, and, consequently, $B_{qijq} = 0$ for all i and j. Setting $k = h = q$ in (9) and applying

$B_{qijq} = 0$, we get

$$p_q(B_{lijq}+B_{qijl}) = 0. \tag{11}$$

Assume $p_q = 0$. Then $c_q = b_q \neq 0$ and, therefore, (9) with $l = h = q$ yields $c_q B_{qijk} = 0$ since $B_{qijq} = 0$. Hence $B_{qijk} = 0$ for all i, j, and k. Setting now $l = q$ in (9) and using the first equation of (10), we obtain $c_q B_{hijk} = 0$, whence $B_{hijk} = 0$ for all h, i, j, and k.

Suppose now that $p_q \neq 0$. Then (11), in view of the first equation of (10), implies

$$B_{qijl} = B_{qjli}. \tag{12}$$

But the second equation of (10) yields $B_{qijk}+B_{qjki}+B_{qkij} = 0$, whence, in view of (12), $B_{qjki}+B_{qjki}+B_{qijk} = 0$. Applying to B_{qijk} the condition (12), we obtain easily $B_{qjki} = 0$ for all j, k, and i. Setting now $h = q$ in (9) and taking into account $B_{qijk} = 0$, we get $p_q B_{lijk} = 0$, which, evidently, completes the proof.

LEMMA 4. *If c_j, p_j, and D_{hij} are numbers satisfying*

$$c_l D_{ijk}+p_i D_{ljk}+p_j D_{ilk}+p_k D_{ijl} = 0, \tag{13}$$

$$D_{ijk} = -D_{ikj}, \qquad D_{ijk}+D_{jki}+D_{kij} = 0, \tag{14}$$

then each $b_j = c_j+2p_j$ is zero or each D_{ijk} is zero.

Proof. Suppose that one of the b's, say b_q, is not zero. Then, by an argument similar to that in the proof of Lemma 3, we obtain $D_{qjq} = 0$ and

$$p_q(D_{ljq}+D_{qjl}) = 0. \tag{15}$$

Assume $p_q = 0$. Then $c_q = b_q \neq 0$ and, therefore, (13) with $l = k = q$ yields $c_q D_{ijq} = 0$ since $D_{qjq} = 0 = D_{jqq}$. Hence $D_{ijq} = 0$ for all i and j. On the other hand, setting $l = i = q$ in (13) and using the first equation of (14), we get $c_q D_{qjk} = 0$, whence $D_{qjk} = 0$ for all j and k. Now, (13) with $l = q$ implies $c_q D_{ijk} = 0$. Thus $D_{ijk} = 0$ for all i, j, and k.

If now $p_q \neq 0$, then (15), in view of the first equation of (14), gives

$$D_{ljq} = D_{qlj}. \tag{16}$$

On the other hand, the second equation of (14) implies $D_{qjk}+D_{jkq}+$ $+D_{kqj} = 0$, which, because of (16), yields $2D_{qjk}-D_{kjq} = 0$. But the last result, in view of (16) and (14), gives $D_{qjk} = 0$ for all j and k. Setting now $i = q$ in (13) and applying $D_{qjk} = 0$, we obtain easily $p_q D_{ljk} = 0$. Thus the lemma is proved.

LEMMA 5. *Let* $\bar{g}_{ij} = (\exp(2p))g_{ij}$. *Then we have* ([6], p. 89 and 90)

$$\left\{\overline{k \atop ij}\right\} = \left\{k \atop ij\right\} + \delta_i^k p_j + \delta_j^k p_i - p^k g_{ij}, \tag{17}$$

$$\bar{C}^h{}_{ijk} = C^h{}_{ijk}, \tag{18}$$

$$\bar{R}_{ij} = R_{ij} + (n-2)(p_{i,j} - p_i p_j) + (p^r{}_{,r} + (n-2)p^r p_r)g_{ij}, \tag{19}$$

where $p_j = \partial_j p$ *and* $p^k = g^{kr} p_r$.

PROPOSITION 1. *Let* g *and* $\bar{g}$ *be two conformally recurrent metrics on* M. *If* g *and* $\bar{g}$ *are conformally related by* $\bar{g} = (\exp(2p))g$, *then* $p_r C^r{}_{ijk} = 0$.

Proof. It is sufficient to prove our assertion in the open subset of M where $C^h{}_{ijk} \neq 0$.

Differentiating (18) covariantly and using (17), we get

$$\begin{aligned}\bar{C}^h{}_{ijk;l} = C^h{}_{ijk,l} + \delta_l^h p_r C^r{}_{ijk} - 2p_l C^h{}_{ijk} - p^h C_{lijk} - p_i C^h{}_{ljk} - p_j C^h{}_{ilk} - \\ - p_k C^h{}_{ijl} + g_{il} p^r C^h{}_{rjk} + g_{jl} p^r C^h{}_{irk} + g_{kl} p^r C^h{}_{ijr},\end{aligned} \tag{20}$$

where the semicolon denotes covariant differentiation with respect to $\bar{g}$.

Since g and $\bar{g}$ are both conformally recurrent by assumption, relation (2) implies $C^h{}_{ijk,l} = a_l C^h{}_{ijk}$ and $\bar{C}^h{}_{ijk;l} = \bar{a}_l \bar{C}^h{}_{ijk}$, where a_j and $\bar{a}$ denote the recurrence vectors of C and $\bar{C}$, respectively.

Hence (20) can be written as

$$\begin{aligned}(\bar{a}_l - a_l)C_{hijk} = g_{hl} p_r C^r{}_{ijk} + g_{il} p^r C_{hrjk} + g_{jl} p^r C_{hirk} + \\ + g_{kl} p^r C_{hijr} - 2p_l C_{hijk} - p_h C_{lijk} - p_i C_{hljk} - p_j C_{hilk} - p_k C_{hijl},\end{aligned} \tag{21}$$

whence, by Lemma 1 and contraction with g^{hl}, we obtain

$$d_r C^r{}_{ijk} = (n-3)p_r C^r{}_{ijk}, \tag{22}$$

where $d_j = \bar{a}_j - a_j$.

Transvecting now (21) with p^h, we get

$$\begin{aligned}d_l p_r C^r{}_{ijk} = -p_l p_r C^r{}_{ijk} - p^r p_r C_{lijk} - p_i p_r C^r{}_{ljk} - \\ - p_j p_r C^r{}_{ilk} - p_k p_r C^r{}_{ijl} - g_{jl} T_{ik} + g_{kl} T_{ij},\end{aligned} \tag{23}$$

where $T_{ij} = p^r p^s C_{rijs}$.

But (23), by transvection with p^k, implies

$$p^s p_s (p_r C^r{}_{ijl} + p_r C^r{}_{jil}) + p_i T_{lj} + p_j T_{il} = -d_l T_{ij}. \tag{24}$$

Permuting the indices i, j, l in (24) cyclically, adding the resulting equations to (24) and using (7), we get

$$(d_l + 2p_l)T_{ij} + (d_i + 2p_i)T_{lj} + (d_j + 2p_j)T_{li} = 0,$$

which, evidently, is of the form (8). If, according to Lemma 2, $d_j = -2p_j$, then (22) implies $(n-1)p_r C^r{}_{ijk} = 0$. Hence $T_{ij} = 0$ at each point.

Substituting the last result into (24), we obtain

$$p^r p_r (p_s C^s{}_{ijl} + p_s C^s{}_{jil}) = 0.$$

Suppose now that at a given point we have $p^r p_r = 0$. Then (23) implies

$$(d_l + p_l) p_r C^r{}_{ijk} + p_i p_r C^r{}_{ljk} + p_j p_r C^r{}_{ilk} + p_k p_r C^r{}_{ijl} = 0.$$

Hence, in view of Lemma 4, we have $d_j = -3p_j$ or $p_r C^r{}_{ijk} = 0$. If $d_j = -3p_j$, then (22) yields $p_r C^r{}_{ijk} = 0$.

Assume now $p_r C^r{}_{ijl} + p_r C^r{}_{jil} = 0$. Then, in view of Lemma 1, we have $p_r C^r{}_{ijl} = p_r C^r{}_{jli}$, which, in virtue of $p_r C^r{}_{ijl} + p_r C^r{}_{jli} + p_r C^r{}_{lij} = 0$, gives $2p_r C^r{}_{jli} - p_r C^r{}_{lji} = 0$ and, consequently, $3p_r C^r{}_{jli} = 0$. Thus our assertion is proved.

3. Main results. We are now in a position to prove Theorems 1-3.

Proof of Theorem 1. It is sufficient to prove our assertion at points where $C^h{}_{ijk} \neq 0$.

Since $p_r C^r{}_{ijk} = 0$, (21) yields

$$(d_l + 2p_l) C_{hijk} + p_h C_{lijk} + p_i C_{hljk} + p_j C_{hilk} + p_k C_{hijl} = 0, \tag{25}$$

whence, using Lemma 3, we get $\bar{a}_j = a_j - 4p_j$.

Hence (25) takes the form

$$p_h C_{lijk} + p_i C_{hljk} + p_j C_{hilk} + p_k C_{hijl} - 2p_l C_{hijk} = 0. \tag{26}$$

Permuting the indices l, j, k in (26) cyclically, adding the resulting equations to (26) and using Lemma 1, we obtain easily (a). But the last result, together with $p_r C^r{}_{ijk} = 0$, implies $p^r p_r = 0$, which completes the proof.

Proof of Theorem 2. Assume now (a). Then, by (6), we have

$$(p_j C_{hilk} + p_k C_{hijl} + p_l C_{hikj}) + (p_h C_{lijk} + p_i C_{hljk} + p_l C_{ihjk}) = 0,$$

whence

$$-4p_l C^h{}_{ijk} = -2p_l C^h{}_{ijk} - p^h C_{lijk} - p_i C^h{}_{ljk} - p_j C^h{}_{ilk} - p_k C^h{}_{ijl}. \tag{27}$$

But as an immediate consequence of (a) we have $p_r C^r{}_{ijk} = 0$, which, together with (27) and (20), yields $\bar{C}^h{}_{ijk;l} + 4p_l C^h{}_{ijk} = C^h{}_{ijk,l}$. Hence $\bar{C}^h{}_{ijk;l} = (a_l - 4p_l)\bar{C}^h{}_{ijk}$ at points where $\bar{C}^h{}_{ijk} \neq 0$, which, evidently, completes the proof.

Proof of Theorem 3. Since C and $\bar{C}$ are parallel, we may assume that $C^h{}_{ijk} \neq 0 \neq \bar{C}^h{}_{ijk}$ everywhere. Hence, because of (2), $\bar{a}_j = a_j = 0$. But, in view of Theorem 1, the last relation implies $p_j = 0$, which completes the proof.

PROPOSITION 2. *For each $n \geqslant 4$, there exist n-dimensional pairwise conformally related and non-homothetic metrics g, g_1, g_2, g_3 such that g satisfies* (5) *and*

(i) *g, g_1 are essentially conformally recurrent,*

(ii) *g_2 is recurrent,*

(iii) *g_3 is essentially conformally symmetric.*

Proof. As one can easily verify, in the metric (4) the only Christoffel symbols not identically zero are

$$\begin{Bmatrix} \lambda \\ 11 \end{Bmatrix} = -\frac{1}{2} k^{\lambda\omega} Q_{.\omega}, \quad \begin{Bmatrix} n \\ 11 \end{Bmatrix} = \frac{1}{2} Q_{.1}, \quad \begin{Bmatrix} n \\ 1\lambda \end{Bmatrix} = \frac{1}{2} Q_{.\lambda},$$

where the dot denotes partial differentiation with respect to coordinates.

It can be also found [9] that the only components of R_{ij}, $R_{ij,k}$, and C_{hijk} not identically zero are those related to

$$R_{11} = n-2+\exp(x^1), \qquad R_{11,1} = \exp(x^1),$$

$$C_{1\lambda\mu 1} = \left(c_{\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu}\right)\exp(x^1).$$

Moreover, we can easily show that $a_j = \delta_j^1$ and that condition (a) is satisfied for $p = cx^1$, where c is an arbitrary constant. In view of Theorem 2, the metric $\bar{g}_{ij} = \big(\exp(2cx^1)\big) g_{ij}$ is conformally recurrent. $\bar{C}^h{}_{ijk} \neq 0$ everywhere and $\bar{a}_j = a_j - 4p_j = (1-4c)\,\delta_j^1$.

Hence $\bar{g}$ is recurrent if and only if its Ricci tensor $\bar{R}_{ij}$ satisfies

$$\bar{R}_{ij;k} = \bar{a}_k \bar{R}_{ij}. \tag{28}$$

In view of (17), (19), $p_{i,j} = 0$, and $p^r p_r = p^r{}_{,r} = 0$, we get

$$\bar{R}_{ij;k} = R_{ij,k} - 2p_k R_{ij} - p_i R_{jk} - p_j R_{ik} + 4(n-2) p_i p_j p_k.$$

Thus (28) can be written as

$$R_{ij,k} + 2p_k R_{ij} - p_i R_{jk} - p_j R_{ik} - a_k R_{ij} + (n-2) a_k p_i p_j = 0. \tag{29}$$

If now $c = 1$, then (29) is satisfied and the metric $g_2 = \big(\exp(2x^1)\big) g$ is recurrent. If $c = 2$, then we have $R_{11,1} + 2p_1 R_{11} - p_1 R_{11} - p_1 R_{11} - a_1 R_{11} + (n-2) a_1 p_1 p_1 = 3(n-2) \neq 0$. Hence $g_1 = \big(\exp(4x^1)\big) g$ is essentially conformally recurrent. Finally, if $c = \frac{1}{4}$, then $\bar{a}_j = (1-4c)\,\delta_j^1 = 0$ and

$g_3 = (\exp(\frac{1}{2}x^1))g$ is conformally symmetric. Since $\bar{R}_{11;1} \neq 0$, g_3 is essentially conformally symmetric. This completes the proof.

As an immediate consequence of Theorems 1 and 2, we have the following

COROLLARY. *Let (M, g) be conformally recurrent and p a function on M. Then $\bar{g} = (\exp(2p))g$ is conformally recurrent if and only if p satisfies condition* (a).

Remark. Conformally symmetric manifolds with positive definite metrics are necessarily conformally flat or locally symmetric (see [4], Theorem 2). On the other hand, as proved in [5] (see Theorems 2 and 4), for each $q \in \{1, 2, \ldots, n-1\}$ there exist essentially conformally symmetric manifolds with metrics of index q.

Theorem 3 deals therefore with a more general class of Riemannian manifolds than Theorem B. Moreover, it implies Theorem B.

Theorem 3 can be also deduced from Miyazawa's Lemmas 1 and 2 of [8]. Theorem 1.1 of [7] and Theorem 3 of [8] are consequences of Theorem 1.

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