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GRADIENTS OF BOREL FUNCTIONS

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In the last years the problem of characterizing functions (of a point or of a set) having a.e. finite derivatives or having derivatives belonging to the space L^1 was studied by various authors (e.g., L. Albano and N. Fedele, V. Aversa, B. Bongiorno, P. de Lucia, R. Fiorenza, U. Oliveri, B. Pettineo, P. Vetro, H. Wright and W. S. Snyder).

In a recent paper [1] we have considered functions defined on an open bounded subset of R^n, continuous separately with respect to each variable, and we have found conditions under which the gradient exists a.e. and belongs to the space L^p for $p > 0$.

The aim of this paper* is to extend the results of [1] to the case of Borel functions defined on a bounded Borel set. So our results will cover also the case of approximate gradient.

1. Notation. We shall denote by R^n the n-dimensional Euclidean space, by $x = (x_1, x_2, \ldots, x_n)$ its elements, by $\|\cdot\|$ the norm, by $\langle\cdot,\cdot\rangle$ the scalar product in R^n, by R_+ and R_+^n the positive cones in R and R^n (respectively), and by $\{e_j\}$ the canonical base of R^n.

Moreover, for $x \in R^n$ and $\lambda \in R_+^n$ we shall denote by $[x, \lambda]$ the closed interval having x as a point with the smallest coordinates and $x+\lambda$ as a point with the greatest coordinates, and for $\lambda \in R_+^n$ we put

$$r(\lambda) = (\max_{1\leqslant j\leqslant n} \langle\lambda, e_j\rangle)^{-n} \prod_{j=1}^{n} \langle\lambda, e_j\rangle .$$

So for each $x \in R^n$ and $\lambda \in R_+^n$ the number $r(\lambda)$ is equal to the parameter of regularity of the interval $[x, \lambda]$.

For each $X \subset R^n$ and $s \in \{1, 2, \ldots, n\}$, $\mu_s(X)$ will denote the s-dimensional Lebesgue measure of X. For a real-valued function f defined on a set $X \subset R^n$, we shall denote by $\operatorname{grad} f(x)$ ($\operatorname{grad}_{\mathrm{ap}} f(x)$) the gradient (the approximate gradient) of f at x.

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For each interval $[x, \lambda]$ such that $x \in X$, $x+\langle\lambda, e_j\rangle e_j \in X$ for $j \in \{1, 2, \ldots, n\}$, and for every $p > 0$ we put

$$F_{p,j}(x) = |f(x+\langle\lambda, e_j\rangle e_j)-f(x)|^p \cdot |\langle\lambda, e_j\rangle|^{n-p}.$$

2. Lemmas.

LEMMA 1. *Let E be a μ_n-measurable subset of R^n and let $a \in (0,1)$. For almost every $x \in E$ there exists $c(x) > 0$ such that if $h \in (0, c(x))$ is a number fulfilling the condition $x+he_i \in E$ for some $i \in \{1, 2, \ldots, n\}$, then there exists an interval $[x, \lambda]$ having the parameter of regularity greater than a and such that*

$$x+\langle\lambda, e_j\rangle e_j \in E \text{ for every } j \in \{1, 2, \ldots, n\} \quad \text{and} \quad h = \max_{1\leqslant j\leqslant n}\langle\lambda, e_j\rangle.$$

Proof. Let $\tilde{E}$ be the subset of E consisting of the points x fulfilling the following condition: for every $j \in \{1, 2, \ldots, n\}$ the set

$$E(x, j) = \{t \in R_+:\ x+te_j \in E\}$$

is μ_1-measurable and has zero as a right-hand point of density. It is well known (see [3]) that $\mu_n(E-\tilde{E}) = 0$. If $x \in \tilde{E}$, then there exists $c(x) > 0$ such that for every $\gamma \in (0, c(x))$ and $j \in \{1, 2, \ldots, n\}$ we have

$$\mu_1\{t \in (0, \gamma):\ x+te_j \in E\} > (1-a^{1/(n-1)})\gamma.$$

Let $h \in (0, c(x))$ be such that $x+he_i \in E$ for some $i \in \{1, 2, \ldots, n\}$. Then we have

$$\{t \in (a^{1/(n-1)}h, h):\ x+te_j \in E\} \neq \emptyset \quad \text{for every } j \in \{1, 2, \ldots, n\}.$$

Let $\lambda_j \in (a^{1/(n-1)}h, h)$ $(j \neq i)$ be such that $x+\lambda_j e_j \in E$ and let λ be the vector having λ_j as the j-th coordinate and h as the i-th coordinate. It is easy to verify that the interval $[x, \lambda]$ satisfies the required condition.

Let $X \subset R^n$ be a Borel set and let $f\colon X \to R$ be a Borel function. For any real number $k \geqslant 0$ and natural number m we denote by $X_{m,k,f}$ the set of all points $x \in X$ fulfilling the following condition:

$$|f(x+he_j)-f(x)| \leqslant hk$$

for every $h \in (0, m^{-1})$ and for $j \in \{1, 2, \ldots, n\}$ such that $x+he_j \in X$.

LEMMA 2. *$X_{m,k,f}$ is a μ_n-measurable set.*

Proof. Observe that $X \times (0, m^{-1})$ is a Borel set in R^{n+1}. Put

$$B_{m,j} = \{(x, h) \in X \times (0, m^{-1}):\ x+he_j \in X\} \quad \text{for } j \in \{1, 2, \ldots, n\}.$$

It is easy to see that the function $\psi(x, h) = \varphi_X(x+he_j)$, where φ_X is the characteristic function of X, is a Borel function. $B_{m,j}$ is a Borel set, since

$$B_{m,j} = \{(x, h):\ \psi(x, h) = 1\}.$$

Also $h^{-1}|f(x+he_j)-f(x)|$ is a Borel function on $B_{m,j}$. Let $\Delta_{m,j,k}$ be the projection of the set

$$\{(x, h) \in B_{m,j}\colon h^{-1}|f(x+he_j)-f(x)| > k\}$$

onto R^n. $\Delta_{m,j,k}$ is a μ_n-measurable set (see [2]) and

$$X_{m,k,f} = X - \bigcup_{j=1}^{n} \Delta_{m,j,k}.$$

Let now $\chi\colon X \to R$ be a non-negative μ_n-measurable function. For any natural number m we denote by $X_{m,\chi,f}$ the set of all points $x \in X$ fulfilling the following condition:

$$|f(x+he_j)-f(x)| \leqslant h\chi(x)$$

for every $h \in (0, m^{-1})$ and for $j \in \{1, 2, \ldots, n\}$ such that $x+he_j \in X$.

LEMMA 3. *$X_{m,\chi,f}$ is a μ_n-measurable set.*

Proof. First step. Let χ be a simple function,

$$\chi(x) = \sum_p k_p \varphi_{X_p}(x)$$

(where φ_{X_p} is the characteristic function of X_p). Then we have

$$X_{m,\chi,f} = \bigcup_p (X_p \cap X_{m,k_p,f})$$

and we apply Lemma 2.

Second step. Let χ be a bounded function and let $\{g_s(x)\}$ be a decreasing sequence of simple functions pointwise convergent to $\chi(x)$. Then we have

$$X_{m,\chi,f} = \bigcap_{s\in N} X_{m,g_s,f}$$

and we apply the first step.

Third step. If χ is an unbounded function, then we put

$$\chi_s(x) = \min\big(\chi(x), s\big).$$

We have

$$X_{m,\chi,f} = \bigcup_{s\in N} X_{m,\chi_s,f}$$

and we apply the second step.

3. Existence theorems.

THEOREM 1. *If $X \subset R^n$ is a bounded Borel set and $f\colon X \to R$ is a Borel function, then the following conditions are equivalent:*

(i) grad$f$ *exists a.e. in X.*

(ii) *For almost every $x \in X$, for every $a \in (0,1)$ and for every $p \in R_+$ there exist two positive numbers $K_{a,p}(x)$ and $\eta(x)$ such that*

$$F_{p,j}([x,\lambda]) \leqslant K_{a,p}(x)\mu_n([x,\lambda]) \tag{1}$$

holds for all $j \in \{1,2,\ldots,n\}$ and for each $\lambda \in R_+^n$ satisfying $\|\lambda\| < \eta(x)$, $r(\lambda) > a$ and $x + \langle\lambda, e_j\rangle e_j \in X$ for every $j \in \{1,2,\ldots,n\}$.

(iii) *For every $\sigma > 0$, for every $p \in R_+$ and for every $a \in (0,1)$ there exist a μ_n-measurable set $X_\sigma \subset X$ and two positive numbers $M_{\sigma,p,a}$ and τ_σ such that*

$$\mu_n(X - X_\sigma) < \sigma \tag{2}$$

and

$$\sum_h F_{p,j}([x_h,\lambda_h]) \leqslant M_{\sigma,p,a} \tag{3}$$

holds for every $j \in \{1,2,\ldots,n\}$ and for each finite sequence $\{[x_h,\lambda_h]\}$ of pairwise disjoint intervals satisfying

$$r(\lambda_h) > a, \qquad \|\lambda_h\| < \tau_\sigma, \qquad x_h \in X_\sigma, \qquad x_h + \langle\lambda_h, e_j\rangle e_j \in X \tag{4}$$

for $j \in \{1,2,\ldots,n\}$.

Proof. (i) $\Rightarrow$ (ii). Let $x \in X$ be such a point that $\operatorname{grad} f(x)$ exists and is finite. Put

$$H(x) = \max(2\|\operatorname{grad} f(x)\|, 1), \qquad K_{a,p}(x) = \frac{1}{a}H^p(x).$$

Let $\eta(x) > 0$ be such that the inequality

$$|f(x + \langle\lambda, e_j\rangle e_j) - f(x)| \leqslant H(x)\langle\lambda, e_j\rangle \tag{5}$$

holds for every $\lambda \in R_+^n$ satisfying the conditions

$$\|\lambda\| < \eta(x), \qquad x + \langle\lambda, e_j\rangle e_j \in X \text{ for } j \in \{1,2,\ldots,n\}.$$

In virtue of (5) we have, for every $j \in \{1,2,\ldots,n\}$,

$$|f(x + \langle\lambda, e_j\rangle e_j) - f(x)|^p \, |\langle\lambda, e_j\rangle|^{n-p} \leqslant H^p(x)\,|\langle\lambda, e_j\rangle|^n.$$

Then, if $r(\lambda) > a$, we have

$$F_{p,j}([x,\lambda]) \leqslant H^p(x) \prod_{i=1}^n \langle\lambda, e_i\rangle a^{-1} = K_{a,p}(x)\mu_n([x,\lambda]).$$

(ii) $\Rightarrow$ (iii). Using Lemma 2 we have

$$\mu_n(X - X_{m,m,f}) \to 0 \qquad \text{for } m \to \infty.$$

Hence for $\sigma > 0$ there exists m_σ such that

$$\mu_n(X - X_{m_\sigma,m_\sigma,f}) < \sigma.$$

Put $X_\sigma = X_{m_\sigma, m_\sigma, f}$ and $\tau_\sigma = (m_\sigma)^{-1}$. Let I be an interval containing X and such that the distance between its boundary and X is greater than 1. Put

$$M_{\sigma,p,a} = \frac{1}{a}(m_\sigma)^p \mu_n(I).$$

Then for each finite sequence $\{[x_h, \lambda_h]\}$ of pairwise disjoint intervals satisfying (4) we have

$$\sum_h F_{p,f}([x_h, \lambda_h]) \leqslant (m_\sigma)^p \sum_h |\langle\lambda_h, e_j\rangle|^n \leqslant \frac{1}{a}(m_\sigma)^p \sum_h \mu_n([x_h, \lambda_h])$$
$$\leqslant \frac{1}{a}(m_\sigma)^p \mu_n(I) = M_{\sigma,p,a}.$$

(iii) $\Rightarrow$ (i). Let Y_j be the set of points in which f has an infinite right-hand upper derivative or infinite right-hand lower derivative with respect to x_j. Since f is a Borel function, Y_j is a μ_n-measurable set; moreover, in virtue of the Denjoy-Young-Saks theorem, the finite derivative f'_{x_j} exists a.e. in $X - Y_j$. Therefore, $\operatorname{grad} f$ exists and is finite a.e. in $X - \bigcup_{j=1}^{n} Y_j$. We shall prove, under the assumption of (iii), that $\mu_n(Y_j) = 0$ for every $j \in \{1, 2, \ldots, n\}$.

Suppose that there exists j such that

$$\mu_n(Y_j) > 0. \tag{6}$$

Let $0 < \sigma < \frac{1}{2}\mu_n(Y_j)$. Then in virtue of (2) we have

$$\mu_n(Y_j \cap X_\sigma) > 0. \tag{7}$$

Let $a \in (0, 1)$ be fixed. From Lemma 1 it follows that for almost every $x \in Y_j \cap X_\sigma$ there exists a sequence $\{[x, \lambda_\nu]\}$ of intervals such that

$$\begin{gathered} r(\lambda_\nu) > a, \quad x + \langle\lambda_\nu, e_i\rangle e_i \in X, \quad \|\lambda_\nu\| \to 0, \ \langle\lambda_\nu, e_i\rangle \leqslant \langle\lambda_\nu, e_j\rangle \\ \text{for } i \in \{1, 2, \ldots, n\}, \\ |f(x + \langle\lambda_\nu, e_j\rangle e_j) - f(x)| > (\lambda_\nu, e_j)\left(\frac{2M_{\sigma,p,a}}{\mu_n(Y_j \cap X_\sigma)}\right)^{1/p}. \end{gathered} \tag{8}$$

Then in virtue of Vitali's theorem there exists a finite family of pairwise disjoint intervals $\{[x_h, \lambda_h]\}$ of described type and such that

$$\sum_h \mu_n([x_h, \lambda_h]) > \frac{1}{2}\mu_n(Y_j \cap X_\sigma).$$

At last, from (8) it follows that

$$\sum_h F_{p,j}([x_h, \lambda_h]) > M_{\sigma,p,a},$$

which contradicts (3).

THEOREM 2. *Let E be a μ_n-measurable subset of R^n and let $f\colon E \to R$ be a μ_n-measurable function. The finite approximate gradient of f exists a.e. in E if and only if there exists a sequence $\{C_m\}$ of closed sets such that*

$$\mu_n\big(E - \bigcup_m C_m\big) = 0 \tag{9}$$

and $\operatorname{grad}(f|C_m)$ *exists and is finite on C_m for every natural m.*

This follows immediately from the theorem of Whitney (see [4]).

The following theorem is an immediate consequence of Theorems 1 and 2.

THEOREM 3. *Let E and f be as in Theorem 2. The finite approximate gradient of f exists a.e. in E if and only if the following condition is fulfilled:*

(iii)$_{\mathrm{ap}}$ *For every $\sigma > 0$, for every $p \in R_+$ and for every $a \in (0,1)$ there exist a μ_n-measurable set E_σ and two positive numbers $M_{\sigma,p,1}$ and τ_σ such that*

$$E_\sigma \subset E \quad \text{and} \quad \mu_n(E - E_\sigma) < \sigma$$

and inequality (3) *holds for every $j \in \{1, 2, \ldots, n\}$ and for each finite sequence $\{[x_h, \lambda_h]\}$ of pairwise disjoint intervals such that $r(\lambda_h) > a$, $\|\lambda_h\| < \tau_\sigma$, $x_h \in E_\sigma$ for each h and $x_h + \langle \lambda_h, e_j \rangle e_j \in E_\sigma$ for each j.*

4. Integrability theorems.

THEOREM 4. *Let X and f be as in Theorem 1. For each $p \in R_+$ the following conditions are equivalent:*

(i)′ $\operatorname{grad} f \in L^p(X)$.

(ii)′ *There exist a set $\tilde{X} \subset X$ with $\mu_n(X - \tilde{X}) = 0$ and a μ_n-measurable function $\sigma(x)\colon X \to R_+$ such that for every $a \in (0,1)$ there exists a positive constant $M_{a,p}$ such that*

$$\sum_h F_{p,j}([x_h, \lambda_h]) \leqslant M_{a,p}$$

holds for each $j \in \{1, 2, \ldots, n\}$ and for each finite sequence $\{[x_h, \lambda_h]\}$ of pairwise disjoint intervals satisfying $r(\lambda_h) > a$, $\|\lambda_h\| < \sigma(x_h)$, $x_h \in \tilde{X}$, $x_h + \langle \lambda_h, e_j \rangle e_j \in X$ for every h and every $j \in \{1, 2, \ldots, n\}$.

(iii)′ *Condition* (iii) *holds and the constant $M_{\sigma,p,a}$ is independent of σ.*

Proof. (i)′ $\Rightarrow$ (ii)′. Let I be an interval including X and such that the distance between its boundary and X is greater than 1. Put

$$M_{a,p} = \frac{1}{a}\Big[2^p \sum_{j=1}^{n} \int_X |f'_{x_j}(t)|^p \, d\mu_n(t) + n + \mu_n(I)\Big] \tag{10}$$

and consider a pairwise disjoint sequence of closed sets $\{C_m\}$ such that

$$\mu_n\big(X - \bigcup_m C_m\big) = 0$$

and $\operatorname{grad}(f|C_m)$ is continuous for each m. Let A_m be an open set containing C_m and satisfying

$$\mu_n(A_m - C_m) < 2^{-(m+2)}\Big/\Big(\sum_{i=1}^{n} \max_{x\in C_m} |f'_{x_i}|^p + 1\Big).$$

Since $\operatorname{grad} f$ is continuous on the compact set C_m, $|f'_{x_j}|^p$ is uniformly continuous on C_m for every $j \in \{1, 2, \ldots, n\}$. Let σ_m be a positive number smaller than the distance between C_m and the boundary of A_m and such that for every $x, y \in C_m$ with $\|x-y\| < \sigma_m$ we have

$$\big||f'_{x_j}(x)|^p - |f'_{x_j}(y)|^p\big| < 2^{-(m+2)}/\mu_n(A_m) \quad \text{for } j \in \{1, 2, \ldots, n\}.$$

If $\{[x_h, \lambda_h]\}$ is a finite sequence of non-overlapping intervals such that $x_h \in C_m$ and $\|\lambda_h\| < \sigma_m$ for each $j \in \{1, 2, \ldots, n\}$, then

$$\begin{aligned}
(11)\quad \sum_h |f'_{x_j}(x_h)|^p \mu_n([x_h, \lambda_h]) &\leqslant \sum_h \Big(\min_{x\in C_m \cap [x_h, \lambda_h]} |f'_{x_j}(x)|^p\Big)\mu_n([x_h, \lambda_h]) + 2^{-(m+2)} \\
&\leqslant \sum_h \Big(\min_{x\in C_m \cap [x_h, \lambda_h]} |f'_{x_j}(x)|^p\Big)\mu_n([x_h, \lambda_h] \cap C_m) + \\
&\quad + \max_{x\in C_m} |f'_{x_j}(x)|^p \mu_n(A_m - C_m) + 2^{-(m+2)} \\
&\leqslant \int_{C_m} |f'_{x_j}|^p + 2^{-(m+1)}.
\end{aligned}$$

The function

$$\chi(x) = \max\{2 \max_{1\leqslant j\leqslant n} |f'_{x_j}(x)|, 1\}$$

is μ_n-measurable, so, by Lemma 3, $X_{m,\chi,f}$ is a μ_n-measurable set and

$$\mu_n\big(X - \bigcup_{m'} X_{m',\chi,f}\big) = 0.$$

Put

$$\tilde{X} = \big(\bigcup_m C_m\big) \cap \big(\bigcup_{m'} X_{m',\chi,f}\big)$$

and let

$$\sigma(x) = \min\big\{\sigma_m, [\inf\{m' : x \in X_{m',\chi,f}\}]^{-1}\big\}, \qquad x \in C_m.$$

Fix $a \in (0, 1)$. Let $\{[x_h, \lambda_h]\}$ be a finite sequence of pairwise disjoint intervals such that $x_h \in \tilde{X}$, $\|\lambda_h\| < \sigma(x_h)$, and $r(\lambda_h) > a$ for each h and $x_h + \langle\lambda_h, e_j\rangle e_j \in X$ for each $j \in \{1, 2, \ldots, n\}$. From (10) and (11), after not difficult computations, we obtain

$$\sum_h F_{p,j}([x_h, \lambda_h]) \leqslant \frac{1}{a}\Big(\sum_h \mu_n([x_h, \lambda_h])\chi^p(x_h)\Big) \leqslant M_{a,p}.$$

(ii)′ $\Rightarrow$ (iii)′. Observe that for every $\sigma > 0$ there exists $m \in N$ such that

$$\mu_n\big(\{x \in \tilde{X}\colon\ \sigma(x) \leqslant m^{-1}\}\big) < \sigma.$$

Put

$$X_\sigma = \{x \in \tilde{X}\colon\ \sigma(x) > m^{-1}\}, \qquad \tau_\sigma = m^{-1}, \qquad M_{\sigma,p,a} = M_{a,p}.$$

(iii)′ $\Rightarrow$ (i)′. In virtue of Theorem 1, $\operatorname{grad} f$ exists and is finite a.e. in X. Using the method similar to that in [1] (proof of (γ) $\Rightarrow$ (α)) we infer that for every $\sigma > 0$ there exists a closed set $\Gamma_\sigma \subset X$ such that $\mu_n(X - \Gamma_\sigma) < \sigma$ and

$$\int_{\Gamma_\sigma} |f'_{x_j}(t)|^p d\mu_n(t) \leqslant M_{\sigma,p,a} + \sigma\left(2 + \frac{1}{a}(\operatorname{diam} X + \sigma)^n\right). \tag{12}$$

Since $M_{\sigma,p,a}$ is independent of σ, the required inequality follows when $\sigma \to 0$ in (12).

THEOREM 5. *Let E and f be as in Theorem 2. For each $p \in R_+$ the following conditions are equivalent*:

(i)″ $\operatorname{grad}_{\mathrm{ap}} f \in L^p(E)$.

(ii)″ *There exists an increasing sequence $\{C_m\}$ of closed sets satisfying* (9) *and such that, for each $m \in N$, f fulfils on C_m condition* (ii)′ *and the constant $M_{a,p}$ is independent of m.*

(iii)″ *There exists an increasing sequence $\{C_m\}$ of closed sets satisfying* (9) *and such that, for each $m \in N$, f fulfils on C_m condition* (iii)$_{\mathrm{ap}}$ *and the constant $M_{\sigma,p,a}$ is independent of σ and m.*

Proof. (i)″ $\Rightarrow$ (ii)″. From Theorem 2 it follows that there exists an increasing sequence $\{C_m\}$ of closed sets satisfying (9) and such that $f|C_m$ has a finite gradient on C_m. Obviously,

$$\operatorname{grad}(f|C_m)(x) = \operatorname{grad}_{\mathrm{ap}} f(x) \quad \text{for almost every } x \in C_m.$$

Now it suffices to apply Theorem 4.

(ii)″ $\Rightarrow$ (iii)″. For every $m \in N$ and for every $\sigma > 0$ put $(C_m)_\sigma = \tilde{C}_m$.

(iii)″ $\Rightarrow$ (i)″. Using Theorem 4, for every $\sigma \in R_+$, for every $a \in (0, 1)$, and for every $m \in N$ we have

$$\int_{(C_m)_\sigma} |(f|C_m)'_{x_j}(t)|^p d\mu_n(t) \leqslant M_{\sigma,p,a} \quad \text{for } j \in \{1, 2, \ldots, n\}.$$

But $M_{\sigma,p,a}$ is independent of σ. Hence

$$\int_{C_m} |(f|C_m)'_{x_j}(t)|^p d\mu_n(t) \leqslant M_{\sigma,p,a}$$

for every $j \in \{1, 2, \ldots, n\}$, for every $a \in (0, 1)$ and for every $m \in N$. Moreover, since $\mathrm{grad}(f|C_m)(x) = \mathrm{grad}_{\mathrm{ap}} f(x)$ for almost every $x \in C_m$, the conclusion follows from (9) in virtue of the independence of $M_{\sigma,p,a}$ from m.

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