

SOME ALGEBRAIC THEOREMS ON VECTOR-VALUED FORMS  
AND THEIR GEOMETRIC APPLICATIONS

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**Preface.** The study of those properties of submanifolds in  $E^n$  which depend on the concept of a *type number* makes a nice piece of the classical metric differential geometry.

Properties of a hypersurface  $V_n$  in  $E^{n+1}$  related to its type number (i.e., to the rank of the second fundamental form) were investigated by Thomas [14]. Further, Allendoerfer has introduced in [1] a generalized concept of the type with respect to a *system* of second fundamental forms, and has obtained a number of theorems for a Riemannian submanifold  $V_n$  isometrically embedded into  $E^{n+p}$ . One can summarize Allendoerfer's results as follows:

**THEOREM A.** *If the first normal space of a simply connected  $V_n \subset E^{n+p}$  is of  $q \leq p$  dimensions at every point and if  $V_n$  is of type  $\geq 3$  at every point, then  $V_n$  can be imbedded into  $E^{n+q}$  and this imbedding is unique to within a rigid motion.*

**THEOREM B.** *If the type of  $V_n$  is  $\geq 4$  at every point, the Codazzi equations are consequences of the Gauss equations.*

As concerns the "rigidity" part of Theorem A, Chern has shown its purely algebraic nature in [6] (cf. also [8], Note 17). An algebraic nature of the "integrability" part of Theorem A and, particularly, of Theorem B was stated by the author in [11] (cf. Lemmas 1-4). In the same paper, algebraic theorems led to further generalizations of the previous geometric results.

In the present work we are going to show that the concept of type as well as the corresponding algebraic theorems can be generalized essentially. Namely, while the former results made sense for *linear* vector-valued forms only, the generalized ones are valid for *arbitrary* vector-valued forms. The Main Theorem of this paper corresponds to geometric Theorem B. It is worth mentioning that the Main Theorem generalizes also the classical *Cartan's lemmas* in a sense that, roughly speaking, the

*independent vector variables are replaced by arbitrary vector-valued forms.* First three sections are devoted to the formulation and proof of that theorem.

A general algebraic "integrability theorem" and also "rigidity theorem" is dealt with in Sections 4 and 5.

Finally, in the last Section 6 we develop a corresponding geometric theory. We show that all results by Allendoerfer [1] are true in a more general situation, exceeding the scope of the metric differential geometry. We obtain a number of theorems for *metric connections in Riemannian vector bundles*.

**1. Formulation of the Main Theorem.** Let  $V, F$  be (non-trivial) vector spaces over the field  $R$  of real numbers. Let  $L^p(V, F)$  denote the space of all  $F$ -valued  $p$ -linear forms on  $V$  and  $\wedge^p(V, F)$  (or  $S^p(V, F)$ ) the subspace of all antisymmetric (or symmetric) forms in  $L^p(V, F)$ ,  $p = 1, 2, \dots$ . In addition, for  $p = 0$ , we set

$$(1.1) \quad L^0(V, F) = \wedge^0(V, F) = S^0(V, F) = F.$$

In the case  $F = R$  we introduce the usual notation

$$(1.2) \quad \begin{aligned} \wedge^{*p} V &= \wedge^p(V, R), & S^{*p} V &= S^p(V, R), \\ \wedge^p V &= \wedge^p(V^*, R), & S^p V &= S^p(V^*, R). \end{aligned}$$

There are canonical isomorphisms between  $L^p(V, F)$ ,  $\wedge^p(V, F)$ ,  $S^p(V, F)$  and  $\text{Hom}(\otimes^p V, F)$ ,  $\text{Hom}(\wedge^p V, F)$ ,  $\text{Hom}(S^p V, F)$ , respectively (cf. [4]).

Let  $E$  be another vector space over  $R$ . For any  $\alpha \in \wedge^p(V, E)$  and  $\beta \in \wedge^q(V, F)$  we define the *product*  $\alpha \times \beta \in \wedge^{p+q}(V, E \otimes F)$  as

$$(1.3) \quad \begin{aligned} (\alpha \times \beta)(x_1, \dots, x_{p+q}) \\ = \sum_{\varrho \in \mathfrak{S}_{p,q}} \text{sgn } \varrho \cdot \alpha(x_{\varrho(1)}, \dots, x_{\varrho(p)}) \otimes \beta(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) \end{aligned}$$

for any  $x_1, \dots, x_{p+q} \in V$ ,

where  $\mathfrak{S}_{p,q}$  denotes the set of all  $(p, q)$ -shuffles; specifically,

$$(1.4) \quad \mathfrak{S}_{p,q} = \{\varrho \in \mathfrak{S}_{p+q} \mid \varrho(1) < \dots < \varrho(p) \text{ and } \varrho(p+1) < \dots < \varrho(p+q)\}$$

(cf. [3], Chapter 4, and [2], Chapitre II).

Similarly, for any  $\alpha \in S^p(V, E)$  and  $\beta \in S^q(V, F)$  we define the *product*  $\alpha \times \beta \in S^{p+q}(V, E \otimes F)$  as follows:

$$(1.5) \quad \begin{aligned} (\alpha \times \beta)(x_1, \dots, x_{p+q}) \\ = \sum_{\varrho \in \mathfrak{S}_{p,q}} \alpha(x_{\varrho(1)}, \dots, x_{\varrho(p)}) \otimes \beta(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}). \end{aligned}$$

In particular, for  $h \in \wedge^{*p} V$  and  $\beta \in \wedge^q(V, F)$  we get  $h \times \beta \in \wedge^{p+q}(V, F)$ , and for  $h \in S^{*p} V$  and  $\beta \in S^q(V, F)$  we get  $h \times \beta \in S^{p+q}(V, F)$  with respect to the canonical isomorphism  $R \otimes F \cong F$ .

**Remark.** Our operations make sense for  $p \geq 0, q \geq 0$ . In particular, in the case  $p = 0, q = 0$  we have  $\alpha \in E, \beta \in F, \alpha \times \beta = \alpha \otimes \beta$ .

A linear map  $f: \wedge^p V \otimes E \rightarrow F$  is called of *type  $t$*  if

(i) there is a subspace  $V_0 \subset V$  of dimension  $t$  such that the restriction of  $f$  to  $\wedge^p V_0 \otimes E$  is injective,

(ii)  $t$  is the maximum number of that property.

Each subspace  $V_0 \subset V$  satisfying (i) and (ii) will be called a *distinguished subspace with respect to  $f$* .

In particular, a  $d$ -tuple  $(\alpha_1, \dots, \alpha_d)$  of forms  $\alpha_i \in \wedge^p(V, F)$  can be considered as a linear map  $f: \wedge^p V \otimes R^d \rightarrow F$ . The  $d$ -tuple  $(\alpha_1, \dots, \alpha_d)$  is said to be of *type  $t$*  if the corresponding map  $f$  is of type  $t$ .

**Notation.**  $t = t(\alpha_1, \dots, \alpha_d)$ .

Similarly, we define the type of a linear map  $g: S^p V \otimes E \rightarrow F$  and the type of a  $d$ -tuple of forms  $(\alpha_1, \dots, \alpha_d), \alpha_i \in S^p(V, F)$ .

**Remark.** The above-mentioned definitions of type generalize those of [11] and [8] (cf. Note 17).

**LEMMA 1.** *Let  $\alpha_i \in \wedge^p(V, F)$  (or  $\alpha_i \in S^p(V, F)$ ) for  $i = 1, \dots, d$  and  $t(\alpha_1, \dots, \alpha_d) = t$ . Then there is a  $t$ -dimensional subspace  $V_0 \subset V$ , called a distinguished subspace of  $(\alpha_1, \dots, \alpha_d)$ , with the following property: for any linearly independent vectors  $e_1, \dots, e_k \in V_0$  the vectors  $\alpha_j(e_{i_1}, \dots, e_{i_p})$  ( $j = 1, \dots, d, 1 \leq i_1 < \dots < i_p \leq k$ ) are linearly independent in  $F$ .*

**Proof** is obvious.

Let us introduce canonical projections

$$(1.6) \quad \text{Alt}: F \otimes F \rightarrow \wedge^2 F, \quad \text{Sym}: F \otimes F \rightarrow S^2 F$$

by the rules  $\text{Alt}(f \otimes f') = f \otimes f' - f' \otimes f$ ,  $\text{Sym}(f \otimes f') = f \otimes f' + f' \otimes f$ . Let  $R^{d \times d}$  denote the vector space of all real  $(d \times d)$ -matrices. Now we can formulate the basic algebraic theorem:

**MAIN THEOREM.** *Let  $p \geq q \geq 0$  be integers and  $(\omega_1, \dots, \omega_d), (\eta_1, \dots, \eta_d)$  two collections of forms such that either  $\omega_i \in \wedge^p(V, F), \eta_i \in \wedge^q(V, F)$ ,  $i = 1, \dots, d$ , or  $\omega_i \in S^p(V, F), \eta_i \in S^q(V, F)$ ,  $i = 1, \dots, d$ .*

*Suppose  $t(\eta_1, \dots, \eta_d) \geq p + q + 1$ .*

(I) *If  $\omega_i \in \wedge^p(V, F), \eta_i \in \wedge^q(V, F)$  and  $\text{Alt} \circ \left( \sum_{i=1}^d \omega_i \times \eta_i \right) = 0$  (or  $\text{Sym} \circ \left( \sum_{i=1}^d \omega_i \times \eta_i \right) = 0$ ), then there is a unique element  $\|h_i^j\| \in \wedge^{p-q}(V, R^{d \times d})$  such that*

$$\omega_i = \sum_{j=1}^d h_i^j \times \eta_j, \quad i = 1, \dots, d.$$

Moreover,  $h^i_j = (-1)^q h^j_i$  (or  $h^i_j = (-1)^{q+1} h^j_i$ ) for  $i, j = 1, \dots, d$ .

(II) If  $\omega_i \in S^p(V, F)$ ,  $\eta_i \in S^q(V, F)$  and  $\text{Alt} \circ \left( \sum_{i=1}^d \omega_i \times \eta_i \right) = 0$  (or  $\text{Sym} \circ \left( \sum_{i=1}^d \omega_i \times \eta_i \right) = 0$ ), then there is a unique element  $\|h^j_i\| \in S^{p-q}(V, R^{d \times d})$  such that

$$\omega_i = \sum_{j=1}^d h^j_i \times \eta_j, \quad i = 1, \dots, d.$$

Moreover,  $h^i_j = h^j_i$  (or  $h^i_j = -h^j_i$ ) for  $i, j = 1, \dots, d$ .

Remark. For  $p = q = 0$  we obtain two well-known versions of Cartan's lemma. In fact, in this case we have  $\omega_i, \eta_i \in F$ ,  $\omega_i \times \eta_i = \omega_i \otimes \eta_i$ ,

$$\text{Alt} \circ \left( \sum_{i=1}^d \omega_i \times \eta_i \right) = \sum_{i=1}^d \omega_i \wedge \eta_i,$$

$$\text{Sym} \circ \left( \sum_{i=1}^d \omega_i \times \eta_i \right) = \sum_{i=1}^d \omega_i \eta_i.$$

Moreover,  $t(\eta_1, \dots, \eta_d) \geq 1$  means that  $\eta_1, \dots, \eta_d$  are linearly independent elements of  $F$  (see [5]).

We shall present a complete proof of the first assertion only; the others can be proved by a similar argument. We are going to state the assertion as a special theorem. Firstly, for  $\omega \in \wedge^p(V, F)$ ,  $\eta \in \wedge^q(V, F)$ , we put  $\omega \cdot \eta = \text{Alt} \circ (\omega \times \eta)$ , and for  $h \in \wedge^{*p} V$ ,  $\eta \in \wedge^q(V, F)$  we put  $h \wedge \eta = h \times \eta$ . Clearly,

$$(1.7) \quad (\omega \cdot \eta)(x_1, \dots, x_{p+q}) \\ = \sum_{e \in \mathfrak{S}_{p,q}} \text{sgn } e \cdot \omega(x_{e(1)}, \dots, x_{e(p)}) \wedge \eta(x_{e(p+1)}, \dots, x_{e(p+q)}),$$

$$(1.8) \quad (h \wedge \eta)(x_1, \dots, x_{p+q}) \\ = \sum_{e \in \mathfrak{S}_{p,q}} \text{sgn } e \cdot h(x_{e(1)}, \dots, x_{e(p)}) \cdot \eta(x_{e(p+1)}, \dots, x_{e(p+q)}).$$

We can easily see that if  $(f_1, \dots, f_m)$  is a basis of  $F$ ,

$$\omega = \sum_{i=1}^m \omega^i \otimes f_i, \quad \eta = \sum_{i=1}^m \eta^i \otimes f_i,$$

then

$$\omega \cdot \eta = \sum_{i,j=1}^m (\omega^i \wedge \eta^j) \otimes (f_i \wedge f_j),$$

$$h \wedge \eta = \sum_{i,j=1}^m (h \wedge \eta^j) \otimes f_j.$$

On the right-hand sides we have here usual exterior products (cf. [2]).

**THEOREM 1.1.** *Let  $(\omega_1, \dots, \omega_d), (\eta_1, \dots, \eta_d)$  be two collections of forms,  $\omega_i \in \wedge^p(V, F), \eta_i \in \wedge^q(V, F)$  for  $i = 1, \dots, d, p \geq q$ . Suppose  $t(\eta_1, \dots, \eta_d) \geq p + q + 1$ . If  $\omega_1 \cdot \eta_1 + \dots + \omega_d \cdot \eta_d = 0$ , then there is a unique matrix-valued  $(p - q)$ -form  $\|h_i^j\|$  ( $i, j = 1, \dots, d$ ) such that*

$$(1.9) \quad \omega_i = \sum_{j=1}^d h_i^j \wedge \eta_j, \quad i = 1, \dots, d.$$

Moreover,  $h_j^i = (-1)^q h_i^j$  for  $i, j = 1, \dots, d$ .

Proof is rather complicated and it will be divided into several steps.

**2. The case  $t(\eta_1, \dots, \eta_d) = \dim V$ .**

Our assumptions are:  $n = t \geq p + q + 1$  and (with respect to (1.7))

$$(2.1) \quad \sum_{i=1}^d \sum_{\varrho \in \mathfrak{S}_{p,q}} (\operatorname{sgn} \varrho) \omega_i(x_{\varrho(1)}, \dots, x_{\varrho(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) = 0$$

for any  $x_1, \dots, x_{p+q} \in V$ .

For  $k \geq 1$ , let  $V_{(k)}$  denote the *Stiefel manifold of all (regular)  $k$ -frames*  $(x_1, \dots, x_k)$  on  $V$  and, for  $k = 0$ , put  $V_{(0)} = 0 \in V$ . Let us introduce  $F$ -valued functions  $\omega_{i,\varrho}$  and  $\eta_{j,\sigma}$  ( $i, j = 1, \dots, d; \varrho, \sigma \in \mathfrak{S}_{p,q}$ ) on the manifold  $V_{(p+q)}$  as follows:

$$(2.2) \quad \begin{aligned} \omega_{i,\varrho}((x_1, \dots, x_{p+q})) &= \omega_i(x_{\varrho(1)}, \dots, x_{\varrho(p)}), \\ \eta_{j,\sigma}((x_1, \dots, x_{p+q})) &= \eta_j(x_{\sigma(p+1)}, \dots, x_{\sigma(p+q)}), \end{aligned}$$

$(x_1, \dots, x_{p+q}) \in V_{(p+q)}.$

Then we can re-write (2.1) in the form

$$(2.3) \quad \sum_{i=1}^d \sum_{\varrho \in \mathfrak{S}_{p,q}} (\operatorname{sgn} \varrho) \omega_{i,\varrho}(u) \wedge \eta_{i,\varrho}(u) = 0, \quad u \in V_{(p+q)}.$$

(Note that if the vectors  $x_1, \dots, x_{p+q}$  are linearly dependent, the relation (2.1) becomes trivial.)

According to Lemma 1 (where  $V_0 = V, k = p + q$ ), the vectors  $\eta_{i,\varrho}(u)$  ( $i = 1, \dots, d; \varrho \in \mathfrak{S}_{p,q}$ ) form a linearly independent system in  $F$  for any  $u \in V_{(p+q)}$ .

Suppose  $\mathfrak{S}_{p,q}$  be ordered in a fixed way. According to Cartan's lemma, we get a unique expression

$$(2.4) \quad \omega_{i,\varrho} = \sum_{j=1}^d \sum_{\sigma \in \mathfrak{S}_{p,q}} a_{i,\varrho}^{j,\sigma} \eta_{j,\sigma}, \quad i = 1, \dots, d; \varrho \in \mathfrak{S}_{p,q},$$

where  $a_{i,\varrho}^{j,\sigma}$  are well-defined real functions on  $V_{(p+q)}$ .

Moreover, if we express the functions  $(\operatorname{sgn} \varrho) \omega_{i,\varrho}$  instead of  $\omega_{i,\varrho}$ , the corresponding matrix of coefficients is symmetric, i.e.,

$$(2.5) \quad (\operatorname{sgn} \varrho) a_{i,\varrho}^{j,\sigma} = (\operatorname{sgn} \sigma) a_{j,\sigma}^{i,\varrho}.$$

LEMMA 2.  $a_{i,\varrho}^{j,\sigma} = 0$  whenever  $\{\sigma(p+1), \dots, \sigma(p+q)\} \not\subset \{\varrho(1), \dots, \varrho(p)\}$ .

Proof. Let  $u = (x_1, \dots, x_{p+q}) \in V_{(p+q)}$  and  $\varrho \in \mathfrak{S}_{p,q}$ . Suppose  $\bar{\sigma} \in \mathfrak{S}_{p,q}$  be such that the difference set  $\{\bar{\sigma}(p+1), \dots, \bar{\sigma}(p+q)\} \setminus \{\varrho(1), \dots, \varrho(p)\}$  is not empty and contains an integer  $m$ . Define a new frame  $u' = (x'_1, \dots, x'_{p+q}) \in V_{(p+q)}$  putting  $x'_l = x_l$  for  $l \neq m$  and  $x'_m = x_{p+q+1}$ , where  $x_{p+q+1} \in V$  is a new vector linearly independent of  $x_1, \dots, x_{p+q}$ . Then  $\omega_{i,\varrho}(u') = \omega_{i,\varrho}(u)$  for  $i = 1, \dots, d$  and hence

$$(2.6) \quad \sum_{j=1}^d \sum_{\sigma \in \mathfrak{S}_{p,q}} a_{i,\varrho}^{j,\sigma}(u') \eta_{j,\sigma}(u') = \sum_{j=1}^d \sum_{\sigma \in \mathfrak{S}_{p,q}} a_{i,\varrho}^{j,\sigma}(u) \eta_{j,\sigma}(u).$$

Let us divide  $\mathfrak{S}_{p,q}$  into two parts  $\mathfrak{A}$  and  $\mathfrak{B}$ :

$$\mathfrak{A} = \{\gamma \in \mathfrak{S}_{p,q} \mid \gamma(p+l) \neq m \text{ for } l = 1, \dots, q\},$$

$$\mathfrak{B} = \{\delta \in \mathfrak{S}_{p,q} \mid \delta(p+l) = m \text{ for some } l \in \{1, \dots, q\}\}.$$

Now, according to Lemma 1, the vectors  $\eta_{j,\gamma}(u') = \eta_{j,\gamma}(u)$  ( $j = 1, \dots, d$ ;  $\gamma \in \mathfrak{A}$ ) and the vectors  $\eta_{j,\delta}(u')$ ,  $\eta_{j,\delta}(u)$  ( $j = 1, \dots, d$ ;  $\delta \in \mathfrak{B}$ ) form together a linearly independent system. From (2.6) we get

$$\begin{aligned} \sum_{j=1}^d \left\{ \sum_{\gamma \in \mathfrak{A}} [a_{i,\varrho}^{j,\gamma}(u') - a_{i,\varrho}^{j,\gamma}(u)] \eta_{j,\gamma}(u) + \sum_{\delta \in \mathfrak{B}} a_{i,\varrho}^{j,\delta}(u') \eta_{j,\delta}(u') - \right. \\ \left. - \sum_{\delta \in \mathfrak{B}} a_{i,\varrho}^{j,\delta}(u) \eta_{j,\delta}(u) \right\} = 0, \end{aligned}$$

and thus  $a_{i,\varrho}^{j,\delta}(u) = 0$  for any  $\delta \in \mathfrak{B}$ ,  $i, j = 1, \dots, d$ . In particular,  $\bar{\sigma} \in \mathfrak{B}$ , hence the result follows.

Denote by  $J: V_{(p+q)} \rightarrow V_{(p)}$  canonical projection of Stiefel manifolds given by  $J((x_1, \dots, x_{p+q})) = (x_1, \dots, x_p)$ . Let  $\varepsilon \in \mathfrak{S}_{p,q}$  denote the identity transformation and  $\bar{\varepsilon} \in \mathfrak{S}_{p,q}$  the transformation given by

$$(2.7) \quad \bar{\varepsilon}(l) = \begin{cases} l, & l = 1, \dots, p-q, \\ q+l, & l = p-q+1, \dots, p, \end{cases}$$

$$\bar{\varepsilon}(p+l) = p-q+l, \quad l = 1, \dots, q.$$

Then  $a_{i,\varepsilon}^{j,\bar{\varepsilon}}(u') = a_{i,\varepsilon}^{j,\bar{\varepsilon}}(u)$  whenever  $J(u') = J(u)$ . Thus we can define real-valued functions  $a_i^j$  on  $V_{(p)}$  ( $i, j = 1, \dots, d$ ) by the formula

$$(2.8) \quad a_i^j \circ J = a_{i,\varepsilon}^{j,\bar{\varepsilon}}.$$

LEMMA 3. If  $u = (x_1, \dots, x_{p+q}) \in V_{(p+q)}$ , then

$$a_i^j((x_1, \dots, x_p)) = (-1)^q a_j^i((x_1, \dots, x_{p-q}, x_{p+1}, \dots, x_{p+q}))$$

for any  $i, j = 1, \dots, d$ .

Proof. Let  $u = (x_1, \dots, x_{p+q}) \in V_{(p+q)}$  and put  $u^* = (x_1, \dots, x_{p-q}, x_{p+1}, \dots, x_{p+q}, x_{p-q+1}, \dots, x_p)$ . Then

$$(2.9) \quad \omega_{i,\bar{\varepsilon}}(u) = \omega_{i,\varepsilon}(u^*) = \omega_i(x_1, \dots, x_{p-q}, x_{p+1}, \dots, x_{p+q}),$$

$$(2.10) \quad \eta_{j,\varepsilon}(u) = \eta_{j,\bar{\varepsilon}}(u^*) = \eta_j(x_{p+1}, \dots, x_{p+q}).$$

According to (2.4) and (2.9), we obtain

$$(2.11) \quad \sum_{j=1}^d \sum_{\sigma \in \mathfrak{S}_{p,q}} a_{i,\bar{\varepsilon}}^{j,\sigma}(u) \eta_{j,\sigma}(u) = \sum_{j=1}^d \sum_{\sigma \in \mathfrak{S}_{p,q}} a_{i,\varepsilon}^{j,\sigma}(u^*) \eta_{j,\sigma}(u^*).$$

Since  $\eta_{j,\sigma}(u)$  as well as  $\eta_{j,\sigma}(u^*)$  form a linearly independent system of vectors, the coefficients at  $\eta_{j,\varepsilon}(u)$  and  $\eta_{j,\bar{\varepsilon}}(u^*)$  in (2.11) are the same, according to (2.10), i.e.,  $a_{i,\bar{\varepsilon}}^{j,\varepsilon}(u) = a_{i,\varepsilon}^{j,\bar{\varepsilon}}(u^*)$ . Since  $\text{sgn } \bar{\varepsilon} = (-1)^{q^2} = (-1)^q$ , we obtain from (2.5)  $a_{i,\bar{\varepsilon}}^{j,\varepsilon} = (-1)^q a_{j,\varepsilon}^{i,\bar{\varepsilon}}$ . Consequently, we get

$$\begin{aligned} a_i^j((x_1, \dots, x_p)) &= a_{i,\bar{\varepsilon}}^{j,\varepsilon}(u) = (-1)^q a_{j,\varepsilon}^{i,\bar{\varepsilon}}(u) = (-1)^q a_{j,\varepsilon}^{i,\bar{\varepsilon}}(u^*) \\ &= (-1)^q a_j^i((x_1, \dots, x_{p-q}, x_{p+1}, \dots, x_{p+q})). \end{aligned}$$

LEMMA 4. If  $v = (x_1, \dots, x_p) \in V_{(p)}$  and  $\delta$  is a permutation of the set  $\{p-q+1, \dots, p\}$ , then

$$a_i^j((x_1, \dots, x_{p-q}, x_{\delta(p-q+1)}, \dots, x_{\delta(p)})) = a_i^j(v).$$

Proof. Complete the frame  $v$  by new vectors  $x_{p+1}, \dots, x_{p+q}$  to a frame  $u \in V_{(p+q)}$  and then use Lemma 3.

Let us consider a canonical projection of Stiefel manifolds  $Q: V_{(p)} \rightarrow V_{(p-q)}$  given by  $Q((x_1, \dots, x_p)) = (x_1, \dots, x_{p-q})$ . (Recall that in the case  $p = q$ , we have  $Q(v) = 0$  = the null vector of  $V$  for any  $v \in V_{(p)}$ .)

LEMMA 5. We have  $a_i^j(v') = a_i^j(v)$  ( $i, j = 1, \dots, d$ ) whenever  $v, v' \in V_{(p)}$ ,  $Q(v') = Q(v)$ .

Proof. Put  $v = (x_1, \dots, x_{p-q}, x_{p-q+1}, \dots, x_p)$  and  $v' = (x_1, \dots, x_{p-q}, x'_{p-q+1}, \dots, x'_p)$ .

Suppose that we have already proved

$$a_i^j((x_1, \dots, x_{p-q}, x'_{p-q+1}, \dots, x'_k, x''_{k+1}, \dots, x''_p)) = a_i^j(v),$$

where  $p-q \leq k < p$  and  $x''_{k+1}, \dots, x''_p$  are vectors such that

$$v_k = (x_1, \dots, x_{p-q}, x'_{p-q+1}, \dots, x'_k, x''_{k+1}, \dots, x''_p) \in V_{(p)}.$$

Let us distinguish two cases:

(a) If  $x'_{k+1}$  is a linear combination of vectors of  $v_k$ ,

$$x'_{k+1} = \sum_{i=1}^{p-q} a_i x_i + \sum_{i=p-q+1}^k b_i x'_i + \sum_{i=k+1}^p c_i x''_i,$$

then this expression is unique and at least one  $c_i$  is non-zero. According to Lemma 4, we can suppose that  $c_{k+1} \neq 0$ . Then

$$v_{k+1} = (x_1, \dots, x_{p-q}, x'_{p-q+1}, \dots, x'_k, x'_{k+1}, x''_{k+2}, \dots, x''_p) \in V_{(p)}.$$

Completing  $v_{k+1}$  to a frame  $u_{k+1} \in V_{p+q}$  by addition of new vectors  $x_{p+1}, \dots, x_{p+q}$ , we obtain, by Lemma 3,

$$(2.12) \quad a_i^j(v_{k+1}) = (-1)^q a_j^i(x_1, \dots, x_{p-q}, x_{p+1}, \dots, x_{p+q}) = a_i^j(v_k).$$

(b) If  $x'_{k+1}$  is linearly independent of vectors of  $v_k$ , we can find vectors  $x_{p+1}, \dots, x_{p+q} \in V$  such that

$$(x_1, \dots, x_{p-q}, x'_{p-q+1}, \dots, x'_k, x'_{k+1}, x''_{k+1}, \dots, x''_p, x_{p+1}, \dots, x_{p+q}) \in V_{(p+q+1)},$$

because  $\dim V \geq p+q+1$ . Now, to complete the proof it suffices to observe that (2.12) is still valid.

On the basis of Lemma 5 we can define *real-valued functions*  $g_i^j$  on  $V_{p-q}$  ( $i, j = 1, \dots, d$ ) by the formula

$$(2.13) \quad g_i^j \circ Q = a_i^j.$$

In particular, in the case  $p = q$ ,  $g_i^j$  are well-determined constants. Put

$$\tilde{\mathfrak{S}}_{p,q} = \{\delta \in \mathfrak{S}_{p,q} \mid \{\delta(p+1), \dots, \delta(p+q)\} \subset \{1, \dots, p\}\}.$$

Then there is a natural 1-1 map  $*$ :  $\tilde{\mathfrak{S}}_{p,q} \rightarrow \mathfrak{S}_{p-q,q}$  defined as follows: to any permutation  $\delta \in \tilde{\mathfrak{S}}_{p,q}$  we ascribe a permutation  $\delta^* \in \mathfrak{S}_{p-q,q}$  such that  $\delta^*(l) = \delta(l)$  ( $l = 1, \dots, p-q$ ), and  $\delta^*(p-q+l) = \delta(p+l)$  ( $l = 1, \dots, q$ ).

According to Lemma 2 and formula (2.4),

$$(2.14) \quad \omega_{i,\varepsilon} = \sum_{j=1}^d \sum_{\delta \in \tilde{\mathfrak{S}}_{p,q}} a_{i,\varepsilon}^{j,\delta} \eta_{j,\delta} \quad (i = 1, \dots, d)$$

holds on  $V_{(p+q)}$ . The values  $\omega_{i,\varepsilon}(v)$  and  $\eta_{j,\delta}(v)$  ( $v \in V_{(p+q)}$ ) depend only on  $J(v) \in V_{(p)}$ . Thus we can define  $F$ -valued functions  $\bar{\omega}_i$  and  $\bar{\eta}_{j,\varrho}$  ( $i, j = 1, \dots, d$ ;  $\varrho \in \mathfrak{S}_{p-q,q}$ ) on  $V_{(p)}$  by the rules

$$\bar{\omega}_i \circ J = \omega_{i,\varepsilon}, \quad \bar{\eta}_{j,\delta^*} \circ J = \eta_{j,\delta} \quad (\delta \in \tilde{\mathfrak{S}}_{p,q}).$$

Clearly,

$$(2.15) \quad \begin{aligned} \bar{\omega}_i((x_1, \dots, x_p)) &= \omega_i(x_1, \dots, x_p), \\ \bar{\eta}_{j,\varrho}((x_1, \dots, x_p)) &= \eta_j(x_{\varrho(p-q+1)}, \dots, x_{\varrho(p)}), \end{aligned} \quad (x_1, \dots, x_p) \in V_{(p)}.$$



According to (2.14), there are real-valued functions  $a_i^{j,e}$  ( $i, j = 1, \dots, d$ ;  $\varrho \in \mathfrak{S}_{p-q,q}$ ) on  $V_{(p)}$  such that

$$(2.16) \quad \bar{\omega}_i = \sum_{j=1}^d \sum_{\varrho \in \mathfrak{S}_{p-q,q}} a_i^{j,e} \bar{\eta}_{j,e} \quad (i = 1, \dots, d).$$

Moreover,  $a_i^{j,\delta^*} \circ J = a_i^{j,\delta}$  for any  $\delta \in \tilde{\mathfrak{S}}_{p,q}$ . In particular, we get  $a_i^{j,e} \circ J = a_i^{j,\bar{e}}$ , where in the symbol  $a_i^{j,\bar{e}}$  the index  $\bar{e}$  denotes the identity permutation in  $\mathfrak{S}_{p-q,q}$ . Hence,  $a_i^{j,\bar{e}} = a_i^{j,e}$  on  $V_{(p)}$ .

For  $v \in V_{(p)}$ ,  $v = (x_1, \dots, x_p)$  and  $\varrho \in \mathfrak{S}_{p-q,q}$ , let us put  $\varrho \cdot v = (x_{\varrho(1)}, \dots, x_{\varrho(p)})$ . Now,

$$\bar{\omega}_i(\varrho \cdot v) = (\text{sgn } \varrho) \bar{\omega}_i(v) \quad \text{and} \quad \bar{\eta}_{j,e}(v) = \bar{\eta}_{j,\bar{e}}(\varrho \cdot v),$$

according to (2.15).  $a_i^{j,e}(v)$  is the coefficient at  $\bar{\eta}_{j,e}(v)$  in formula (2.16) for  $\bar{\omega}_i(v)$ , and  $a_i^{j,\bar{e}}(\varrho \cdot v)$  is the coefficient at  $\bar{\eta}_{j,\bar{e}}(\varrho \cdot v)$  in formula (2.16) for  $\bar{\omega}_i(\varrho \cdot v) = (\text{sgn } \varrho) \bar{\omega}_i(v)$ . Therefore,

$$\begin{aligned} a_i^{j,e}(v) &= (\text{sgn } \varrho) a_i^{j,\bar{e}}(\varrho \cdot v) = (\text{sgn } \varrho) a_i^j(\varrho \cdot v) \\ &= (\text{sgn } \varrho) g_i^j((x_{\varrho(1)}, \dots, x_{\varrho(p-q)})). \end{aligned}$$

Hence and from (2.15) and (2.16) we get

$$\begin{aligned} (2.17) \quad \omega_i(x_1, \dots, x_p) &= \sum_{j=1}^d \sum_{\varrho \in \mathfrak{S}_{p-q,q}} (\text{sgn } \varrho) g_i^j((x_{\varrho(1)}, \dots, x_{\varrho(p-q)})) \eta_j(x_{\varrho(p-q+1)}, \dots, x_{\varrho(p)}) \\ &\quad \text{for any } (x_1, \dots, x_p) \in V_{(p)}. \end{aligned}$$

$g_i^j((x_1, \dots, x_{p-q}))$  is the (uniquely determined) coefficient at  $\eta_j(x_{p-q+1}, \dots, x_p)$  in formula (2.17).

Hence one can see that each function  $g = g_i^j$  satisfies:

- (i)  $g((x_{\sigma(1)}, \dots, x_{\sigma(p-q)})) = (\text{sgn } \sigma) g((x_1, \dots, x_{p-q}))$  for any  $(x_1, \dots, x_{p-q}) \in V_{(p-q)}$ ,  $\sigma \in \mathfrak{S}_{p-q}$ .
- (ii)  $g((\lambda x_1, x_2, \dots, x_{p-q})) = \lambda g((x_1, \dots, x_{p-q}))$  for any  $(x_1, \dots, x_{p-q}) \in V_{(p-q)}$  and  $\lambda \in R$ ,  $\lambda \neq 0$ .

(iii) If  $x_1, y_1, x_2, \dots, x_{p-q} \in V$  are linearly independent, then  $g((x_1 + y_1, x_2, \dots, x_{p-q})) = g((x_1, \dots, x_{p-q})) + g((y_1, x_2, \dots, x_{p-q}))$ .

(iv) If  $(x_1, \dots, x_{p-q}) \in V_{(p-q)}$  and  $y_1$  depends linearly on  $x_2, \dots, x_p$ , then  $g((x_1 + y_1, x_2, \dots, x_{p-q})) = g((x_1, \dots, x_{p-q}))$ .

Now it is not difficult to see that there exists a unique system of forms  $h_i^j \in \wedge^{p-q} V$  ( $i, j = 1, \dots, d$ ) such that

$$h_i^j(x_1, \dots, x_{p-q}) = g_i^j((x_1, \dots, x_{p-q})) \quad \text{for any } (x_1, \dots, x_{p-q}) \in V_{(p-q)}.$$

According to (1.8) and (2.17),

$$(2.18) \quad \omega_i = \sum_{j=1}^d h_i^j \wedge \eta_j \quad (i = 1, \dots, d)$$

holds on each frame  $(x_1, \dots, x_p) \in V_{(p)}$ . On the other hand, if  $(x_1, \dots, x_p)$  is a linearly dependent  $d$ -tuple, both sides of (2.18) are identically zero. Hence (2.18) holds identically.

To complete the proof of our special theorem, we must show that  $h_j^i = (-1)^a h_i^j$  for  $i, j = 1, \dots, d$ . But the last formula follows from Lemma 3 and Lemma 5.

We can summarize:

**THEOREM 2.1.** *The assertion of Theorem 1.1 is true in the case of  $V$  being a distinguished space of the collection of forms  $(\eta_1, \dots, \eta_d)$ .*

**3. The general case of Theorem 1.1.** Firstly, we must introduce some new algebraic notions and operations.

Let  $V_0 \subset V$  be a fixed subspace and  $k, l \geq 0$  integers. Consider the space  $\wedge^{*k} V \otimes \wedge^{*l} V_0$ . For any  $a = 1, \dots, k$  we define a restriction map

$$(3.1) \quad i_a: \wedge^{*k} V \otimes \wedge^{*l} V_0 \rightarrow \wedge^{*(k-a)} V \otimes \wedge^{*a} V_0 \otimes \wedge^{*l} V_0$$

as follows: for  $h \in \wedge^{*k} V \otimes \wedge^{*l} V_0$ ,  $x_1, \dots, x_{k-a} \in V$  and  $x_{k-a+1}, \dots, x_k, x_{k+1}, \dots, x_{k+l} \in V_0$ , put

$$\begin{aligned} i_a(h)(x_1, \dots, x_{k-a}; x_{k-a+1}, \dots, x_k; x_{k+1}, \dots, x_{k+l}) \\ = h(x_1, \dots, x_{k-a}, i(x_{k-a+1}), \dots, i(x_k); x_{k+1}, \dots, x_{k+l}), \end{aligned}$$

where  $i: V_0 \rightarrow V$  is the inclusion map. In addition, put  $i_{k+1} = 0$ .

Each space  $\wedge^{*(a+l)} V_0$  can be identified with a subspace of  $\wedge^{*a} V_0 \otimes \wedge^{*l} V_0$ . Now, a space  $D^{k,l}$  ( $k, l \geq 0$ ) will be defined by the condition

$$(3.2) \quad D^{k,l} = \{h \in \wedge^{*k} V \otimes \wedge^{*l} V_0 \mid i_a(h) \in \wedge^{*(k-a)} V \otimes \wedge^{*(a+l)} V_0\} \\ (a = 1, \dots, k).$$

Clearly,  $D^{k,0} = \wedge^{*k} V$  and  $i_1(D^{k,l-1}) \subset D^{k-1,l}$  for  $k, l \geq 1$ . For any vector space  $E$  over  $R$ , we denote by the same symbol  $i_1$  the induced map

$$(3.3) \quad i_1: D^{k,l-1} \otimes E \rightarrow D^{k-1,l} \otimes E$$

acting trivially on  $E$ . Put  $i_a = (i_1)^a$  for  $a = 1, \dots, k$ .

Let us define a composition law

$$\cap: D^{k,l} \times \wedge^q(V, F) \rightarrow D^{k,l+q} \otimes F$$

as follows: for  $h \in D^{k,l}$ ,  $\eta \in \wedge^q(V, F) = \wedge^{*q}V \otimes F$ ,  $x_1, \dots, x_k \in V$  and  $x_{k+1}, \dots, x_{k+l+q} \in V_0$  we put

$$(3.4) \quad (h \cap \eta)(x_1, \dots, x_k; x_{k+1}, \dots, x_{k+l}, x_{k+l+1}, \dots, x_{k+l+q}) \\ = \sum_{\sigma \in \mathfrak{S}_{k+l,q}} (\text{sgn } \sigma) h(x_{\sigma(1)}, \dots; \dots, x_{\sigma(k+l)}) \eta(x_{\sigma(k+l+1)}, \dots, x_{\sigma(k+l+q)}).$$

The right-hand side of (3.4) makes sense, because  $x_{\sigma(k+l+1)}, \dots, x_{\sigma(k+l+q)} \in V_0$  for any  $\sigma \in \mathfrak{S}_{k+l,q}$ . It is a routine to show that actually  $h \cap \eta \in D^{k,l+q} \otimes F$ .

Further, we define a composition law

$$\times : (D^{k,l} \otimes F) \times \wedge^q(V, F) \rightarrow D^{k,l+q} \otimes (F \wedge F)$$

as follows: for  $a \in D^{k,l} \otimes F$ ,  $\eta \in \wedge^q(V, F)$  we put

$$(3.5) \quad (a \times \eta)(x_1, \dots, x_k; x_{k+1}, \dots, x_{k+l}, x_{k+l+1}, \dots, x_{k+l+q}) \\ = \sum_{\sigma \in \mathfrak{S}_{k+l,q}} (\text{sgn } \sigma) a(x_{\sigma(1)}, \dots; \dots, x_{\sigma(k+l)}) \wedge \eta(x_{\sigma(k+l+1)}, \dots, x_{\sigma(k+l+q)}).$$

Let us mention obvious rules

$$(3.6) \quad i_a(h \cap \eta) = i_a(h) \cap \eta, \quad i_a(a \times \eta) = i_a(a) \times \eta.$$

In a special case of  $h \in D^{k,0} = \wedge^{*k}V$ ,  $\eta \in \wedge^q(V, F)$  and  $a \in D^{k,0} \otimes F = \wedge^k(V, F)$ , we have

$$(3.7) \quad h \cap \eta = i_q(h \wedge \eta), \quad a \times \eta = i_q(a \cdot \eta).$$

(The new operation  $\times$  is not to be confused with that of (1.3)).

Let  $E$  be an arbitrary vector space over  $R$  (in particular, one of the spaces  $R, F, F \wedge F$ ). For  $a \in D^{k,l} \otimes E$  and  $z \in V$  we define the *inner product*  $z \lrcorner a$ : for  $k = 0$  put  $z \lrcorner a = 0$ , and for  $k > 0$  define  $z \lrcorner a \in D^{k-1,l} \otimes E$  by the formula

$$(3.8) \quad (z \lrcorner a)(y_1, \dots, y_{k-1}; x_1, \dots, x_l) = a(z, y_1, \dots, y_{k-1}; x_1, \dots, x_l) \\ \text{for } y_1, \dots, y_{k-1} \in V, x_1, \dots, x_l \in V_0.$$

LEMMA 6. Let  $a \in D^{k,l} \otimes F$ ,  $\eta \in \wedge^q(V, F)$  and  $h \in D^{k,l}$ ,  $k \geq 1$ . Then, for any  $z \in V$ ,

- (i)  $z \lrcorner (a \times \eta) = (z \lrcorner a) \times \eta + (-1)^{k+l} i_1(a) \times (z \lrcorner \eta)$ ,
- (ii)  $z \lrcorner (h \cap \eta) = (z \lrcorner h) \cap \eta + (-1)^{k+l} i_1(h) \cap (z \lrcorner \eta)$ .

Proof. If  $\omega$  and  $\eta$  are real-valued forms on  $V$  of degree  $p$  and  $q$ , respectively, then  $z \lrcorner (\omega \wedge \eta) = (z \lrcorner \omega) \wedge \eta + (-1)^p \omega \wedge (z \lrcorner \eta)$  (see, for instance, [13], p. 21). Further, if  $\omega \in \wedge^{k+l}(V, F)$ ,  $\eta \in \wedge^q(V, F)$  and  $\bar{h} \in \wedge^{*k+l}V$ , then, using an arbitrary basis  $(f_1, \dots, f_m)$  of  $F$ , we easily get

$$(3.9) \quad z \lrcorner (\omega \cdot \eta) = (z \lrcorner \omega) \cdot \eta + (-1)^{k+l} i_1(\omega) \cdot (z \lrcorner \eta), \\ z \lrcorner (\bar{h} \wedge \eta) = (z \lrcorner \bar{h}) \wedge \eta + (-1)^{k+l} i_1(\bar{h}) \wedge (z \lrcorner \eta).$$

Let us remark that any  $\alpha \in D^{k,l} \otimes F$  can be written as  $i_l(\omega)$ ,  $\omega \in \wedge^{k+l}(V, F)$ , and any  $h \in D^{k,l}$  can be written as  $i_l(\bar{h})$ ,  $\bar{h} \in \wedge^{*k+l}V$ . Now, it is sufficient to apply to (3.9) the map  $i_l \circ i_q$  and to take into account (3.6) and (3.7).

Remark that for  $\eta \in \wedge^p(V, F)$ ,  $\eta' \in \wedge^q(V, F)$  and  $h \in D^{k,l}$  there is

$$(3.10) \quad \eta \cdot \eta' = (-1)^{pq+1} \eta' \cdot \eta,$$

$$(3.11) \quad (h \cap \eta) \times \eta' = h \cap (\eta \cdot \eta') \quad (\text{associative law}).$$

Proof is routine.

In the subsequent considerations we always suppose that  $V_0$  is a distinguished space of a collection of forms  $(\eta_1, \dots, \eta_d)$ ,  $\eta_i \in \wedge^q(V, F)$ .

LEMMA 7. Let  $\eta_1, \dots, \eta_d \in \wedge^q(V, F)$  and let  $r, a$  be integers such that  $1 \leq a \leq r$ . Suppose  $t(\eta_1, \dots, \eta_d) \geq r + q$ . If  $h^j \in D^{a, r-a}$  ( $j = 1, \dots, d$ ) are such that  $i_1(h^j) = 0$ , then  $\sum_{j=1}^d h^j \cap \eta_j = 0$  implies  $h^1 = \dots = h^d = 0$ .

Proof. We have  $h^j \cap \eta_j \in D^{a, r+q-a}$  ( $j = 1, \dots, d$ ) and, for any  $x_1, \dots, x_a \in V$ ,  $x_{a+1}, \dots, x_r, x_{r+1}, \dots, x_{r+q} \in V_0$ ,

$$\begin{aligned} (h^j \cap \eta_j)(x_1, \dots, x_a; x_{a+1}, \dots, x_{r+q}) \\ = \sum_{\sigma \in \mathfrak{S}_{r,q}} (\text{sgn } \sigma) h^j(x_{\sigma(1)}, \dots; \dots, x_{\sigma(r)}) \eta_j(x_{\sigma(r+1)}, \dots, x_{\sigma(r+q)}). \end{aligned}$$

Consider a decomposition  $\mathfrak{S}_{r,q} = \mathfrak{A} \cup \mathfrak{B}$  defined in a way such that

$$\mathfrak{A} = \{\sigma \in \mathfrak{S}_{r,q} \mid \{\sigma(1), \dots, \sigma(r)\} \supset \{1, \dots, a\}\}.$$

Then

$$\begin{aligned} 0 &= \sum_{j=1}^d (h^j \cap \eta_j)(x_1, \dots, x_a; x_{a+1}, \dots, x_{r+q}) \\ &= \sum_{j=1}^d \sum_{\sigma \in \mathfrak{A}} h^j(x_{\sigma(1)}, \dots; \dots, x_{\sigma(r)}) \eta_j(x_{\sigma(r+1)}, \dots, x_{\sigma(r+q)}) + \\ &\quad + \sum_{j=1}^d \sum_{\sigma \in \mathfrak{B}} i_1(h^j)(x_{\sigma(1)}, \dots; \dots, x_{\sigma(r)}) \eta_j(x_{\sigma(r+1)}, \dots, x_{\sigma(r+q)}). \end{aligned}$$

The second sum is identically zero. Further, if  $x_{a+1}, \dots, x_{r+q} \in V_0$  are linearly independent ( $\dim V_0 \geq r + q$ !), the vectors  $\eta_j(x_{\sigma(r+1)}, \dots, x_{\sigma(r+q)})$  ( $j = 1, \dots, d; \sigma \in \mathfrak{A}$ ) form a linearly independent system in  $F$ , and the coefficients  $h^j(x_{\sigma(1)}, \dots; \dots, x_{\sigma(r)}) = h^j(x_1, \dots, x_a; x_{\sigma(a+1)}, \dots, x_{\sigma(r)})$  are zero. Hence, the assertion follows immediately.

THEOREM 3.1. Let  $\eta_j \in \wedge^q(V, F)$  ( $j = 1, \dots, d$ ), and  $t = t(\eta_1, \dots, \eta_d) \geq 2q + 1$ . Then for any integer  $a \in \langle 0, t - 2q - 1 \rangle$  and any integer  $p \in \langle q + a, t - q - 1 \rangle$  the following is true: if  $\alpha_i \in D^{a, p-a} \otimes F$  ( $i = 1, \dots, d$ ) and  $\sum_{i=1}^d \alpha_i \times \eta_i$

$= 0$ , then there is a unique system of elements  $h_i^j \in D^{a,p-q-a}$ ,  $h_j^i = (-1)^q h_i^j$  ( $i, j = 1, \dots, d$ ) such that

$$\alpha_i = \sum_{j=1}^d h_i^j \cap \eta_j \quad (i = 1, \dots, d).$$

**Proof.** We proceed by induction on  $a$ . For  $a = 0$  and any  $p \in \langle q, t-q-1 \rangle$  our assertion is nothing but Theorem 2.1.

Suppose now that Theorem 3.1 is true for any couple  $(a', p')$ , where  $a' \leq a < t-2q-1$  and  $p' \in \langle q+a', t-q-1 \rangle$ . Choose an integer  $p \in \langle q+a+1, t-q-1 \rangle$  and let  $\alpha_i \in D^{a+1,p-a-1} \otimes F$  be given such that

$$(3.12) \quad \sum_{i=1}^d \alpha_i \times \eta_i = 0.$$

For any  $z \in V$  we have in  $D^{a,p+q-a} \otimes (F \wedge F)$  by Lemma 6 the following identity:

$$(3.13) \quad z \lrcorner (\alpha_i \times \eta_i) = (z \lrcorner \alpha_i) \times \eta_i + (-1)^p \cdot i_1(\alpha_i) \times (z \lrcorner \eta_i).$$

From (3.12) it follows that

$$\sum_{i=1}^d i_1(\alpha_i) \times \eta_i = 0, \quad i_1(\alpha_i) \in D^{a,p-a} \otimes F,$$

and, according to the induction assumption, there is a unique system of elements  $h_i^j \in D^{a,p-q-a}$ ,  $h_j^i = (-1)^q h_i^j$  ( $i, j = 1, \dots, d$ ) such that

$$i_1(\alpha_i) = \sum_{j=1}^d h_i^j \cap \eta_j.$$

Thus (3.13) and (3.12) imply

$$\sum_{i=1}^d (z \lrcorner \alpha_i) \times \eta_i + (-1)^p \sum_{i,j=1}^d (h_i^j \cap \eta_j) \times (z \lrcorner \eta_i) = 0,$$

whence taking into account (3.10), (3.11) and the relations  $h_i^j = (-1)^q h_j^i$ , we get

$$\begin{aligned} \sum_{i=1}^d (z \lrcorner \alpha_i) \times \eta_i - (-1)^p \sum_{i,j=1}^d [h_i^j \cap (z \lrcorner \eta_i)] \times \eta_j \\ = \sum_{i=1}^d [(z \lrcorner \alpha_i) - (-1)^{p-q} \sum_{j=1}^d h_i^j \cap (z \lrcorner \eta_j)] \times \eta_i = 0. \end{aligned}$$

The terms in brackets are elements of the space  $D^{a,p-a-1} \otimes F$  for  $i = 1, \dots, d$ .

According to the induction assumption (concerning the couple  $(a, p-1)$ ), we have a unique decomposition of the elements

$$(3.14) \quad \alpha_i[z] = (z \sqcup \alpha_i) - (-1)^{p-q} \sum_{j=1}^d h_i^j \cap (z \sqcup \eta_j)$$

in the form

$$(3.15) \quad \alpha_i[z] = \sum_{j=1}^d h_i^j[z] \cap \eta_j,$$

where

$$(3.16) \quad h_i^j[z] \in D^{a, p-q-a-1}, \quad h_j^i[z] = (-1)^q h_i^j[z] \quad (i, j = 1, \dots, d).$$

For  $a \geq 1$ , (3.15) implies also

$$(3.17) \quad i_1(\alpha_i[z]) = \sum_{j=1}^d i_1(h_i^j[z]) \cap \eta_j,$$

and such a decomposition is *unique*, because

$$\sum_{i=1}^d i_1(\alpha_i[z]) \times \eta_i = 0, \quad i_1(\alpha_i[z]) \in D^{a-1, p-a},$$

and we can use the induction assumption concerning the couple  $(a-1, p-1)$ .

On the other hand, from the identity

$$i_1(\alpha_i) = \sum_{j=1}^d h_i^j \cap \eta_j$$

we get (using part (ii) of Lemma 6)

$$z \sqcup i_1(\alpha_i) = \sum_{j=1}^d (z \sqcup h_i^j) \cap \eta_j + (-1)^{p-q} \sum_{j=1}^d i_1(h_i^j) \cap (z \sqcup \eta_j)$$

whence, according to (3.14),

$$(3.18) \quad i_1(\alpha_i[z]) = \sum_{j=1}^d (z \sqcup h_i^j) \cap \eta_j.$$

By comparison of (3.17) and (3.18) we get

$$(3.19) \quad i_1(h_i^j[z]) = z \sqcup h_i^j.$$

Further, from (3.14) and (3.15) it follows that

$$(3.20) \quad z \sqcup \alpha_i = \sum_{j=1}^d h_i^j[z] \cap \eta_j + (-1)^{p-q} \sum_{j=1}^d h_i^j \cap (z \sqcup \eta_j),$$

whence, using part (ii) of Lemma 6 and (3.19), we obtain by a routine calculation

$$\begin{aligned} 0 &= z' \sqcup (z \sqcup \alpha_i) + z \sqcup (z' \sqcup \alpha_i) \\ &= \sum_{j=1}^d (z' \sqcup h_i^j[z] + z \sqcup h_i^j[z']) \cap \eta_j \quad (i = 1, \dots, d). \end{aligned}$$

Moreover, (3.19) yields

$$i_1(z' \sqcup h_i^j[z] + z \sqcup h_i^j[z']) = 0 \quad (i, j = 1, \dots, d).$$

Thus the assumptions of Lemma 7 are satisfied, and we obtain

$$(3.21) \quad z' \sqcup h_i^j[z] + z \sqcup h_i^j[z'] = 0 \quad (z, z' \in V; i, j = 1, \dots, d).$$

Let us define elements  $h_i^{*j} \in V^* \otimes D^{a, p-q-a-1}$  by the relation

$$h_i^{*j}(z; z_1, \dots, z_a; x_1, \dots, x_{p-q-a-1}) = h_i^j[z](z_1, \dots, z_a; x_1, \dots, x_{p-q-a-1}).$$

According to (3.21) and (3.16), we can consider  $h_i^{*j}$  as elements of  $\bigwedge^{*(q+1)} V \otimes \bigwedge^{*(p-q-a-1)} V_0$  and, according to (3.19), we can see that  $h_i^{*j} \in D^{a+1, p-q-a-1}$  and

$$(3.22) \quad i_1(h_i^{*j}) = h_i^j, \quad z \sqcup h_i^{*j} = h_i^j[z].$$

Also, according to (3.16),  $h_j^{*i} = (-1)^q h_i^{*j}$  ( $i, j = 1, \dots, d$ ). Taking into account (3.20) and (3.22), we get

$$z \sqcup \alpha_i = \sum_{j=1}^d (z \sqcup h_i^{*j}) \cap \eta_j + (-1)^{p-q} \sum_{j=1}^d i_1(h_i^{*j}) \cap (z \sqcup \eta_j),$$

i.e.,

$$z \sqcup \alpha_i = z \sqcup \left( \sum_{j=1}^d h_i^{*j} \cap \eta_j \right) \quad (i = 1, \dots, d).$$

Finally, we obtain

$$\alpha_i = \sum_{j=1}^d h_i^{*j} \cap \eta_j \quad (i = 1, \dots, d)$$

which is the existence part of our theorem.

It remains to complete the uniqueness part: suppose that

$$\alpha_i = \sum_{j=1}^d \tilde{h}_i^j \cap \eta_j \quad (i = 1, \dots, d)$$

for some elements  $\tilde{h}_i^j \in D^{a+1, p-q-a-1}$ . We get

$$i_1(\alpha_i) = \sum_{j=1}^d i_1(\tilde{h}_i^j) \cap \eta_j,$$

and from the uniqueness of the last decomposition we get  $i_1(\tilde{h}_i^j) = h_i^j$ . Hence

$$\sum_{j=1}^d (h_i^{*j} - \tilde{h}_i^j) \cap \eta_j = 0 \quad \text{and} \quad i_1(h_i^{*j} - \tilde{h}_i^j) = 0 \quad \text{for } i, j = 1, \dots, d.$$

According to Lemma 7,  $h_i^{*j} - \tilde{h}_i^j = 0$ , which completes the proof.

**THEOREM 3.2.** *Let  $\eta_1, \dots, \eta_d \in \wedge^q(V, F)$ ,  $t(\eta_1, \dots, \eta_d) \geq 2q+1$ , and let  $p, k$  be integers such that  $0 \leq k \leq q \leq p \leq t-q-1$ . If for  $i = 1, \dots, d$*

$$a_i \in D^{p-q+k, q-k} \otimes F \quad \text{and} \quad \sum_{i=1}^d a_i \times \eta_i = 0,$$

*then there is a unique set of elements  $h_i^j \in \wedge^{*p-q}V$ ,  $h_j^i = (-1)^q h_i^j$  ( $i, j = 1, \dots, d$ ) such that*

$$a_i = i_{q-k} \left\{ \sum_{j=1}^d h_i^j \wedge \eta_j \right\}.$$

*In particular, for  $k = q$  we get Theorem 1.1.*

We start with some preliminaries. Put  $\mathfrak{B} = \mathfrak{S}_{p,q} \times \mathfrak{S}_{p-q,q}$ .

**LEMMA 8.** *To any couple  $(\varrho, \sigma) \in \mathfrak{B}$  there is exactly one couple  $(\tilde{\varrho}, \tilde{\sigma}) \in \mathfrak{B}$  such that*

- (a)  $\{\tilde{\varrho}(p+1), \dots, \tilde{\varrho}(p+q)\} = \{(\varrho \circ \sigma)(p-q+1), \dots, (\varrho \circ \sigma)(p)\},$
- (b)  $\{\varrho(p+1), \dots, \varrho(p+q)\} = \{(\tilde{\varrho} \circ \tilde{\sigma})(p-q+1), \dots, (\tilde{\varrho} \circ \tilde{\sigma})(p)\}.$

*Moreover,*

- (c)  $\text{sgn } \tilde{\varrho} \cdot \text{sgn } \tilde{\sigma} = (-1)^q \text{sgn } \varrho \cdot \text{sgn } \sigma,$
- (d)  $\tilde{\varrho} \circ \tilde{\sigma} = \varrho \circ \sigma$  on the subset  $\{1, \dots, p-q\},$
- (e)  $(\tilde{\varrho}, \tilde{\sigma}) = (\varrho, \sigma).$

**Proof.** The permutations  $\tilde{\varrho}$  is uniquely determined by (a). Now,  $\varrho(p+1) < \dots < \varrho(p+q)$  and  $\{\varrho(p+1), \dots, \varrho(p+q)\} \subset \{\tilde{\varrho}(1), \dots, \tilde{\varrho}(p)\}$  according to (a). Thus  $\tilde{\sigma}$  is uniquely determined by (b). Further, let us define permutation  $\delta, \tilde{\delta} \in \mathfrak{S}_{p,q}$  by the relations

$$\begin{aligned} \{\delta(1), \dots, \delta(p+q)\} &= \{(\varrho \circ \sigma)(1), \dots, (\varrho \circ \sigma)(p), \varrho(p+1), \dots, \varrho(p+q)\}, \\ \{\tilde{\delta}(1), \dots, \tilde{\delta}(p+q)\} &= \{(\tilde{\varrho} \circ \tilde{\sigma})(1), \dots, (\tilde{\varrho} \circ \tilde{\sigma})(p), \tilde{\varrho}(p+1), \dots, \tilde{\varrho}(p+q)\}. \end{aligned}$$

According to (a) and (b),  $\text{sgn } \tilde{\delta} = (-1)^{q^2} \text{sgn } \delta$ . Further, we get easily  $\text{sgn } \delta = \text{sgn } \varrho \cdot \text{sgn } \sigma$ ,  $\text{sgn } \tilde{\delta} = \text{sgn } \tilde{\varrho} \cdot \text{sgn } \tilde{\sigma}$  and hence (c) follows. (d) and (e) are trivial.

**Notation.** Put  $f(\varrho, \sigma) = (\tilde{\varrho}, \tilde{\sigma})$  for  $(\varrho, \sigma) \in \mathfrak{B}$ . According to (e), the map  $f: \mathfrak{B} \rightarrow \mathfrak{B}$  is one-to-one.

**Proof of Theorem 3.2.** We proceed by induction on  $k$ . For  $k = 0$  our statement is a consequence of Theorem 3.1 (here we put  $q = p + a$ ,



see also formula (3.7)). Suppose now that Theorem 3.2 is true for any  $0 \leq k' \leq k$  where  $k < q$ . Let

$$\alpha_i \in D^{p-q+k+1, q-k-1} \otimes F, \quad \sum_{i=1}^d \alpha_i \times \eta_i = 0.$$

Consider a decomposition  $\mathfrak{S}_{p,q} = \mathfrak{U}_1 \cup \mathfrak{U}_2$ , where

$$\mathfrak{U}_1 = \{\varrho \in \mathfrak{S}_{p,q} \mid \{\varrho(1), \dots, \varrho(p)\} \supset \{1, \dots, p-q+k+1\}\},$$

and put  $\mathfrak{B}_i = \mathfrak{U}_i \times \mathfrak{S}_{p-q,q}$ , i.e., we have a decomposition  $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2$ . We can easily see from (a) and (b) that  $f(\mathfrak{B}_1) \subset \mathfrak{B}_2$ . Hence, the complement  $\mathfrak{B}'$  of  $f(\mathfrak{B}_1)$  in  $\mathfrak{B}_2$  is *invariant* by  $f$ .

Take a system of vectors  $x_1, \dots, x_{p+q}$ , where  $x_1, \dots, x_{p-q+k+1} \in V$  and the rest is in  $V_0$ . Now,

$$\sum_{i=1}^d (\alpha_i \times \eta_i)(x_1, \dots; \dots, x_{p+q}) = 0,$$

i.e.,

$$(3.23) \quad \sum_{i=1}^d \sum_{\varrho \in \mathfrak{U}_1} (\text{sgn } \varrho) \alpha_i(x_{\varrho(1)}, \dots; \dots, x_{\varrho(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) + \\ + \sum_{i=1}^d \sum_{\varrho \in \mathfrak{U}_2} (\text{sgn } \varrho) i_1(\alpha_i)(x_{\varrho(1)}, \dots; \dots, x_{\varrho(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) = 0.$$

Since

$$\sum_{i=1}^d i_1(\alpha_i) \times \eta_i = 0, \quad i_1(\alpha_i) \in D^{p-q+k, q-k} \otimes F,$$

we get, according to the induction assumption,

$$\cdot i_1(\alpha_i) = i_{q-k} \left( \sum_{j=1}^d h_i^j \wedge \eta_j \right), \quad \text{where } h_i^j \in \wedge^{*p-q} V, \quad h_i^j = (-1)^q h_j^i \\ (i, j = 1, \dots, d).$$

Thus, we can re-write the second term of (3.23) in the form

$$(3.24) \quad \sum_{i,j=1}^d \sum_{(\varrho, \sigma) \in \mathfrak{B}_2} \text{sgn } \varrho \cdot \text{sgn } \sigma \cdot h_i^j(x_{(\varrho \circ \sigma)(1)}, \dots, x_{(\varrho \circ \sigma)(p-q)}) \cdot \\ \cdot \eta_j(x_{(\varrho \circ \sigma)(p-q+1)}, \dots, x_{(\varrho \circ \sigma)(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}).$$

Now, according to (c) and (d) of Lemma 8, for any  $(\varrho, \sigma) \in \mathfrak{B}_2$ , we have

$$\sum_{i,j=1}^d \text{sgn } \varrho \cdot \text{sgn } \sigma \cdot h_i^j(x_{(\varrho \circ \sigma)(1)}, \dots, x_{(\varrho \circ \sigma)(p-q)}) \cdot \\ \cdot \eta_j(x_{(\varrho \circ \sigma)(p-q+1)}, \dots, x_{(\varrho \circ \sigma)(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)})$$

$$\begin{aligned}
&= (-1)^q \sum_{i,j=1}^d \operatorname{sgn} \tilde{\varrho} \cdot \operatorname{sgn} \tilde{\sigma} \cdot h_i^j(x_{(\tilde{\varrho} \circ \tilde{\sigma})(1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q)}) \cdot \\
&\quad \cdot \eta_j(x_{\tilde{\varrho}(p+1)}, \dots, x_{\tilde{\varrho}(p+q)}) \wedge \eta_i(x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q+1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p)}) \\
&= - \sum_{i,j=1}^d \operatorname{sgn} \tilde{\varrho} \cdot \operatorname{sgn} \tilde{\sigma} \cdot h_j^i(x_{(\tilde{\varrho} \circ \tilde{\sigma})(1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q)}) \cdot \\
&\quad \cdot \eta_i(x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q+1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p)}) \wedge \eta_j(x_{\tilde{\varrho}(p+1)}, \dots, x_{\tilde{\varrho}(p+q)}).
\end{aligned}$$

Now,  $\mathfrak{B}_2 = f(\mathfrak{B}_1) \cup \mathfrak{B}'$ ,  $f(\mathfrak{B}') = \mathfrak{B}'$ . Hence, we can re-write (3.24) in the form

$$\begin{aligned}
&- \sum_{(\varrho, \sigma) \in f(\mathfrak{B}_1)} \sum_{i,j=1}^d \operatorname{sgn} \tilde{\varrho} \cdot \operatorname{sgn} \tilde{\sigma} \cdot h_i^j(x_{(\tilde{\varrho} \circ \tilde{\sigma})(1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q)}) \cdot \\
&\quad \cdot \eta_j(x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q+1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p)}) \wedge \eta_i(x_{\tilde{\varrho}(p+1)}, \dots, x_{\tilde{\varrho}(p+q)}) + \\
&+ \frac{1}{2} \sum_{(\varrho, \sigma) \in \mathfrak{B}'} \sum_{i,j=1}^d \operatorname{sgn} \varrho \cdot \operatorname{sgn} \sigma \cdot h_i^j(x_{(\varrho \circ \sigma)(1)}, \dots, x_{(\varrho \circ \sigma)(p-q)}) \cdot \\
&\quad \cdot \eta_j(x_{(\varrho \circ \sigma)(p-q+1)}, \dots, x_{(\varrho \circ \sigma)(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) - \\
&- \frac{1}{2} \sum_{(\varrho, \sigma) \in \mathfrak{B}'} \sum_{i,j=1}^d \operatorname{sgn} \tilde{\varrho} \cdot \operatorname{sgn} \tilde{\sigma} \cdot h_i^j(x_{(\tilde{\varrho} \circ \tilde{\sigma})(1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q)}) \cdot \\
&\quad \cdot \eta_j(x_{(\tilde{\varrho} \circ \tilde{\sigma})(p-q+1)}, \dots, x_{(\tilde{\varrho} \circ \tilde{\sigma})(p)}) \wedge \eta_i(x_{\tilde{\varrho}(p+1)}, \dots, x_{\tilde{\varrho}(p+q)}) \\
&= - \sum_{\tilde{\varrho} \in \mathfrak{U}_1} \operatorname{sgn} \tilde{\varrho} \sum_{i,j=1}^d (h_i^j \wedge \eta_j)(x_{\tilde{\varrho}(1)}, \dots, x_{\tilde{\varrho}(p)}) \wedge \eta_i(x_{\tilde{\varrho}(p+1)}, \dots, x_{\tilde{\varrho}(p+q)}).
\end{aligned}$$

Consequently, (3.23) takes the form

$$\begin{aligned}
&\sum_{i=1}^d \sum_{\varrho \in \mathfrak{U}_1} \operatorname{sgn} \varrho \cdot \alpha_i(x_{\varrho(1)}, \dots; \dots, x_{\varrho(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) - \\
&- \sum_{i,j=1}^d \sum_{\varrho \in \mathfrak{U}_1} \operatorname{sgn} \varrho (h_i^j \wedge \eta_j)(x_{\varrho(1)}, \dots, x_{\varrho(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) = 0,
\end{aligned}$$

i.e.,

$$(3.25) \quad \sum_{\varrho \in \mathfrak{U}_1} \sum_{i=1}^d (\operatorname{sgn} \varrho) \tilde{\alpha}_i(x_{\varrho(1)}, \dots; \dots, x_{\varrho(p)}) \wedge \eta_i(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) = 0,$$

where

$$\tilde{\alpha}_i = \alpha_i - i_{q-k-1} \left\{ \sum_{j=1}^d h_i^j \wedge \eta_j \right\} \quad (i = 1, \dots, d).$$

Remark that for  $\varrho \in \mathfrak{U}_1$  we always have

$$\{\varrho(p+1), \dots, \varrho(p+q)\} \subset \{p-q+k+2, \dots, p+q\},$$

and hence

$$x_{e(p+1)}, \dots, x_{e(p+q)} \in V_0.$$

If  $x_{p-q+k+2}, \dots, x_{p+q}$  are linearly independent, then the system of vectors  $\eta_i(x_{e(p+1)}, \dots, x_{e(p+q)})$  ( $i = 1, \dots, d; \varrho \in \mathfrak{A}_1$ ) is linearly independent in  $F$ , and we can apply Cartan's lemma to (3.25).

For  $l = p+k+1, \dots, p+q$  consider the subsets  $\mathfrak{A}^{(l)} = \{\varrho \in \mathfrak{A}_1 \mid \varrho(p+q) = l\}$  and  $\mathfrak{B}^{(l)} = \bigcup \mathfrak{A}^{(i)} \ (p+k+1 \leq i \leq l)$ . Obviously,  $\mathfrak{B}^{(p+k+1)} \subset \dots \subset \mathfrak{B}^{(p+q)} = \mathfrak{A}_1$  and  $\mathfrak{B}^{(p+k+1)}$  consists of one element only.

**LEMMA 9.** *If  $x_1, \dots, x_{p-q+k+1} \in V$ , and  $x_{p-q+k+2}, \dots, x_{p+q} \in V_0$  are linearly independent, then for any  $l \in \{p+k+1, \dots, p+q\}$  the vectors  $\tilde{a}_i(x_1, \dots, \dots, x_p)$  ( $i = 1, \dots, d$ ) are linear combinations of the vectors  $\eta_j(x_{e(p+1)}, \dots, \dots, x_{e(p+q)})$  ( $j = 1, \dots, d; \varrho \in \mathfrak{B}^{(l)}$ ).*

**Proof.** We proceed by induction on  $l$  (where  $l$  is decreasing from  $p+q$  to  $p+k+1$ ). For  $l = p+q$  the result follows immediately from Cartan's lemma. Suppose that we have already proved (for any admissible collection of vectors  $(x_1, \dots, x_{p+q})$ ) that

$$\begin{aligned} (3.26) \quad \tilde{a}_i(x_1, \dots; \dots, x_p) &= \sum_{j=1}^d \sum_{\varrho \in \mathfrak{B}^{(l)}} a_{i,\varrho}^j \eta_j(x_{e(p+1)}, \dots, x_{e(p+q)}) \\ &= \sum_{j=1}^d \sum_{\varrho \in \mathfrak{B}^{(l-1)}} a_{i,\varrho}^j \eta_j(x_{e'(p+1)}, \dots, x_{e(p+q)}) + \\ &\quad + \sum_{j=1}^d \sum_{\varrho \in \mathfrak{A}^{(l)}} a_{i,\varrho}^j \eta_j(x_{e(p+1)}, \dots, x_{e(p+q-1)}, x_l), \end{aligned}$$

where  $a_{i,\varrho}^j$  are some constants.

Suppose now the collection  $(x_1, \dots, x_{p+q})$  to be fixed. Since  $\dim V_0 \geq 2q+1$ , one can complete vectors  $x_{p-q+k+2}, \dots, x_{p+q}$  by a new vector  $x'_l \in V_0$  to a linearly independent system. Then we can write, analogously to (3.26),

$$\begin{aligned} (3.27) \quad \tilde{a}_i(x_1, \dots; \dots, x_p) &= \sum_{j=1}^d \sum_{\varrho \in \mathfrak{B}^{(l-1)}} b_{i,\varrho}^j \eta_j(x_{e(p+1)}, \dots, x_{e(p+q)}) + \\ &\quad + \sum_{j=1}^d \sum_{\varrho \in \mathfrak{A}^{(l)}} b_{i,\varrho}^j \eta_j(x_{e(p+1)}, \dots, x_{e(p+q-1)}, x'_l) \end{aligned}$$

for the collection of vectors  $(x_1, \dots, x_p, \dots, x_{l-1}, x'_l, x_{l+1}, \dots, x_{p+q})$ . Since the vectors  $\eta_j(x_{e(p+1)}, \dots, x_{e(p+q)})$  ( $j = 1, \dots, d; \varrho \in \mathfrak{B}^{(l-1)}$ ) and the vectors  $\eta_j(x_{e(p+1)}, \dots, x_{e(p+q-1)}, x_l)$ ,  $\eta_j(x_{e(p+1)}, \dots, x_{e(p+q-1)}, x'_l)$  ( $j = 1, \dots, d; \varrho \in \mathfrak{A}^{(l)}$ ) for together a linearly independent system, we get, by comparison of (3.26) and (3.27),  $a_{i,\varrho}^j = b_{i,\varrho}^j = 0$  for  $\varrho \in \mathfrak{A}^{(l)}$ , and  $a_{i,\varrho}^j = b_{i,\varrho}^j$  for  $\varrho \in \mathfrak{B}^{(l-1)}$ , which completes the induction procedure for Lemma 9.

Finally, we obtain  $\tilde{a}_i = 0$  ( $i = 1, \dots, d$ ) which completes the induction procedure for Theorem 3.2, and, in particular, the proof of Theorem 1.1.

**Remark.** The special case of  $p = 2, q = 1$  of Theorem 1.1 was proved in [11] (cf. Part II, Lemma 4).

**4. A generalized Nijenhuis product.** Let  $L: \wedge^p V \otimes E \rightarrow F$  be a linear map and  $\omega \in \wedge^q(V, E)$ . We define a *generalized Nijenhuis product*  $\lrcorner$  as follows:  $\omega \lrcorner L \in \wedge^{p+q}(V, F)$  is determined by the formula

$$(4.1) \quad (\omega \lrcorner L)(x_1, \dots, x_{p+q}) \\ = \sum_{\varrho \in \mathfrak{S}_{p,q}} (\text{sgn } \varrho) L(x_{\varrho(1)}, \dots, x_{\varrho(p)}; \omega(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)})).$$

Similarly, for  $L: S^p V \otimes E \rightarrow F$  and  $\omega \in S^q(V, E)$ , we define

$$(4.2) \quad (\omega \lrcorner L)(x_1, \dots, x_{p+q}) \\ = \sum_{\varrho \in \mathfrak{S}_{p,q}} L(x_{\varrho(1)}, \dots, x_{\varrho(p)}; \omega(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)})).$$

**THEOREM 4.1.** *Let  $L: \wedge^p V \otimes E \rightarrow F$  (or  $L: S^p V \otimes E \rightarrow F$ ) be a linear map of type  $t \geq p + q$  and  $\omega \in \wedge^q(V, E)$  (or  $\omega \in S^q(V, E)$ ) a vector-valued form. Then  $\omega \lrcorner L = 0$  implies  $\omega = 0$ .*

**Proof.** We prove the first assertion only.

(i) Suppose first that  $\dim V = t$ , i.e., that  $V$  is a distinguished space with respect to  $L$ . Let in (4.1) the vectors  $x_1, \dots, x_{p+q} \in V$  be linearly independent and let  $\mathfrak{S}' \subset \mathfrak{S}_{p,q}$  be the subset of all  $(p, q)$ -shuffles such that  $\omega(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) \neq 0$  in  $E$ . Since  $L: \wedge^p V \otimes E \rightarrow F$  is injective, the vectors  $L(x_{\varrho(1)}, \dots, x_{\varrho(p)}; \omega(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}))$ ,  $\varrho \in \mathfrak{S}'$ , form a linearly independent system in  $F$ , and for  $\varrho \notin \mathfrak{S}'$  the corresponding  $L$ 's are zero. And since  $\omega \lrcorner L = 0$ , we obtain  $\mathfrak{S}' = \emptyset$ , and  $\omega = 0$  follows easily.

(ii) General case. Let  $V_0 \subset V$  be a distinguished space of  $L$ . According to (i),  $\omega(x_1, \dots, x_q) = 0$  if  $x_1, \dots, x_q \in V_0$ . Suppose that we have already proved  $\omega(x_1, \dots, x_k, x_{k+1}, \dots, x_q) = 0$  whenever  $x_1, \dots, x_k \in V$ ;  $x_{k+1}, \dots, x_q \in V_0$ . Let  $(x_1, \dots, x_q, x_{q+1}, \dots, x_{q+p})$  be a collection of linearly independent vectors such that  $x_1, \dots, x_{k+1} \in V$ , and  $x_{k+2}, \dots, x_q, x_{q+1}, \dots, x_{q+p} \in V_0$ . Let us define a decomposition  $\mathfrak{S}_{p,q} = \mathfrak{A} \cup \mathfrak{B}$  as follows:

$$\mathfrak{A} = \{\varrho \in \mathfrak{S}_{p,q} \mid \{\varrho(p+1), \dots, \varrho(p+q)\} \supset \{x_1, \dots, x_{k+1}\}\}.$$

According to the induction assumption,  $\omega(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) = 0$  for any  $\varrho \in \mathfrak{B}$ , and hence (4.1) and  $\omega \lrcorner L = 0$  imply

$$\sum_{\varrho \in \mathfrak{A}} (\text{sgn } \varrho) L(x_{\varrho(1)}, \dots, x_{\varrho(p)}; \omega(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)})) = 0.$$

Here  $x_{\varrho(1)}, \dots, x_{\varrho(p)} \in V_0$  for any  $\varrho \in \mathfrak{A}$  and, by the same argument as in (i), we show that  $\omega(x_{\varrho(p+1)}, \dots, x_{\varrho(p+q)}) = 0$  for any  $\varrho \in \mathfrak{A}$ . In particular,  $\omega(x_1, \dots, x_{k+1}, \dots, x_q) = 0$ . Now the equality  $\omega = 0$  on  $V$  follows by induction.

**5. An algebraic rigidity theorem.** Let  $\times$  denote the operation defined in Section 1. We can state

**THEOREM 5.1.** *Let  $\bar{\eta}_i, \eta_i \in \wedge^q(V, F)$  ( $i = 1, \dots, d$ ) and  $t(\eta_1, \dots, \eta_d) \geq 2q+1$ . If*

$$\text{Alt} \circ \left( \sum_{i=1}^d \bar{\eta}_i \times \bar{\eta}_i \right) = \text{Alt} \circ \left( \sum_{i=1}^d \eta_i \times \eta_i \right) \quad (q \text{ odd})$$

or

$$\text{Sym} \circ \left( \sum_{i=1}^d \bar{\eta}_i \times \bar{\eta}_i \right) = \text{Sym} \circ \left( \sum_{i=1}^d \eta_i \times \eta_i \right) \quad (q \text{ even}),$$

then there is a unique orthogonal matrix  $s = \|s_i^j\|$  of degree  $d$  such that

$$\bar{\eta}_i = \sum_{j=1}^d s_i^j \eta_j \quad (i = 1, \dots, d).$$

**Proof.** For  $q = 1$  the proof is a slight modification of that of Theorem 1, Note 17, [8] (cf. also [11], Part II, Lemma 3). In the general case, one can follow the same idea, taking advantage of the technique of Section 2. We shall not enter details here.

**Remark.** One can easily see that, if  $\eta_1, \dots, \eta_d$  are arbitrary in  $\wedge^q(V, F)$

$$\text{Alt} \circ \left( \sum_{i=1}^d \eta_i \times \eta_i \right) = 0 \quad \text{for } q \text{ even}$$

and

$$\text{Sym} \circ \left( \sum_{i=1}^d \eta_i \times \eta_i \right) = 0 \quad \text{for } q \text{ odd}.$$

**6. Applications to Riemannian vector bundles.** In the following we consider a fixed differentiable manifold  $M$  and various vector bundles  $E, F, \dots$  over  $M$ , all of class  $C^\infty$ . (The reader is advised to consult the books [3], [7] and [13].)

Notations of the preceding sections concerning vectors and linear maps can be naturally modified for sections of vector bundles and bundle morphisms. For instance,  $\wedge^p(M, F)$  will denote a vector bundle with the fibre  $\wedge^p(T_m(M), F_m)$  at each point  $m \in M$ .

If  $E$  is a vector bundle over  $M$ , then  $E$  will denote the *vector space* of all global sections of  $E$ .

Let  $E$  and  $F$  be vector bundles over  $M$  provided with linear connections  $\nabla^E$  and  $\nabla^F$ , respectively (see [7]).

We can canonically define a linear connection  $\nabla^{(E,F)}$  in the vector bundle  $E \otimes F$  by the rule

$$(6.1) \quad \nabla_x^{(E,F)}(e \otimes f) = \nabla_x^E(e) \otimes f + e \otimes \nabla_x^F(f) \quad (e \in E, f \in F, x \in T(M)).$$

For any vector field  $X \in T(M)$  and any section  $h$  of  $E \otimes F$ ,  $\nabla_X^{(E,F)}(h)$  is a section of  $E \otimes F$ .

Also, we can canonically define a linear connection  $\nabla^{[E,F]}$  in the vector bundle  $\text{Hom}(E, F)$  as follows: for  $\alpha \in \text{Hom}(E, F)$ ,  $e \in E$  and  $x \in T(M)$  we put

$$(6.2) \quad (\nabla_x^{[E,F]}\alpha)(e) = \nabla_x^F[\alpha(e)] - \alpha(\nabla_x^E(e)).$$

For a vector field  $X \in T(M)$  and any  $\alpha \in \text{Hom}(E, F)$ , (6.2) defines a covariant derivative  $\nabla_X^{[E,F]}\alpha \in \text{Hom}(E, F)$ .

If  $\nabla^1$  is a linear connection without torsion on  $M$  (i.e., in the tangent bundle  $T(M)$ ) and  $\nabla^F$  is a linear connection in  $F$ , then we have a canonical linear connection  $\nabla = \nabla^{p,F}$  in each vector bundle  $\wedge^p(M, F)$  ( $p = 0, 1, \dots$ ) given as follows: for  $\alpha \in \wedge^p(M, F)$ ,  $X_1, \dots, X_p \in T(M)$  and  $x \in T(M)$  we put

$$(6.3) \quad (\nabla_x \alpha)(X_1, \dots, X_p) = \nabla_x^F \{ \alpha(X_1, \dots, X_p) \} - \sum_{i=1}^p \alpha(X_1, \dots, \nabla_x^1 X_i, \dots, X_p).$$

For a vector field  $X \in T(M)$  and  $\alpha \in \wedge^p(M, F)$ , (6.3) defines a covariant derivative  $\nabla_X^{p,F} \alpha \in \wedge^p(M, F)$ .

The *Bianchi map*  $D^{p,F}$  is a linear map of  $\wedge^p(M, F)$  into  $\wedge^{p+1}(M, F)$  ( $p = 0, 1, \dots$ ) given as

$$(6.4) \quad (D^{p,F} \alpha)(X_1, \dots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^j (\nabla_{X_j}^{p,F} \alpha)(X_1, \dots, \hat{X}_j, \dots, X_{p+1})$$

for any vector fields  $X_1, \dots, X_{p+1} \in T(M)$ .

We write simply  $D$  instead of  $D^{p,F}$  if there is no risk of confusion.

If  $F$  is a trivial line bundle  $M \times R$ ,  $\wedge^p(M, F) = \wedge^p(M)$ , then  $D\alpha = -d\alpha$ , where  $d$  is the usual exterior derivative on differential forms (see [12]).

**PROPOSITION 6.1.** (i) *The Bianchi map  $D: \wedge^p(M, F) \rightarrow \wedge^{p+1}(M, F)$  is independent of the auxiliary linear connection  $\nabla^1$ .*

(ii) *For  $\alpha \in \wedge^p(M, F)$  and  $\beta \in \wedge^q(M, F)$  we have  $D(\alpha \times \beta) = D\alpha \times \beta + (-1)^p \alpha \times D\beta$ , where  $\times$  is the operation defined by (1.3).*

**Proof.** Let  $m \in M$  be a fixed point. Let us choose local sections  $\xi_1, \dots, \xi_d$  of  $F$  with the following properties: (a)  $\xi_1, \dots, \xi_d$  are defined and linearly independent in a neighbourhood  $U$  of  $m$ ,  $d = \dim F$ ; (b)  $\nabla_x^F \xi_i = 0$  for any  $x \in T_m(M)$  and  $i = 1, \dots, d$ .

With respect to the local frame  $(\xi_1, \dots, \xi_d)$  one can write

$$\alpha = \sum_{i=1}^d \alpha^i \otimes \xi_i, \quad \beta = \sum_{j=1}^d \beta^j \otimes \xi_j,$$

where  $\alpha^i$  ( $\beta^j$ ) are ordinary  $p$ -forms ( $q$ -forms), respectively, defined on  $U$ . Now, we can see that

$$D\alpha = \sum_{i=1}^d D\alpha^i \otimes \xi_i = - \sum_{i=1}^d d\alpha^i \otimes \xi_i$$

is independent of  $\nabla^1$ . Further,

$$\alpha \times \beta = \sum_{i,j=1}^d (\alpha^i \wedge \beta^j) \otimes (\xi_i \otimes \xi_j),$$

and, since  $\nabla_x^{(F,F)}(\xi_i \otimes \xi_j) = 0$  for any  $x \in T_m(M)$  (see (6.1)), we get at  $m$

$$D(\alpha \times \beta) = - \sum_{i,j=1}^d d(\alpha^i \wedge \beta^j) \otimes (\xi_i \otimes \xi_j).$$

The rest follows from the well-known formula

$$d(\alpha^i \wedge \beta^j) = d\alpha^i \wedge \beta^j + (-1)^p \alpha^i \wedge d\beta^j.$$

Let now  $E, F$  be vector bundles over  $M$  provided with fibre metrics  $\langle, \rangle_E, \langle, \rangle_F$ , respectively. Then to any vector-valued form  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$  we can define a dual form  $\alpha^* \in \Lambda^p(M, \text{Hom}(F, E))$  by

$$(6.5) \quad \langle \alpha^*(x_1, \dots, x_p)(f), e \rangle_E = \langle \alpha(x_1, \dots, x_p)(e), f \rangle_F$$

which is to be valid for any  $m \in M, e \in E_m, f \in F_m, x_1, \dots, x_p \in T_m(M)$ .

Further, let be given metric connections  $\nabla^E, \nabla^F$  in the Riemannian vector bundles  $E, F$ , respectively (see [7]).

Then canonical connections  $\nabla^{[E,F]}, \nabla^{[F,E]}$  are induced in the vector bundles  $\text{Hom}(E, F), \text{Hom}(F, E)$ , respectively, according to (6.2).

**PROPOSITION 6.2.** *If  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$  and  $\alpha^*$  is the corresponding dual form, then  $D\alpha^* = 0 \Leftrightarrow D\alpha = 0$ .*

**Proof.** The connections  $\nabla^E, \nabla^F$  are metric, i.e.,

$$x \langle e, e' \rangle_E = \langle \nabla_x^E(e), e' \rangle_E + \langle e, \nabla_x^E(e') \rangle_E,$$

$$x \langle f, f' \rangle_F = \langle \nabla_x^F(f), f' \rangle_F + \langle f, \nabla_x^F(f') \rangle_F$$

for any vector  $x \in T(M)$  and any sections  $e, e' \in E, f, f' \in F$ . Take an auxiliary  $\nabla^1$ . Using (6.2)-(6.5), we infer, by a routine calculation, that

$$(6.6) \quad \langle (Da)(x_1, \dots, x_{p+1})(e), f \rangle_F = \langle (Da^*)(x_1, \dots, x_{p+1})(f), e \rangle_E$$

for any  $x_1, \dots, x_{p+1} \in T(M)$ ,  $e \in E, f \in F$  over the same point  $m \in M$ . Consequently,  $Da^* = (Da)^*$ , and hence the result follows.

Let us introduce the following notation: for any  $\beta \in \Lambda^p(M, \text{Hom}(F, E))$  and any section  $f$  of  $F$  defined on a subset  $U \subset M$  let  $\beta(f)$  denote the section of  $\Lambda^p(M, E)$  defined on  $U$  by the rule  $\beta(f)(X_1, \dots, X_p) = \beta(X_1, \dots, X_p)(f)$ .

**PROPOSITION 6.3.** *Let  $\{\xi_1, \dots, \xi_d\}$  be a local orthonormal frame in the bundle  $F$  and let  $A_i$  be local sections of  $\Lambda^p(M, E)$  defined by  $A_i = \alpha^*(\xi_i)$  ( $i = 1, \dots, d$ ). Then*

$$(6.7) \quad (Da^*)(\xi_i) = DA_i - \sum_{j=1}^d \omega_i^j \wedge A_j \quad (i = 1, \dots, d),$$

where  $\omega_i^j$  are local sections of  $T^*(M)$  defined by the formula

$$(6.8) \quad \omega_i^j(x) = \langle \nabla_x^F \xi_i, \xi_j \rangle_F \quad (i, j = 1, \dots, d)$$

and the exterior product  $\wedge$  in (6.8) is taken in the sense of (1.8).

**Proof.** According to (6.2) and (6.3) (taking an auxiliary  $\nabla^1$ ), we get

$$\begin{aligned} [(\nabla_x \alpha^*)(X_1, \dots, X_p)](\xi_i) &= \nabla_x^E \{ \alpha^*(X_1, \dots, X_p)(\xi_i) \} - \\ &\quad - \alpha^*(X_1, \dots, X_p)(\nabla_x^F \xi_i) - \sum_{i=1}^p \alpha^*(X_1, \dots, \nabla_x^1 X_i, \dots, X_p)(\xi_i) \\ &= \nabla_x^E (A_i(X_1, \dots, X_p)) - \sum_{j=1}^d \langle \nabla_x^F \xi_i, \xi_j \rangle_F A_j(X_1, \dots, X_p) - \\ &\quad - \sum_{i=1}^p A_i(X_1, \dots, \nabla_x^1 X_i, \dots, X_p) \\ &= (\nabla_x^{p,E} A_i)(X_1, \dots, X_p) - \sum_{j=1}^d \omega_i^j(x) A_j(X_1, \dots, X_p). \end{aligned}$$

Now, (6.7) follows easily according to (6.4) and (1.8).

\*

Until further notice we shall suppose only that on vector bundles  $E, F$  are given fixed fibre metrics  $\langle, \rangle_E, \langle, \rangle_F$ , respectively.



Let be given  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$ . For any two sections  $e, e' \in E$  let us define a (real-valued) form  $\langle \alpha \cdot \alpha \rangle [e, e'] \in \Lambda^{*2p}(M)$  by the rule

$$(6.9) \quad \langle \alpha \cdot \alpha \rangle [e, e'] (x_1, \dots, x_{2p}) \\ = \sum_{\varrho \in \mathfrak{S}_{p,p}} \text{sgn } \varrho \langle \alpha(x_{\varrho(1)}, \dots, x_{\varrho(p)})(e), \alpha(x_{\varrho(p+1)}, \dots, x_{\varrho(2p)})(e') \rangle_F.$$

Hence, a *linear operator*  $\langle \alpha \cdot \alpha \rangle$  is defined ascribing a differentiable function to any two sections  $e, e' \in E$  and any  $2p$ -vector fields  $X_1, \dots, X_{2p} \in T(M)$ .  $\langle \alpha \cdot \alpha \rangle$  is of tensor character, i.e., at each point  $m \in M$  it depends on the corresponding vectors only.

**PROPOSITION 6.4.** *If  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$ , then  $\langle \alpha \cdot \alpha \rangle [e', e] = (-1)^p \langle \alpha \cdot \alpha \rangle [e, e']$  for any  $e, e' \in E$ .*

**Proof.** Let us consider a 1-1 map  $\sim : \mathfrak{S}_{p,p} \rightarrow \mathfrak{S}_{p,p}$  given as follows: for  $\varrho \in \mathfrak{S}_{p,p}$ ,  $\tilde{\varrho}$  is defined by  $\tilde{\varrho}(i) = \varrho(p+i)$ ,  $\tilde{\varrho}(p+i) = \varrho(i)$  ( $i = 1, \dots, p$ ). Then  $\text{sgn } \tilde{\varrho} = (-1)^p \text{sgn } \varrho$  and, according to (6.9),

$$\begin{aligned} \langle \alpha \cdot \alpha \rangle [e, e'] (x_1, \dots, x_{2p}) \\ = \sum_{\varrho \in \mathfrak{S}_{p,p}} (-1)^p \text{sgn } \tilde{\varrho} \langle \alpha(x_{\tilde{\varrho}(1)}, \dots, x_{\tilde{\varrho}(p)})(e'), \alpha(x_{\tilde{\varrho}(p+1)}, \dots, x_{\tilde{\varrho}(2p)})(e) \rangle_F \\ = (-1)^p \langle \alpha \cdot \alpha \rangle [e', e] (x_1, \dots, x_{2p}). \end{aligned}$$

Consider the given fibre metric  $\langle, \rangle_E$  on the vector bundle  $E$ . A natural fibre metric  $\langle, \rangle_{E \otimes E}$  on  $E \otimes E$  is defined by the rule

$$\langle X_1 \otimes Y_1, X_2 \otimes Y_2 \rangle_{E \otimes E} = \langle X_1, X_2 \rangle_E \langle Y_1, Y_2 \rangle_E \quad (X_i, Y_i \in T(M)).$$

Let us make the following convention: for  $A \in \Lambda^r(M, E)$  and  $e \in E$ , the *scalar product*  $\langle A, e \rangle_E$  is an element of  $\Lambda^{*r}(M)$  given by

$$(6.10) \quad \langle A, e \rangle_E (X_1, \dots, X_r) = \langle A(X_1, \dots, X_r), e \rangle_E \quad (X_1, \dots, X_r \in T(M)).$$

Taking the vector bundle  $E \otimes E$  instead of  $E$ , we can state the following

**PROPOSITION 6.5.** *Let  $\times$  denote the operation defined by (1.3), and let  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$ . In any neighbourhood  $U \subset M$ , where a local orthonormal frame  $(\xi_1, \dots, \xi_d)$  of  $E$  is defined, the following is true:*

$$(6.11) \quad \begin{aligned} \langle \alpha \cdot \alpha \rangle [e, e'] &= \frac{1}{2} \langle \text{Alt} \circ \left( \sum_{i=1}^d A_i \times A_i \right), e \otimes e' \rangle_{E \otimes E} \quad (p \text{ odd}), \\ \langle \alpha \cdot \alpha \rangle [e, e'] &= \frac{1}{2} \langle \text{Sym} \circ \left( \sum_{i=1}^d A_i \times A_i \right), e \otimes e' \rangle_{E \otimes E} \quad (p \text{ even}), \\ A_i &= \alpha^*(\xi_i) \quad (i = 1, \dots, d). \end{aligned}$$

**Proof.** According to (6.9) and (6.5),

$$\begin{aligned}
& \langle \alpha \cdot \alpha \rangle [e, e'](X_1, \dots, X_{2p}) \\
&= \sum_{i=1}^d \sum_{e \in \mathfrak{E}_{p,p}} \operatorname{sgn} \varrho \langle \alpha(X_{e(1)}, \dots, X_{e(p)})(e), \xi_i \rangle_F \langle \alpha(X_{e(p+1)}, \dots, X_{e(2p)})(e'), \xi_i \rangle_F \\
&= \sum_{i=1}^d \sum_{e \in \mathfrak{E}_{p,p}} \operatorname{sgn} \varrho \langle A_i(X_{e(1)}, \dots, X_{e(p)}), e \rangle_E \langle A_i(X_{e(p+1)}, \dots, X_{e(2p)}), e' \rangle_E \\
&= \sum_{i=1}^d \sum_{e \in \mathfrak{E}_{p,p}} \operatorname{sgn} \varrho \langle A_i(X_{e(1)}, \dots, X_{e(p)}) \otimes A_i(X_{e(p+1)}, \dots, X_{e(2p)}), e \otimes e' \rangle_{E \otimes E},
\end{aligned}$$

and hence

$$(6.12) \quad \langle \alpha \cdot \alpha \rangle [e, e'] = \sum_{i=1}^d \langle A_i \times A_i, e \otimes e' \rangle_{E \otimes E}.$$

Our assertion is now a consequence of Proposition 6.4.

Now, in addition, let be yet given

- (a) a linear connection  $\nabla^1$  without torsion on  $M$ ,
- (b) a metric connection  $\nabla^E$  in the Riemannian vector bundle  $E$ .

For any  $\alpha \in \Lambda^p(M, \operatorname{Hom}(E, F))$  and any vector field  $X \in T(M)$  we define the *covariant derivative*  $\nabla_X \langle \alpha \cdot \alpha \rangle$  as follows: for any sections  $e, e' \in E$  and any vector fields  $X_1, \dots, X_{2p} \in T(M)$  we put

$$\begin{aligned}
(6.13) \quad & (\nabla_X \langle \alpha \cdot \alpha \rangle) \{[e, e'](X_1, \dots, X_{2p})\} \\
&= X \{ \langle \alpha \cdot \alpha \rangle [e, e'](X_1, \dots, X_{2p}) \} - \langle \alpha \cdot \alpha \rangle [\nabla_X^E e, e'](X_1, \dots, X_{2p}) - \\
&- \langle \alpha \cdot \alpha \rangle [e, \nabla_X^E e'](X_1, \dots, X_{2p}) - \sum_{i=1}^{2p} \langle \alpha \cdot \alpha \rangle [e, e'](X_1, \dots, \nabla_X^1 X_i, \dots, X_{2p}).
\end{aligned}$$

Clearly, the covariant derivative  $\nabla_X \langle \alpha \cdot \alpha \rangle$  is a linear operator of the same type as  $\langle \alpha \cdot \alpha \rangle$ .

The *Bianchi image*  $D \langle \alpha \cdot \alpha \rangle$  of  $\langle \alpha \cdot \alpha \rangle$  is defined by the formula

$$\begin{aligned}
(6.14) \quad & (D \langle \alpha \cdot \alpha \rangle) [e, e'](X_1, \dots, X_{2p}, X_{2p+1}) \\
&= \sum_{i=1}^{2p+1} (-1)^i (\nabla_{X_i} \langle \alpha \cdot \alpha \rangle) [e, e'](X_1, \dots, \hat{X}_i, \dots, X_{2p+1})
\end{aligned}$$

for any sections  $e, e' \in E$  and any vector fields  $X_1, \dots, X_{2p+1} \in T(M)$ . One can see that  $D \langle \alpha \cdot \alpha \rangle$  is a multilinear map on vectors at each point  $m \in M$ , and it is *independent of an auxiliary*  $\nabla^1$ .

**PROPOSITION 6.6.** *Let  $(\xi_1, \dots, \xi_d)$  be an orthonormal frame of the bundle  $F$  defined in a neighbourhood  $U \subset M$ . Put  $A_i = \alpha^*(\xi_i)$  ( $i = 1, \dots, d$ ),*

where  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$ . Then, on the neighbourhood  $U$ ,  $D\langle \alpha \cdot \alpha \rangle = 0$  holds if and only if

$$\text{Alt} \circ \left( \sum_{i=1}^d DA_i \times A_i \right) = 0 \quad (p \text{ odd}),$$

or

$$\text{Sym} \circ \left( \sum_{i=1}^d DA_i \times A_i \right) = 0 \quad (p \text{ even}).$$

**Proof.** Let  $m \in U$  be a fixed point. Remark that for any  $A \in \Lambda^p(M, E)$  and any section  $e \in E$  such that  $\nabla_x^E e = 0$  for any  $x \in T_m(M)$  we have  $\langle A, e \rangle_E \in \Lambda^{*p}(M)$  (cf. (6.10)) and

$$(6.15) \quad D\langle A, e \rangle_E = \langle DA, e \rangle_E \quad \text{at } m.$$

Further, if  $e' \in E$  is another section such that  $\nabla_x^E e' = 0$  for any  $x \in T_m(M)$ , we get, according to (6.3), (6.4), (6.13) and (6.14),

$$(D\langle \alpha \cdot \alpha \rangle)[e, e'] = D\{\langle \alpha \cdot \alpha \rangle[e, e']\} \quad \text{at } m.$$

If we apply (6.15) to the form

$$A = \sum_{i=1}^d A_i \times A_i \in \Lambda^{2p}(M, E \otimes E)|_U$$

and to the section  $e \otimes e'$  (where  $\nabla_x^{(E, E)}(e \otimes e') = 0$  for any  $x \in T_m(M)$ ), we get from (6.12)

$$D\{\langle \alpha \cdot \alpha \rangle[e, e']\} = \left\langle \sum_{i=1}^d D(A_i \times A_i), e \otimes e' \right\rangle_{E \otimes E} \quad \text{at } m.$$

Consequently,

$$(D\langle \alpha \cdot \alpha \rangle)[e, e'] = \left\langle \sum_{i=1}^d D(A_i \times A_i), e \otimes e' \right\rangle_{E \otimes E} \quad \text{at } m.$$

The last relation depends only on the values of  $e, e'$  at  $m$ , and since  $m \in U$  is arbitrary, it is an identity for any two sections  $e, e'$  of  $E$  defined on  $U$ .

Hence  $D\langle \alpha \cdot \alpha \rangle = 0$  holds on  $U$  if and only if

$$\sum_{i=1}^d D(A_i \times A_i) = 0.$$

Now,

$$(6.16) \quad \sum_{i=1}^d D(A_i \times A_i) = \frac{1}{2} \sum_{i=1}^d (\text{Alt} \circ D(A_i \times A_i) + \text{Sym} \circ D(A_i \times A_i)).$$

According to (3.16), for  $A \in \Lambda^p(M, E)$  and  $B \in \Lambda^q(M, E)$ ,

$$\text{Alt} \circ (A \times B) = (-1)^{pq+1} \text{Alt} \circ (B \times A),$$

and, similarly,

$$\text{Sym} \circ (A \times B) = (-1)^{pq} \text{Sym} \circ (B \times A).$$

Hence

$$(6.17) \quad \begin{aligned} \text{Alt} \circ (A_i \times DA_i) &= -\text{Alt} \circ (DA_i \times A_i), \\ \text{Sym} \circ (A_i \times DA_i) &= \text{Sym} \circ (DA_i \times A_i). \end{aligned}$$

According to Proposition 6.1,  $D(A_i \times A_i) = DA_i \times A_i + (-1)^p A_i \times DA_i$ , and (6.16) and (6.17) yield

$$\begin{aligned} \sum_{i=1}^d D(A_i \times A_i) &= \frac{1}{2} \sum_{i=1}^d \{ [1 - (-1)^p] \text{Alt} \circ (DA_i \times A_i) + \\ &\quad + [1 + (-1)^p] \text{Sym} \circ (DA_i \times A_i) \}. \end{aligned}$$

Consequently,

$$\sum_{i=1}^d D(A_i \times A_i) = \begin{cases} \text{Alt} \circ \left( \sum_{i=1}^d DA_i \times A_i \right) & (p \text{ odd}), \\ \text{Sym} \circ \left( \sum_{i=1}^d DA_i \times A_i \right) & (p \text{ even}), \end{cases}$$

and hence the result follows.

**PROPOSITION 6.7.** *Let be given metric connections  $\nabla^E, \nabla^F$  in Riemannian vector bundles  $E, F$ , respectively. Then, for any  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$ ,  $Da = 0$  implies  $D\langle \alpha \cdot \alpha \rangle = 0$ .*

**Proof.** Suppose  $p$  to be odd (for  $p$  even the proof is similar). According to Proposition 6.2,  $Da = 0$  implies  $Da^* = 0$  and, according to (6.7),  $Da^* = 0$  implies locally

$$DA_i = \sum_{j=1}^d \omega_i^j \wedge A_j \quad (i = 1, \dots, d),$$

where  $\omega_j^i = -\omega_i^j$  according to (6.8). Now (see (1.7) and (1.8)),

$$\begin{aligned} \text{Alt} \circ \left( \sum_{i=1}^d DA_i \times A_i \right) &= \text{Alt} \circ \left( \sum_{i,j=1}^d (\omega_i^j \wedge A_j) \times A_i \right) = \sum_{i,j=1}^d \omega_i^j \wedge (A_j \cdot A_i) \\ &= \sum_{i,j=1}^d \omega_j^i \wedge (A_i \cdot A_j) = (-1)^p \sum_{i,j=1}^d \omega_i^j \wedge (A_j \cdot A_i), \end{aligned}$$

because  $A_i \cdot A_j = (-1)^{p+1} A_j \cdot A_i$ . Hence, it follows that

$$\sum_{i,j=1}^d \omega_i^j \wedge (A_j \cdot A_i) = 0,$$

i.e.,

$$\text{Alt} \circ \left( \sum_{i=1}^d DA_i \times A_i \right) = 0.$$

Now to complete the proof we use Proposition 6.6.

\*

In the following let be given two Riemannian vector bundles  $E, F$  over  $M$  and a metric connection  $\nabla^E$  in  $E$ . (If necessary, an auxiliary connection  $\nabla^1$  on  $M$  will be chosen without comment.)

**Definition 1.** Let  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$ . A metric connection  $\nabla^F$  in  $F$  will be called  $\alpha$ -complementary if  $D\alpha = 0$  with respect to the connections  $\nabla^E$  and  $\nabla^F$  (see (6.2)-(6.4)).

To any  $\beta \in \Lambda^p(M, \text{Hom}(F, E))$  a canonical bundle morphism  $[\beta]: \Lambda^p(M) \otimes F \rightarrow E$  is defined by the rule

$$(6.18) \quad [\beta](X_1 \wedge \dots \wedge X_p \otimes f) = \beta(X_1, \dots, X_p)(f) \\ \text{for any } X_1, \dots, X_p \in T(M), f \in E.$$

We also write  $[\beta](X_1, \dots, X_p; f)$  instead of  $[\beta](X_1 \wedge \dots \wedge X_p \otimes f)$ .

**Definition 2.** A form  $\beta \in \Lambda^p(M, \text{Hom}(F, E))$  is called of type  $t \geq t_0$  if the bundle morphism  $[\beta]$  is of type  $t \geq t_0$  at each point  $m \in M$  (see Section 1).

**Notation.**  $t(\beta) \geq t_0$ .

**PROPOSITION 6.8.** Let  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$  be such that  $t(\alpha^*) \geq p+1$ . Then if a  $\alpha$ -complementary connection  $\nabla^F$  in  $F$  exists, it is unique.

**Proof.** Let  $\nabla^F, \tilde{\nabla}^F$  be two  $\alpha$ -complementary connections in  $F$  and let  $\nabla, \tilde{\nabla}(D, \tilde{D})$  denote the corresponding covariant derivatives (Bianchi maps) acting on the form  $\alpha^* \in \Lambda^p(M, \text{Hom}(F, E))$ . The difference tensor  $T = \tilde{\nabla}^F - \nabla^F$  is a bundle morphism,  $T: T(M) \otimes F \rightarrow F$ . According to (6.2) and (6.3), we get easily

$$(6.19) \quad (\nabla_X \alpha^* - \tilde{\nabla}_X \alpha^*)(X_1, \dots, X_p)(f) = \alpha^*(X_1, \dots, X_p)(T(X \otimes f))$$

for any vector fields  $X_1, \dots, X_p, X \in T(M)$  and any section  $f \in F$ .

Let now  $f$  be fixed. Define a 1-form  $T^f \in \Lambda^1(M, F)$  by  $T^f(X) = T(X \otimes f), X \in T(M)$ . Then we can re-write (6.19) in the form

$$(\nabla_X \alpha^* - \tilde{\nabla}_X \alpha^*)(X_1, \dots, X_p)(f) = [\alpha^*](X_1, \dots, X_p; T^f(X)).$$

Taking Proposition 6.2 into account, we get from (6.4)

$$0 = (D\alpha^* - \tilde{D}\alpha^*)(X_1, \dots, X_p)(f) \\ = \sum_{j=1}^{p+1} (-1)^j [\alpha^*](X_1, \dots, \tilde{X}_j, \dots, X_{p+1}; T^f(X_j))$$

and, consequently, the Nijenhuis product  $T^f \lrcorner [a^*] = 0$  (cf. (4.1)). According to Theorem 4.1,  $T^f = 0$ .

**THEOREM 6.1 (Existence Theorem).** *Let  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$  be such that  $t(\alpha^*) \geq 2p + 2$ . Then a (unique)  $\alpha$ -complementary connection  $\nabla^F$  exists if and only if  $D\langle \alpha \cdot \alpha \rangle = 0$ .*

**Proof.** If a connection  $\nabla^F$  exists such that  $D\alpha = 0$ , then  $D\langle \alpha \cdot \alpha \rangle = 0$  according to Proposition 6.7. Conversely, let  $D\langle \alpha \cdot \alpha \rangle = 0$ . Choose a neighbourhood  $U \subset M$  in which a local orthonormal frame  $(\xi_1, \dots, \xi_d)$  of  $F$  exists. Put  $A_i = \alpha^*(\xi_i)$  ( $i = 1, \dots, d$ ), as usually. According to Proposition 6.6 and part (I) of the Main Theorem,

$$DA_i = \sum_{j=1}^d \omega_i^j \wedge A_j \quad (i = 1, \dots, d),$$

where  $\omega_i^j$  ( $i, j = 1, \dots, d$ ) are uniquely determined 1-forms satisfying  $\omega_j^i = -\omega_i^j$  on  $U$ . Let us determine a connection  $\nabla^F$  in  $F|_U$  in a way such that the covariant derivatives  $\nabla_x^F \xi_i$  are given by formula (6.8). Clearly,  $\nabla^F$  is metric due to the antisymmetry of  $\|\omega_j^i\|$ . According to (6.7), we get  $D\alpha^* = 0$  on  $U$ , i.e.,  $\nabla^F$  is an  $\alpha$ -complementary connection in the neighbourhood  $U$ . Using a suitable open covering of  $M$  and the uniqueness property (see Proposition 6.8), we can complete the proof.

**THEOREM 6.2 (Rigidity Theorem).** *Let  $\alpha, \beta \in \Lambda^p(M, \text{Hom}(E, F))$  be such that  $t(\alpha^*) \geq 2p + 1$  and  $\langle \beta \cdot \beta \rangle = \langle \alpha \cdot \alpha \rangle$ . Then  $\alpha, \beta$  are equivalent in the following sense: there is a bundle isometry  $\Phi: F \rightarrow F$  such that  $[\beta] = \Phi \circ [\alpha]$ .*

**Remark.** Here  $[\alpha]$  and  $[\beta]$  are defined similarly as in (6.18), with the interchanging of  $E$  and  $F$ .

We state beforehand

**PROPOSITION 6.9.** *If  $\alpha \in \Lambda^p(M, \text{Hom}(E, F))$ ,  $t(\alpha^*) \geq p$ , then the bundle morphism  $[\alpha]: \wedge^p(M) \otimes E \rightarrow F$  is onto.*

**Proof.** Suppose the contrary. Then there is a point  $m \in M$  and a vector  $f \in F_m$  such that  $[\alpha](\sigma \otimes e)$  is orthogonal to  $f$  for any  $\sigma \in \wedge^p(T_m(M))$ ,  $e \in E_m$ . Let  $V_0 \subset T_m(M)$  be a distinguished space with respect to  $[\alpha^*]$  and let  $x_1, \dots, x_p \in V_0$  be linearly independent vectors; put  $\sigma_0 = x_1 \wedge \dots \wedge x_p$ . Then

$$\langle [\alpha](\sigma_0 \otimes e), f \rangle_F = \langle [\alpha^*](\sigma_0 \otimes f), e \rangle_E = 0 \quad \text{for any } e \in E_m,$$

and hence  $[\alpha^*](\sigma_0 \otimes f) = 0$ , a contradiction to the injectivity of  $[\alpha^*]$  on the space  $\wedge^p(V_0) \otimes F_m$ .

**Proof of Theorem 6.2.** Let  $(\xi_1, \dots, \xi_d)$  be an orthonormal frame of  $F$  defined in a neighbourhood  $U \subset M$ . Put  $A_i = \alpha^*(\xi_i)$ ,  $B_i = \beta^*(\xi_i)$  ( $i = 1, \dots, d$ ). According to Proposition 6.5, the relation  $\langle \beta \cdot \beta \rangle = \langle \alpha \cdot \alpha \rangle$

implies

$$\text{Alt} \circ \left( \sum_{i=1}^d B_i \times B_i \right) = \text{Alt} \circ \left( \sum_{i=1}^d A_i \times A_i \right) \quad \text{for } p \text{ odd,}$$

or

$$\text{Sym} \circ \left( \sum_{i=1}^d B_i \times B_i \right) = \text{Sym} \circ \left( \sum_{i=1}^d A_i \times A_i \right) \quad \text{for } p \text{ even.}$$

From Theorem 5.1 it follows that

$$B_i = \sum_{j=1}^d s_i^j A_j \quad (i = 1, \dots, d),$$

where  $s_i^j$  ( $i, j = 1, \dots, d$ ) are differentiable functions on  $U$  and the matrix  $\|s_i^j\|$  is orthogonal at each point  $m \in U$ . Let  $\Phi$  be an isometry of the restricted bundle  $F|_U$  onto itself given by

$$\Phi^{-1}(\xi_i) = \sum_{j=1}^d s_i^j \xi_j \quad (i = 1, \dots, d).$$

We get  $\beta^*(\xi_i) = \alpha^*(\Phi^{-1}(\xi_i))$  for  $i = 1, \dots, d$ , i.e.,  $\beta^*(f) = \alpha^*(\Phi^{-1}(f))$  for any section  $f$  of  $F|_U$ . Using the duality formula (6.5) and the property  $\langle \Phi(f), \Phi(f') \rangle_F = \langle f, f' \rangle_F$ , we obtain our assertion for the subset  $U \subset M$ .

Now, we can construct an open covering  $\{U_i\}_{i \in I}$  of  $M$  and a system of bundle isometries  $\Phi_i: F|_{U_i} \rightarrow F|_{U_i}$  satisfying  $[\beta] = \Phi_i \circ [\alpha]$ . In any intersection  $U_i \cap U_j$  we get  $\Phi_i \circ \Phi_j^{-1} \circ [\alpha] = [\alpha]$ , and Proposition 6.9 implies that  $\Phi_i \circ \Phi_j^{-1} = \text{identity}$ .

Consequently, the system  $\{\Phi_i\}_{i \in I}$  determines a unique bundle isometry  $\Phi: F \rightarrow F$  (of class  $C^\infty$ ) with the wanted property.

\*

In the rest of this section we turn our attention to the case of a *generalized second fundamental form*.

We use some basic facts of [9]. Let  $E, F$  be Riemannian vector bundles over  $M$  and let  $E \oplus F$  denote their *Riemannian direct sum* (i.e., the fibre metric  $\langle, \rangle_{E \oplus F}$  is defined as a direct sum of the fibre metrics  $\langle, \rangle_E$  and  $\langle, \rangle_F$ ). To any metric connection  $\nabla$  in  $E \oplus F$  we can construct metric connections  $\nabla^E, \nabla^F$  which are *orthogonal projections* of  $\nabla$  into  $E, F$ , respectively. Moreover, we have the following

**PROPOSITION 6.10.** *To any metric connection  $\nabla$  in  $E \oplus F$  there is a unique form  $\alpha \in \Lambda^1(M, \text{Hom}(E, F))$  (called the *second fundamental form* of  $\nabla$ ) such that for any sections  $e \in E, f \in F$  and any vector  $x \in T(M)$  there is*

$$\begin{aligned} \nabla_x e &= \nabla_x^E(e) + \alpha(x)(e), \\ \nabla_x f &= \nabla_x^F(f) - \alpha^*(x)(f), \end{aligned} \quad (6.20)$$

where  $\alpha^* \in \Lambda^1(M, \text{Hom}(F, E))$  is the dual form to  $\alpha$ .

Proof is obvious (cf. also Proposition 7 from [9]).

Consider the curvature tensors  $R \in \Lambda^2(M, \text{Hom}(E \oplus F, E \oplus F))$ ,  $R^E \in \Lambda^2(M, \text{Hom}(E, E))$ ,  $R^F \in \Lambda^2(M, \text{Hom}(F, F))$  of the connections  $\nabla$ ,  $\nabla^E$ ,  $\nabla^F$ , respectively. (Let us remind that  $R(X, Y)\xi = [\nabla_X, \nabla_Y]\xi - \nabla_{[X, Y]}\xi$  for any vector fields  $X, Y \in T(M)$  and any section  $\xi: M \rightarrow E \oplus F$ , and  $R^E, R^F$  are defined in a similar way (cf. [7] and [9]).

Let us introduce a linear operator  $\langle R^E \rangle$  as follows:

$$\langle R^E \rangle[e, e'](X, Y) = \langle R^E(X, Y)e, e' \rangle_E \quad \text{for any } X, Y \in T(M), e, e' \in E.$$

Similarly, we define a linear operator  $\langle R^F \rangle$  (cf. formula (6.9)).

Making use of an auxiliary linear connection  $\nabla^1$  without torsion on  $M$ , one can easily check the second Bianchi identities  $DR^E = 0$ ,  $DR^F = 0$ , i.e.,

$$(6.21) \quad D\langle R^E \rangle = 0, \quad D\langle R^F \rangle = 0.$$

The connection  $\nabla$  is called *flat* if the curvature tensor  $R$  vanishes. Now, we can state

**PROPOSITION 6.11.** *The connection  $\nabla$  is flat if and only if the following integrability conditions hold:*

$$(6.22) \quad Da = Da^* = 0 \quad (\text{Codazzi equations}),$$

$$(6.23) \quad \langle R^E \rangle + \langle \alpha \cdot \alpha \rangle = 0 \quad (\text{Gauss equations}).$$

$$(6.24) \quad \langle R^F \rangle + \langle \alpha^* \cdot \alpha^* \rangle = 0$$

Proof follows easily from the definitions and from (6.20).

**PROPOSITION 6.12.** *If  $\nabla^E, \nabla^F$  are metric connections in  $E, F$ , respectively, and  $\alpha \in \Lambda^1(M, \text{Hom}(E, F))$  satisfies  $t(\alpha^*) \geq 3$ , then (6.24) is a consequence of (6.22) and (6.23).*

Outline of the proof. Let us write out the identity  $D(Da) = 0$ . After a routine (but rather lengthy) calculation we obtain, as a consequence of  $Da = 0$ ,

$$(6.25) \quad \mathfrak{S}\{R^F(X, Y)(\alpha(Z)(e))\} = \mathfrak{S}\{\alpha(Y)(R^E(Z, X)e)\},$$

$$X, Y, Z \in T(M), e \in E,$$

where  $\mathfrak{S}$  denotes the cyclic sum with respect to  $X, Y, Z$  (cf. [10], Proposition 1). Now we get successively, using (6.23), (6.5) and (6.25),

$$\begin{aligned} \mathfrak{S}\{\langle \alpha^* \cdot \alpha^* \rangle[\alpha(Z)(e), f](X, Y)\} &= \mathfrak{S}\{\langle R^E(Z, X)e, \alpha^*(Y)(f) \rangle_E\} \\ &= -\mathfrak{S}\{\langle \alpha(Y)(R^E(Z, X)e), f \rangle_F\} = -\mathfrak{S}\{\langle R^F(X, Y)(\alpha(Z)(e)), f \rangle_F\} \\ &= -\mathfrak{S}\{\langle R^F \rangle[\alpha(Z)(e), f](X, Y)\}. \end{aligned}$$

Put  $Q = \langle R^F \rangle + \langle \alpha^* \cdot \alpha^* \rangle$ ; thus  $\mathfrak{S}\{Q[\alpha(Z)(e), f](X, Y)\} = 0$  (cf. [10],



Proposition 2). For any fixed  $f \in F$  let us define a form  $Q^f \in \Lambda^2(M, F)$  as follows:

$$\langle Q^f(X, Y), f' \rangle_F = Q[f, f'](X, Y) \quad \text{for any } X, Y \in T(M), f' \in F.$$

We get  $\mathfrak{S}\{\langle Q^f(X, Y), \alpha(Z)(e) \rangle_F\} = 0$ , i.e.,

$$\mathfrak{S}\{\langle \alpha^*(Z)(Q^f(X, Y)), e \rangle_E\} = 0$$

and hence

$$\mathfrak{S}\{[\alpha^*](Z; Q^f(X, Y))\} = 0.$$

The last equality means that the Nijenhuis product  $Q^f \lrcorner [\alpha^*]$  vanishes. From Theorem 4.1 we get  $Q^f = 0$  for any section  $f \in F$ , and hence  $Q = 0$  identically (cf. Theorem 1 from [11]).

We say that a metric connection  $\nabla$  in  $E \oplus F$  is an *extension* of a metric connection  $\nabla^E$  in  $E$  if  $\nabla^E$  is the orthogonal projection of  $\nabla$  into  $E$ . Now we can state

**THEOREM 6.3.** *Let  $E, F$  be Riemannian vector bundles over  $M$ , and let  $\nabla^E$  be a given metric connection in  $E$ . Suppose that there is a form  $\alpha \in \Lambda^1(M, \text{Hom}(E, F))$  such that  $t(\alpha^*) \geq 4$  and  $\langle R^E \rangle + \langle \alpha \cdot \alpha \rangle = 0$ . Then there is a unique flat connection  $\nabla$  in  $E \oplus F$  extending  $\nabla^E$  and having  $\alpha$  as its second fundamental form. Moreover, there is a 1-1 correspondence between all flat connections in  $E \oplus F$  extending  $\nabla^E$  and between all bundle isometries of  $F$ .*

**Proof.** Because of  $D\langle R^E \rangle = 0$  we get  $D\langle \alpha \cdot \alpha \rangle = 0$ , and, according to Theorem 6.1 (case  $p = 1$ ), there is a unique  $\alpha$ -complementary connection  $\nabla^F$  in  $F$ . With respect to  $\nabla^E, \nabla^F$ , (6.22) is satisfied. According to Proposition 6.12, we have also (6.24), and hence the connection  $\nabla$  given by formulae (6.20) is flat (and, obviously, metric in  $E \oplus F$ ).

Further, if  $\tilde{\nabla}$  is another flat connection extending  $\nabla^E$ , and  $\beta$  is its second fundamental form, then, according to (6.23),  $\langle \beta \cdot \beta \rangle = \langle \alpha \cdot \alpha \rangle$  and there is a unique bundle isometry  $\Phi_\beta: F \rightarrow F$  such that  $[\beta] = \Phi_\beta \circ [\alpha]$  (see Theorem 6.2). Conversely, if  $\Phi: F \rightarrow F$  is a bundle isometry, there is a unique form  $\beta \in \Lambda^1(M, \text{Hom}(E, F))$  such that  $[\beta] = \Phi \circ [\alpha]$ . Clearly,  $t(\beta^*) = t(\alpha^*) \geq 4$  and  $\langle \beta \cdot \beta \rangle = \langle \alpha \cdot \alpha \rangle$ , i.e.,  $\langle R^E \rangle + \langle \beta \cdot \beta \rangle = 0$ . Hence  $\beta$  determines a unique flat connection  $\nabla$  in  $E \oplus F$  extending  $\nabla^E$  and having  $\beta$  as its second fundamental form, which completes the proof.

**Remark.** In the case where  $E, F$  are  $k$ -th and  $(k+1)$ -th *normal bundle* of a manifold  $M$ , isometrically embedded in a euclidean space  $E^N$ , we obtain the results of Part III of the paper [11]. In particular, if  $E$  is the tangent bundle and  $F$  the *first normal bundle* of  $M \subset E^N$ , we get all the classical results of Allendoerfer [1].

## REFERENCES

- [1] C. B. Allendoerfer, *Rigidity for spaces of class greater than one*, American Journal of Mathematics 61 (1939), p. 633-644.
- [2] D. Bernard, *Sur la géométrie différentielle des G-structures*, Annales de l'Institut Fourier (Grenoble) 10 (1960), p. 151-270.
- [3] R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, New York and London 1964.
- [4] N. Bourbaki, *Éléments de mathématique*, Première Partie, Livre II, Algèbre, Chapitre III; Paris.
- [5] E. Cartan, *Les systèmes différentiels extérieurs*, Paris 1945.
- [6] S. S. Chern, *On a theorem of algebra and its geometrical applications*, The Journal of the Indian Mathematical Society 8 (1944), p. 29-36.
- [7] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I, New York-London 1963.
- [8] — *Foundations of differential geometry*, Vol. II, New York-London-Sydney 1969.
- [9] O. Kowalski, *Immersions of Riemannian manifolds with a given normal bundle structure*, Part I, Czechoslovak Mathematical Journal 19 (94) (1969), p. 676-695.
- [10] — *Immersions of Riemannian manifolds with a given normal bundle structure*, Part II, ibidem 21 (96) (1971), p. 137-156.
- [11] — *Type numbers in the metric differential geometry of higher order*, Journal of Differential Geometry (to appear).
- [12] Y. Matsushima, *Vector bundle valued harmonic forms and immersions of Riemannian manifolds*, preprint.
- [13] S. Sternberg, *Lectures on differential geometry*, New Jersey 1964.
- [14] T. Y. Thomas, *Riemann spaces of class one and their characterization*, Acta Mathematica 67 (1936), p. 169-211.

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