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DIFFERENTIAL MODULES

BY

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In my paper [2] I have shown that a large part of the classical differential geometry is a part of linear algebra, more precisely — of the theory of modules (see also [1]). Many notions (e.g., the notion of a covariant derivative, curvature tensor, torsion tensor, etc.) can be formulated and many theorems can be proved without introducing the notion of a differentiable manifold. Thus the algebraic notion of a module seems to be an adequate tool to investigate many fundamental notions in differential geometry. The algebraic notion of a module is a common generalization of various modules appearing in differential geometry, modules of vector fields, modules of tensor fields, etc.

Of course, the algebraic notion of a module is too general to enable the formulation of all geometric notions and theorems. In this paper I define a notion, called *differential module*, which is less general than the algebraic notion of a module but is also a common generalization of modules of vector fields or tensor fields appearing in differential geometry. Elements of a differential module are functions whose domain is a differentiable manifold (or, more generally, a differential space) and whose values belong to linear spaces. A fundamental property of a differential module is that it has locally a finite basis.

The definition of a differential module is given in Section 2 (p. 52). Section 3 treats of modules of tangent vector fields which yield the simplest examples of differential modules. The main aim of Sections 2, 4, 5 is to give three methods of construction of new differential modules from the given ones (Theorems 2.8, 4.7 and 5.3). The first method (restriction of a module to a subset) is a particular case of the second one (modules induced by mappings) but it is much simpler and, therefore, it is discussed separately. Theorem 6.1 shows that the second method is commutative with the third one (modules of tensor fields). In Section 7 to any differential module $\mathscr{W}$ there is associated a fibre space Q such that elements of $\mathscr{W}$ are identical, roughly speaking, with cross sections in Q.

Sections 8-11 are devoted to study covariant derivatives in differential modules. The main theorems are 9.3, 10.2 and 11.2 which assert that each of the three methods of construction of new differential modules from the given ones permits also to introduce, in a natural way, a covariant derivative in the constructed module, which is uniquely determined by the given covariant derivatives in the given differential modules. Theorem 11.3 is a supplement to Theorem 6.1 on commutativity of the three constructions of modules.

The notion of a differential module and the notion of a differential space of finite dimension, introduced in Section 3, seem to be convenient tools in differential geometry. The notion of a differential space of finite dimension is much more general than that of a differentiable manifold. It enables to generalize a big part of differential geometry to the case of spaces more general than differentiable manifolds, in particular to differentiable manifolds with boundary. For a systematic course in differential geometry, based on the notions of a differential module and differential space of finite dimension see my book [3], to appear.

1. Preliminary definitions. Let M be a topological space and let C be a set of functions (with arbitrary values) defined on M. A function f defined on a set $A \subset M$ is said to be a *local C-function* provided for every $p \in A$ there exist a neighbourhood B of p in the subspace A, and a function $g \in C$ such that $f|B = g|B$. In other words, a function f defined on a set $A \subset M$ is a local C-function provided there exists an open covering $\mathscr{B}$ of the space A, such that for every set $B \in \mathscr{B}$ there exists a function $g_B \in C$ with $f|B = g_B|B$. The set of all local C-functions defined on a fixed set $A \subset M$ will be denoted by C_A.

1.1. *If $A \subset B \subset M$, then $(C_B)_A = C_A$. In particular, $(C_A)_A = C_A$.*

1.2. *If $\mathscr{A}$ is an open covering of M, f is a function defined on M, and $f|A \in C_A$ for every $A \in \mathscr{A}$, then $f \in C_M$.*

A set C of functions defined on a topological space M is said to be *closed with respect to localization* if $C_M = C$. For instance, for every $A \subset M$ the set C_A is closed with respect to localization (see the second part of 1.1).

Let A be an open subset of a topological space M and let $p \in A$. A continuous real function β defined on M is said to *separate* the point p in the set A provided

(1) $$\beta|B = 1 \quad \text{for a neighbourhood } B \text{ of } p$$

and

(2) $$\beta|A_0 = 0 \quad \text{for an open set } A_0 \text{ such that } A \cup A_0 = M.$$

Hence it follows that $B \subset A$.

Let $\mathscr{C}$ be a set of continuous real functions defined on a topological space M. The space M is said to be *$\mathscr{C}$-regular* provided for every open set $A \subset M$ and for every point $p \in A$ there exists a function $\beta \in \mathscr{C}$ such that β separates p in A.

Let $\mathscr{C}$ be a non-empty set of real functions defined on a set M. Consider M as a topological space with the weakest topology such that all functions in $\mathscr{C}$ are continuous. The set $\mathscr{C}$ is said to be a *differential structure on M* provided that

(a) the set $\mathscr{C}$ is closed with respect to localization, i.e., $\mathscr{C} = \mathscr{C}_M$;

(b) the set $\mathscr{C}$ is *closed with respect to composition with smooth functions*, i.e., if $\varphi_1, \ldots, \varphi_n \in \mathscr{C}$ and ω is a smooth real function (= infinitely differentiable function) defined on the n-dimensional Euclidean space E^n, then the composition

$$\omega(\varphi_1(p), \ldots, \varphi_n(p)) \quad \text{for } p \in M \tag{3}$$

is a function in $\mathscr{C}$.

1.3. *If $\mathscr{C}$ is a differential structure on a set M, then $\mathscr{C}$ is a linear algebraic ring over the field E of all real numbers. $\mathscr{C}$ contains all constant real functions on M. The topological space M is $\mathscr{C}$-regular. The composition* (3) *belongs to $\mathscr{C}$ for any $\varphi_1, \ldots, \varphi_n \in \mathscr{C}$ and for every smooth function ω defined on an open set $O \subset E^n$ such that* (3) *is defined for all $p \in M$.*

1.4. *If $\mathscr{C}$ is a differential structure on M and $A \subset M$, then $\mathscr{C}_A$ is a differential structure on A.*

By a *differential space* we shall mean any pair $(M, \mathscr{C})$, where M is a set and $\mathscr{C}$ is a differential structure on M; the set M is then considered as a topological space with the weakest topology such that all real functions in $\mathscr{C}$ are continuous. In the sequel we shall often say "a differential space M" instead of "a differential space $(M, \mathscr{C})$".

If $(M, \mathscr{C})$ is a differential space and $A \subset M$, then $(A, \mathscr{C}_A)$ is also a differential space called a *differential subspace of* $(M, \mathscr{C})$. We shall often say that A is a differential subspace of M. Observe that the differential subspace A considered as a topological space is then a topological subspace of the topological space M.

If $(M, \mathscr{C})$ is a differential space, then all functions in $\mathscr{C}$ are often called *smooth functions on M* because they are an abstract analogue of smooth (= infinitely differentiable) real functions defined on subsets of Euclidean spaces.

The letter E will stand for the set of all real numbers.

Let $(M, \mathscr{C})$ be a differential space. By a *vector tangent to $(M, \mathscr{C})$ at a point $p \in M$* we shall mean any linear mapping $v\colon \mathscr{C} \to E$ such that

$$v(\alpha, \beta) = v(\alpha) \cdot \beta(p) + \alpha(p) \cdot v(\beta) \quad \text{for all } \alpha, \beta \in \mathscr{C}. \tag{4}$$

The real number $v(a)$ is called *the directional derivative of the function* $a \in \mathscr{C}$ *in the direction* v and is often denoted by the symbol $\partial_v a$. Thus (4) can be written as follows

$$(4') \qquad \partial_v(a \cdot \beta) = \partial_v a \cdot \beta(p) + a(p) \cdot \partial_v \beta \quad \text{for all } a, \beta \in \mathscr{C}.$$

The set M_p of all tangent vectors at a given point $p \in M$ is a linear space.

1.4. *If* $(M, \mathscr{C})$ *is a differential space,* $a \in \mathscr{C}$ *and* $a|A = 0$ *for a neighbourhood* A *of a point* $p \in M$*, then* $\partial_v a = 0$ *for every* $v \in M_p$.

Consequently, if functions $a, \beta \in \mathscr{C}$ *are equal on a neighbourhood of a point* $p \in M$*, then* $\partial_v a = \partial_v \beta$ *for every* $v \in M_p$.

Let $(A, \mathscr{C}_A)$ be a differential subspace of a differential space $(M, \mathscr{C})$ and let $p \in A$. If $v \in A_p$, i.e. if v is a vector tangent to A at p, then the formula

$$\bar{v}(a) = v(a|A) \quad \text{for } a \in \mathscr{C}$$

defines a vector $\bar{v} \in M_p$. The mapping which assigns $\bar{v} \in M_p$ to any $v \in A_p$ is a linear monomorphism of the linear space A_p into M_p. We shall identify v with $\bar{v}$. After this identification the following proposition is true:

1.5. *The set* A_p *of all vectors tangent at* $p \in A$ *to a subspace* A *of a differential space* M *is a linear subspace of the linear space* M_p *of all vectors tangent at* p *to* M.

If A *is an open subset of* M*, then* $A_p = M_p$ *for every* $p \in A$.

Let $(M, \mathscr{C})$ and $(N, \mathscr{D})$ be differential spaces. A mapping $f\colon M \to N$ is said to be *smooth* provided $a \circ f \in \mathscr{C}$ for every $a \in \mathscr{D}$. If $f\colon M \to N$ is smooth and $v \in M_p$, then the formula

$$w(a) = v(a \circ f) \quad \text{for } a \in \mathscr{D}$$

defines a vector w tangent to N at $f(p)$. The vector w will be denoted by $\mathbf{d}f(v)$. The mapping

$$\mathbf{d}f\colon \bigcup_{p \in M} M_p \to \bigcup_{q \in N} N_q$$

which assigns the vector $\mathbf{d}f(v)$ to any vector v tangent to M is called the *differential of the mapping* f. Note that the differential of a composition of smooth mappings is the composition of the differentials of the mappings.

Every smooth mapping is continuous.

Let $(M, \mathscr{C})$ and $(N, \mathscr{D})$ be differential spaces. A one-to-one mapping f from M onto N is said to be a *diffeomorphism* provided both mappings $f\colon M \to N$ and $f^{-1}\colon N \to M$ are smooth. Then $(M, \mathscr{C})$ and $(N, \mathscr{D})$ are said to be *diffeomorphic*.

A differential space $(M, \mathscr{C})$ is said to be an *m-dimensional differential manifold* provided every point $p \in M$ has a neighbourhood A such that A

is diffeomorphic with an open subset A of the m-dimensional Euclidean space E^m. More precisely, $(M, \mathscr{C})$ is an m-dimensional differential manifold if for every $p \in M$ there exist a neighbourhood A of p, an open set $O \subset E^m$, and a diffeomorphism f from $(O, \mathscr{E}_O)$ onto $(A, \mathscr{C}_A)$. The symbol $\mathscr{E}$ denotes here the set of all smooth (= infinitely differentiable) functions on E^m and, consequently, $\mathscr{E}_O$ denotes the set of all smooth functions on the open set $O \subset E^m$.

By an *algebraic ring* we mean in this paper a commutative linear (over the field E of real numbers) ring with the unit element 1. If C is an algebraic ring, then a *C-module* (or simply a *module*) is a set $\mathscr{W}$ of elements $V, W, U, \ldots$, on which there are defined addition $V+W \in \mathscr{W}$ $(V, W \in \mathscr{W})$ and multiplication $aV \in \mathscr{W}$ $(a \in C, V \in \mathscr{W})$ satisfying the well-known axioms

$$V+W = W+V, \quad (V+W)+U = V+(W+U),$$
$$\text{if } U+V = U+W, \quad \text{then } V = W,$$
$$a(V+W) = aV+aW, \quad (a+\beta)V = aV+\beta V,$$
$$a\cdot(\beta V) = (a\cdot\beta)\cdot V, \quad 1V = V.$$

A finite sequence $W_1, \ldots, W_n$ of elements of a C-module $\mathscr{W}$ is said to be a *C-basis* (or simply a *basis*) of $\mathscr{W}$ provided every element $W \in \mathscr{W}$ can be uniquely represented as a linear combination $W = a^i W_i$ with $a_i \in C$. The usual summation convention used here will be also used systematically in the sequel.

If C is the set E of real numbers, then the notion of an E-module coincides with the notion of a linear space over E, and the notion of an E-basis coincides with the usual notion of a basis of a linear space.

It follows from the definition that the set E of all real numbers is a subring of every algebraic ring C (identify any real number a with $a \cdot 1 \in C$, where 1 is the unit element in C). Hence it follows that every C-module $\mathscr{W}$ can be also interpreted as an E-module, i.e. as a linear space over the field E of all real numbers.

Let $\mathscr{V}$ and $\mathscr{W}$ be C-modules, C being an algebraic ring. A mapping L: $V \to W$ is said to be *C-linear* if it is *additive*, i.e.

$$L(V_1+V_2) = L(V_1)+L(V_2) \quad \text{for } V_1, V_2 \in \mathscr{V},$$

and *C-homogeneous*, i.e.,

$$L(aV) = aL(V) \quad \text{for } a \in C \text{ and } V \in \mathscr{V}.$$

Interpreting $\mathscr{V}$ and $\mathscr{W}$ as E-modules, we say that a mapping L: $\mathscr{V} \to \mathscr{W}$ is *E-linear* provided that it is additive and *E-homogeneous*, i.e.,

$$L(aV) = aL(V) \quad \text{for } a \in E \text{ and } V \in \mathscr{V}.$$

Suppose that $\mathscr{W}_1, \ldots, \mathscr{W}_n, \mathscr{W}$ are C-modules. A mapping

(5) $$L: \mathscr{W}_1 \times \ldots \times \mathscr{W}_n \to \mathscr{W}$$

is said to be *C-n-linear* (or simply *n-linear*) if the expression $L(W_1, \ldots, W_n)$ is a C-linear function of each of the variables $W_1, \ldots, W_n$, the remaining variables being fixed. C-n-linear mappings (5) will often be called *C-tensors* (or simply *tensors*). Replacing C by E we get the notion of an *E-n-linear* mapping (5) and *E-tensor*. The set of all C-n-linear tensors (5) will be denoted by $\mathfrak{L}_C(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W})$. Consequently, $\mathfrak{L}_E(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W})$ stands for the set of all E-n-linear tensors (5). Both $\mathfrak{L}_C(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W})$ and $\mathfrak{L}_E(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W})$ are C-modules with the obvious definitions of addition and multiplication.

The simplest example of a C-module is the ring C itself.

By a *linear space* we always mean, in this paper, a linear space over the field E of real numbers.

2. Definition of a differential module. Let $(M, \mathscr{C})$ be a differential space and let Φ be a mapping which assigns a linear space $\Phi(p)$ to any point $p \in M$. By a *Φ-field* on M (simply a *field* on M) we shall mean any function W which assigns an element $W(p) \in \Phi(p)$ to any $p \in M$. Sometimes we shall say *linear field* instead of "field" to emphasize that the values of W are elements of linear spaces.

As a rule we will not be interested in examining all Φ-fields on M but only a $\mathscr{C}$-module of certain Φ-fields which should be considered to be smooth. We shall examine mainly these modules W which are closed with respect to localization, i.e. such that

$$\mathscr{W} = \mathscr{W}_M \tag{1}$$

(see p. 46). This hypothesis is a precise formulation of the fact suggested by the intuition that the property "to be smooth" has a local character.

There is no general definition of smooth linear Φ-fields in the case of an arbitrary Φ. However, such a definition is possible for many special Φ's. Moreover, we can investigate some operations on modules of linear fields, whose result is also a module of linear fields, such that when these operations are applied to modules of smooth fields, the result is also a module of smooth fields, or — more precisely — a module of linear fields which our intuition suggests to consider as smooth fields. Constructions of this type will be the subject of Sections 4 and 5.

The following theorem yields also an example of such a construction (the *restriction* to a subset).

2.1. *If $\mathscr{W}$ is a $\mathscr{C}$-module of Φ-fields on a differential space $(M, \mathscr{C})$ and $A \subset M$, then $\mathscr{W}_A$ is a $\mathscr{C}_A$-module of $(\Phi|A)$-fields on A, closed with respect to the localization, i.e. such that*

$$(\mathscr{W}_A)_A = \mathscr{W}_A. \tag{2}$$

Property (2) is a particular part of the second part of 1.1. The examination that $\mathscr{W}_A$ is a $\mathscr{C}_A$-module is left to the reader.

2.2. *For any set $\mathscr{W}_0$ of Φ-fields on $(M, \mathscr{C})$, the set $\mathscr{V}$ of all linear combinations $V = a^i W_i$, where $a^i \in \mathscr{C}$ and $W_i \in \mathscr{W}_0$, is the smallest $\mathscr{C}$-module of Φ-fields which contains $\mathscr{W}_0$ as a subset. The set $\mathscr{V}_M$ is the smallest $\mathscr{C}$-module containing $\mathscr{W}_0$ and closed with respect to localization.*

The first part is obvious. The second part follows from the first one and 2.1 (where $A = M$).

A finite sequence

(3) $$W_1, \ldots, W_m$$

is said to be a *vector basis* of a $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields provided that

(a) for every point $p \in M$ the sequence $W_1(p), \ldots, W_m(p)$ is a basis of the linear space $\Phi(p)$;

(b) (3) is a $\mathscr{C}$-basis of the $\mathscr{C}$-module $\mathscr{W}$.

2.3. *In order that* (3) *be a vector basis of a $\mathscr{C}$-module $\mathscr{W}$ it suffices that* (a) *and the following condition* (b′) *hold:*

(b′) *every $W \in \mathscr{W}$ is a linear combination of the elements* (3) *with coefficients in $\mathscr{C}$.*

Every vector basis of a $\mathscr{C}$-module $\mathscr{W}$ is a $\mathscr{C}$-basis of $\mathscr{W}$ by (b). The converse statement is not true, in general. A $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields can have a $\mathscr{C}$-basis without having any vector basis. On the other hand,

2.4. *If a $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields has a vector basis, then every $\mathscr{C}$-basis of $\mathscr{W}$ is a vector basis of $\mathscr{W}$.*

2.5. *If* (3) *is a $\mathscr{C}$-basis of a $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields, if* $\dim \Phi(p) = m$ *for every $p \in M$, and if for every $p \in M$ and $w \in \Phi(p)$ there exists a $W \in \mathscr{W}$ such that $W(p) = w$, then* (3) *is a vector basis of $\mathscr{W}$.*

The easy proofs of 2.4 and 2.5 are left to the reader.

2.6. *If a $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields on M has a vector basis* (3), *then $\mathscr{W}$ is closed with respect to localization, i.e.* (1) *holds.*

Let (3) be a vector basis of $\mathscr{W}$. It follows from (a) that for any Φ-field W on M there exist real functions $a^1, \ldots, a^m$ such that $W = a^i W_i$. If $W \in \mathscr{W}_M$, then these real functions are in $\mathscr{C}$, and, consequently, $W \in \mathscr{W}$ which proves (1). In fact, for every point $p \in M$ there exist a neighbourhood A of p and a field $V \in \mathscr{W}$ such that $V|A = W|A$. By (b), $V = \beta^i W_i$ for certain functions $\beta^i \in \mathscr{C}$. Since $V|A = W|A$, it follows from (a) that $a^i|A = \beta^i|A \in \mathscr{C}_A$. This implies, by 1.2, that $a^i \in \mathscr{C}_M = \mathscr{C}$.

Let A be a subset of the differential space M. If a sequence $W_1, \ldots, W_m \in \mathscr{W}_A$ is a vector basis of the $\mathscr{C}_A$-module $\mathscr{W}_A$, we also say that $W_1, \ldots, W_m$ is a *vector basis of $\mathscr{W}$ on A*. In other words, a sequence $W_1, \ldots, W_m$ is a *vector basis on A of a $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields* provided that

(a) for every $p \in A$, the sequence $W_1(p), \ldots, W_m(p)$ is a basis of the linear space $\Phi(p)$;

(b) the sequence $W_1, \ldots, W_m$ is a $\mathscr{C}_A$-basis of the $\mathscr{C}_A$-module $\mathscr{W}_A$.

2.7. *If* (3) *is a vector basis on* A *of a* $\mathscr{C}$*-module* $\mathscr{W}$ *of* Φ*-fields and if* $B \subset A$*, then*

$$W_1|B, \ldots, W_m|B \tag{4}$$

is a vector basis of $\mathscr{W}$ *on* B.

The property (a) of the sequence (4) follows directly from the property (a) of (3). It suffices to prove that (4) has the property (b). Let $V \in \mathscr{W}_B$. By (a), there exists a unique sequence of real functions $\beta^1, \ldots, \beta^m$ on M such that $V = \beta^i W_i|B$. We shall prove that the functions β^i are in $\mathscr{C}_B$. In fact, it follows from the identity $(\mathscr{W}_A)_B = \mathscr{W}_B$ (see 1.1) that for any point $p \in B$ there exist a neighbourhood B' of p in the space B and a linear field $W \in \mathscr{W}_A$ such that $V|B' = W|B'$. By (b), $W = a^i W_i$ for certain functions $a^i \in \mathscr{C}_A$. Hence it follows, by (a), that $a^i|B' = \beta^i|B'$ which proves that $\beta^i \in (\mathscr{C}_A)_B = \mathscr{C}_B$ (see 1.1).

A $\mathscr{C}$-module $\mathscr{W}$ of Φ-fields on a differential space $(M, \mathscr{C})$ is said to be a *differential module* (or a *module of finite dimension*) provided that $\mathscr{W}$ is closed with respect to localization (i.e., (1) holds) and $\mathscr{W}$ ***has locally a vector basis*** composed of m fields, i.e., if every point $p \in M$ has a neighbourhood A such that there exists a vector basis $W_1, \ldots, W_m$ of $\mathscr{W}$ on A. The number m is then uniquely determined, since it is the linear dimension of all the linear spaces $\Phi(p)$, $p \in M$. The number m is called the *dimension of* $\mathscr{W}$ and is denoted by $\dim \mathscr{W}$.

2.8. *If* $\mathscr{W}$ *is a differential module of* Φ*-fields on* $(M, \mathscr{C})$ *and* $0 \neq A \subset M$, *then* $\mathscr{W}_A$ *is a differential module on* $(A, \mathscr{C}_A)$ *and* $\dim \mathscr{W} = \dim \mathscr{W}_A$.

This follows from 2.1 and 2.7.

Theorem 2.8 states that the operation of restriction to a subset yields a differential module if it is applied to a differential module. In Sections 4 and 5 we shall introduce other operations which yield differential modules when they are applied to differential modules.

Note the following supplement to 2.5:

2.9. *If* $\mathscr{W}$ *is a differential* $\mathscr{C}$*-module of* Φ*-fields on* $(M, \mathscr{C})$*, then for every* $p \in M$ *and for every* $w \in \Phi(p)$ *there exists a* $W \in \mathscr{W}$ *such that* $w = W(p)$.

Suppose that (3) is a vector basis of $\mathscr{W}$ on a neighbourhood A of p. We have $w = a^i W_i(p)$ for certain real numbers a^i. Since $a^i W_i \in \mathscr{W}_A$, there exists a $W \in \mathscr{W}$ such that $W|B = a^i W_i|B$ for a neighbourhood B of p, $B \subset A$. Hence it follows that $W(p) = w$.

3. Vector fields. Let $(M, \mathscr{C})$ and $(N, \mathscr{D})$ be differential spaces and let $f\colon M \to N$.

By a *vector f-field on* M, *tangent to* N, we shall mean any mapping V which assigns a vector $V(p) \in N_{f(p)}$ to any $p \in M$. By the definition,

$$V\colon M \to \bigcup_{q \in N} N_q. \tag{1}$$

In the case where M is a differential subspace of N and f is the identity mapping, instead of "vector f-field on M, tangent to N", we say *vector field on M, tangent to N*. In the case where $M = N$, $\mathscr{C} = \mathscr{D}$, and f is the identity mapping, V is called a *tangent vector field on M*. By the definition, a vector field on M, tangent to N ($M \subset N$), is a function V which assigns a vector $V(p) \in N_p$ to any $p \in M$. A tangent vector field on M is a function V which assings a vector $V(p) \in M_p$ to any $p \in M$.

Any vector f-field on M, tangent to N, is a Φ-field in the sense defined in Section 2, p. 50, Φ being the function

$$\Phi(p) = N_{f(p)} \quad \text{for } p \in M.$$

If V is a vector f-field on M, tangent to N, and if $a \in \mathscr{D}$, then the symbol $\partial_V a$ will denote the real function defined on M by

(2)
$$\partial_V a(p) = V(p)(a) = \partial_{V(p)} a \quad \text{for } p \in M.$$

The function $\partial_V a$ is the *directional derivative of a with respect to V*. The symbol ∂_V will denote the mapping which assigns the function $\partial_V a$ to every $a \in \mathscr{D}$.

A vector f-field V is said to be *smooth* if f is smooth and $\partial_V \colon \mathscr{D} \to \mathscr{C}$, i.e., if $\partial_V a \in \mathscr{C}$ for every $a \in \mathscr{D}$. If V is a smooth vector field on M, tangent to N, then $\partial_V \colon \mathscr{D} \to \mathscr{C}$ is an E-linear mapping satisfying the following formula on the derivation of the product of two functions

(3)
$$\partial_V a\beta = \partial_V a \cdot (\beta \circ f) + (a \circ f) \cdot \partial_V \beta \quad \text{for } a, \beta \in \mathscr{D}.$$

In the case where $M \subset N$ and f is the identity mapping, formula (3) has the from

(3')
$$\partial_V a\beta = \partial_V a \cdot \beta + a \cdot \partial_V \beta.$$

Conversely, if $f \colon M \to N$ is smooth, then every E-linear mapping from $\mathscr{D}$ into $\mathscr{C}$, satisfying (3), is of the form ∂_V for exactly one smooth vector f-field V. We shall often identify V with ∂_V. We see that after this identification the notion of a smooth vector f-field has two interpretations. In the *pointwise interpretation* it is a mapping of the form (1) such that $\partial_V \colon \mathscr{D} \to \mathscr{C}$. In the *global interpretation* it is an E-linear mapping from $\mathscr{D}$ into $\mathscr{C}$ satisfying (3). Both interpretations are useful.

For any smooth mapping $f \colon M \to N$ the symbol $\mathfrak{W}_f(M, N)$ will denote the set of all smooth vector f-fields on M, tangent to N. It is easy to verify that $\mathfrak{W}_f(M, N)$ is a C-module, because the expression $\partial_V a$ is a $\mathscr{C}$-linear function of V. In the case where $M \subset N$ and f is the identity mapping, we write $\mathfrak{W}(M, N)$ to denote the set (the $\mathscr{C}$-module) of all vector fields on M tangent to N. If $M = N$, $\mathscr{C} = \mathscr{D}$ and f is the identity mapping, we write $\mathfrak{W}(M)$ to denote the set (the $\mathscr{C}$-module) of all tangent vector fields on M.

3.1. *If $V \in \mathfrak{W}_f(M, N)$ and g is a smooth mapping of a differential space $(M', \mathscr{C}')$, then $V \circ g \in \mathfrak{W}_{f\circ g}(M', N)$.*

It is obvious that $V \circ g$ is a vector $f \circ g$-field. The following identity is true for every $a \in \mathscr{D}$

$$\partial_{V\circ g}\, a = (\partial_V a) \circ g,$$

because for every point $p \in M'$,

$$\partial_{V\circ g} a(p) = \partial_{V\circ g(p)} a = \partial_{V(g(p))} a = \partial_V a\big(g(p)\big) = (\partial_V a) \circ g(p).$$

Since $\partial_V a \in \mathscr{C}$, we have $\partial_{V\circ g} a \in \mathscr{C}'$ for every $a \in \mathscr{D}$. This proves that $V \circ g$ is smooth and completes the proof of 3.1.

Assuming g to be the indentity mapping on a subset of M, we get the following corollary.

3.2. *If $V \in \mathfrak{W}_f(M, N)$ and $A \subset M$, then $V|A \in \mathfrak{W}_{f|A}(A, N)$. If $V \in \mathfrak{W}(M, N)$ and $A \subset M \subset N$, then $V|A \in \mathfrak{W}(A, N)$. If $V \in \mathfrak{W}(M)$ and $A \subset M$, then $V|A \in \mathfrak{W}(A, M)$; if, in addition, the set A is open, then $V|A \in \mathfrak{W}(A)$.*

The last two sentences follow from the first, if f is the identity mapping on M.

3.3. *If f is a smooth mapping from $(M, \mathscr{C})$ into $(N, \mathscr{D})$, V is a vector f-field on M, tangent to N, $\mathscr{A}$ is an open covering of M, and, for every $A \in \mathscr{A}$, the vector $f|A$-field $V|A$ is smooth, then V is smooth.*

The following identity is true for every $a \in \mathscr{D}$

$$(\partial_V a)|A = \partial_{V|A}\, a.$$

Hence it follows that $\partial_V a|A \in \mathscr{C}_A$ for every $A \in \mathscr{A}$. This implies, by 1.2, that $\partial_V a \in \mathscr{C}_M = \mathscr{C}$. Thus V is smooth.

3.4. *If $V \in \mathfrak{W}_f(M, N)$ and g is a smooth mapping from $(N, \mathscr{D})$ into a differential space $(N', \mathscr{D}')$, then $\mathbf{d}g \circ V \in \mathfrak{W}_{g\circ f}(M, N')$.*

The symbol $\mathbf{d}g \circ V$ denotes here, obviously, the composition of the mapping $V\colon M \to \bigcup_{q\in N} N_q$ and of the mapping $\mathbf{d}g\colon \bigcup_{q\in N} N_q \to \bigcup_{q\in N'} N'_q$. The following identity is true for every $a \in \mathscr{D}'$

$$\partial_{\mathbf{d}g\circ V}\, a = \partial_V (a \circ g),$$

because for every $p \in M$,

$$\partial_{\mathbf{d}g\circ V} a(p) = \partial_{\mathbf{d}g(V(p))} a = \partial_{V(p)}(a \circ g) = \partial_V (a \circ g)(p).$$

It follows from this identity that $\partial_{\mathbf{d}g\circ V} a \in \mathscr{C}$ for every $a \in \mathscr{D}'$ which proves that the vector $g \circ f$-field $\mathbf{d}g \circ V$ is smooth.

3.5. *If A is an open subset of M, then*

$$\mathfrak{W}_{f|A}(A, N) = \big(\mathfrak{W}_f(M, N)\big)_A.$$

In particular,

$$\mathfrak{W}_f(M, N) = \big(\mathfrak{W}_f(M, N)\big)_M,$$

i.e., $\mathscr{C}$-module $\mathfrak{W}_f(M, N)$ is closed with respect to localization.

If $V \in (\mathfrak{W}_f(M, N))_A$, then there exists an open covering $\mathscr{A}$ of A such that for every $B \in \mathscr{A}$ there exists a $V_B \in \mathfrak{W}_f(M, N)$ with $V|B = V_B|B$. Thus $V|B$ is smooth for every $B \in \mathscr{A}$. By 3.3 (where M should be replaced by A) V is smooth, i.e. $V \in \mathfrak{W}_{f|A}(A, N)$.

On the other hand, if $W \in \mathfrak{W}_{f|A}(A, N)$ and $p \in A$, let β be a real continuous function which separates p in A, that is,

$$\beta|B = 1 \quad \text{for a neighbourhood } B \text{ of } p,$$

$$\beta|A' = 0 \quad \text{for an open set } A' \text{ such that } A \cup A' = M.$$

The vector f-field V defined by

$$V(q) = \begin{cases} \beta(q)W(q) & \text{for } q \in A, \\ 0 & \text{for } q \in A', \end{cases}$$

is smooth by 3.3 since the open sets A and A' form an open covering of M, and the $f|A$-field $V|A$ and the $f|A'$-field $V|A'$ are both smooth. Moreover, $V|B = W|B$. The point $p \in A$ being arbitrary, we infer that $W \in (\mathfrak{W}_f(M, N))_A$. This completes the proof of the first part of 3.5. The second part follows directly from the first one.

We say that vector f-fields $W_1, \ldots, W_n \in \mathfrak{W}_f(M, N)$ are *linearly independent* if, for every $p \in M$, the vectors $W_1(p), \ldots, W_n(p) \in N_{f(p)}$ are linearly independent in the linear space $N_{f(p)}$.

3.6. *If $W_1, \ldots, W_n \in \mathfrak{W}_f(M, N)$ are linearly independent and $W = \beta^i W_i$ for certain real functions β^i on M, then $W \in \mathfrak{W}_f(M, N)$ if and only if $\beta^i \in \mathscr{C}$ for $i = 1, \ldots, n$.*

If $\beta^i \in \mathscr{C}$, then $W \in \mathfrak{W}_f(M, N)$, because $\mathfrak{W}_f(M, N)$, is a $\mathscr{C}$-module.

Suppose that $W \in \mathfrak{W}_f(M, N)$, i.e., W is smooth. Let $p \in M$. Since the vectors $W_1(p), \ldots, W_n(p)$ are linearly independent, there exist functions $\alpha^j \in \mathscr{D}$ such that the determinant of the square matrix

$$W_i(p)(\alpha^j) \quad (i, j = 1, \ldots, n)$$

is different from zero. The functions

$$\gamma_i^j = \partial_{W_i}(\alpha^j), \qquad \gamma^j = \partial_W \alpha^j$$

are in $\mathscr{C}$ and

$$\beta^i \gamma_i^j = \gamma^j. \tag{4}$$

Since the determinant of the system of equations (4) is different from zero at p, it is different from zero in a neighbourhood A of p. Calculating the functions β^i from (4) by the Cramer formula we infer that $\beta^i|A \in \mathscr{C}_A$. This implies, by 1.2, that $\beta^i \in \mathscr{C}$.

3.7. *A sequence $W_1, \ldots, W_n \in \mathfrak{W}_f(M, N)$ is a vector basis of $\mathscr{C}$-module $\mathfrak{W}_f(M, N)$ if and only if it satisfies the condition* (a) *in Section* 2, *i.e., if* *for every* $p \in M$ *the sequence* $W_1(p), \ldots, W_n(p)$ *is a basis of the linear space* $N_{f(p)}$.

In fact, it follows from (a) that for any vector f-field W on M there exists a unique sequence $a^1, \ldots, a^n$ of real functions on M such that $W = a^i W_i$. If $W \in \mathfrak{W}_f(M, N)$, then $a^i \in \mathscr{C}$ for $i = 1, \ldots, n$ on account of 3.6. This proves that the condition (b′) in Section 2 is also satisfied.

A sequence $V_1, \ldots, V_m$ is said to be a *vector basis on a differential space* $(M, \mathscr{C})$ provided it is a vector basis of the $\mathscr{C}$-module $\mathfrak{W}(M)$, i.e. $V_1, \ldots, V_m \in \mathfrak{W}(M)$ and, for every $p \in M$, the sequence $V_1(p), \ldots, V_m(p)$ is a basis of the linear space M_p (see 3.7). It follows from 3.7 that

3.8. *If* $V_1, \ldots, V_n$ *is a vector basis on a differential space* $(N, \mathscr{D})$ *and* $f\colon M \to N$ *is smooth, then the sequence* $V_1 \circ f, \ldots, V_n \circ f$ *is a vector basis of the module* $\mathfrak{W}_f(M, N)$.

We say that a differential space $(M, \mathscr{C})$ *has differential dimension* m and we write

$$\dim(M, \mathscr{C}) = m \text{ or, less precisely, } \dim M = m,$$

if $\dim \mathfrak{W}(M) = m$, i.e., if every point $p \in M$ has a neighbourhood A such that there is a vector basis on $(A, \mathscr{C}_A)$ composed of m vector fields. If the differential dimension of $(M, \mathscr{C})$ exists, then it is uniquely determined by $(M, \mathscr{C})$, because it is the common linear dimension of all linear spaces M_p, $p \in M$. It is not always equal to the topological dimension of the topological space M.

3.9. *In order that* $\dim(M, \mathscr{C}) = m$ *it is necessary and sufficient that*

$$\dim M_p = m \quad \text{for every } p \in M$$

and that one of the following equivalent conditions be satisfied:

(a) *for every* $p \in M$ *and* $v \in M_p$ *there exists a smooth tangent vector field* V *on* M *such that* $V(p) = v$;

(a′) *for every* $p \in M$ *and* $v \in M_p$ *there exists a neighbourhood* A *of* p *and a smooth tangent vector field* V *on* A *such that* $V(p) = v$.

We say that a differential space $(M, \mathscr{C})$ is *of finite dimension* if there is a positive integer m such that $\dim(M, \mathscr{C}) = m$, i.e., if the set $\mathfrak{W}(M)$ of all smooth tangent vector fields on M is a differential module.

4. The module induced by a mapping. Let $(M, \mathscr{C})$ and $(N, \mathscr{D})$ be differential spaces and let $f\colon M \to N$ be smooth. Let Φ be a function which assigns a linear space $\Phi(q)$ to any point $q \in N$. Then $\Phi \circ f$ is a function which assigns the linear space $\Phi(f(p))$ to any point $p \in M$.

If $\mathscr{W}$ is a $\mathscr{D}$-module of Φ-fields on N, then the symbol $\mathscr{W}_f$ will denote the smallest $\mathscr{C}$-module closed with respect to localization and containing all $\Phi \circ f$-fields $W \circ f$, where $W \in \mathscr{W}$. By 2.2,

$$\mathscr{W}_f = \mathscr{V}_M, \tag{1}$$

where $\mathscr{V}$ is the smallest $\mathscr{C}$-module containing all $\Phi\circ f$-fields $W\circ f$, where $W\in\mathscr{W}$. By 2.2, $\mathscr{V}$ is the set of all finite linear combinations

$$a^i\cdot W_i\circ f,\qquad \text{where } a^i\in\mathscr{C} \text{ and } W_i\in\mathscr{W}.$$

Hence it follows that

4.1. *In order that a $\Phi\circ f$-field V belong to $\mathscr{W}_f$ it is necessary and sufficient that, for every $p\in M$, there exist a neighbourhood A of p and a linear combination $a^i\cdot W_i\circ f$ such that*

$$V|A=(a^i\cdot W_i\circ f)|A,\qquad a^i\in\mathscr{C},\ W_i\in\mathscr{W}. \tag{2}$$

Let us examine first a particular case.

4.2. *If $(M,\mathscr{C})$ is a differential subspace of $(N,\mathscr{D})$ and f is the identity mapping, then $\mathscr{W}_f=\mathscr{W}_M$.*

If $V\in\mathscr{W}_M$, then for every point $p\in M$ there exist a neighbourhood A of p and a Φ-field $W\in\mathscr{W}$ such that $V|A=W|A$. Thus the condition (2) is satisfied, i.e., $V\in\mathscr{W}_f$.

Conversely, if $V\in\mathscr{W}_f$, then for every point $p\in M$ there exist a neighbourhood A of p and a linear combination $a^i\cdot W_i\circ f$ such that (2) holds, that is,

$$V|A=(a^i|A)\cdot(W_i|A),\qquad a^i\in\mathscr{C}=\mathscr{C}_M,\ W_i\in\mathscr{W}.$$

Since $W_i|A\in\mathscr{W}_A$, $a^i|A\in\mathscr{C}_A=(\mathscr{D}_M)_A=\mathscr{D}_A$ (see 1.1), and $\mathscr{W}_A$ is a $\mathscr{D}_A$-module, we infer that $V|A\in\mathscr{W}_A$. Since every point $p\in M$ has a neighbourhood A such that $V|A\in\mathscr{W}_A$, we have $V\in\mathscr{W}_M$ on account of 1.2.

To formulate the next theorem, let us consider a smooth mapping g from a differential space $(M',\mathscr{C}')$ into $(M,\mathscr{C})$.

4.3. *The following identity holds*

$$\mathscr{W}_{f\circ g}=(\mathscr{W}_f)_g. \tag{3}$$

For every $W\in\mathscr{W}$ we have $W\circ f\in\mathscr{W}_f$ and $W\circ(f\circ g)=(W\circ f)\circ g\in(\mathscr{W}_f)_g$. Thus $(\mathscr{W}_f)_g$ is a $\mathscr{C}'$-module of linear $\Phi\circ(f\circ g)$-fields that contains all $\Phi\circ(f\circ g)$-fields $W\circ(f\circ g)$ with $W\in\mathscr{W}$ and is closed with respect to localization. Since $\mathscr{W}_{f\circ g}$ is the smallest $\mathscr{C}'$-module with these properties, we have

$$\mathscr{W}_{f\circ g}\subset(\mathscr{W}_f)_g.$$

Conversely, if $V\in(\mathscr{W}_f)_g$, then for every point $p\in M'$ there exist a neighbourhood $A_0\subset M'$ of p and a linear combination $a^i\cdot\overline{W}_i\circ g$ such that

$$V|A_0=(a^i\cdot\overline{W}_i\circ g)|A_0,\qquad a^i\in\mathscr{C}',\ \overline{W}_i\in\mathscr{W}_f.$$

For every field $\overline{W}_i$ there exist a neighbourhood A_i of $g(p)$ in the space M and a linear combination $a_i^j\cdot W_{i,j}\circ f$ such that

$$\overline{W}_i|A_i=(a_i^j\cdot W_{i,j}\circ f)|A_i,\qquad a_i^j\in\mathscr{C},\ W_{i,j}\in\mathscr{W}$$

(no summation with respect to i). The common part A of the open set A_0 and of all the open sets $g^{-1}(A_i)$ is a neighbourhood of p. Denoting by $\beta^{i,j}$ the product of the function a^i and of the function $a_i^j \circ g$, we have

$$V|A = (\beta^{i,j} W_{i,j} \circ (f \circ g))|A, \quad \beta^{i,j} \in \mathscr{C}, \; W_{i,j} \in \mathscr{W}.$$

Thus $V \in \mathscr{W}_{f \circ g}$ by 4.1. This proves that

$$(\mathscr{W}_f)_g \subset \mathscr{W}_{f \circ g}.$$

4.4. *For every set $A \subset M$,*

$$\mathscr{W}_{f|A} = (\mathscr{W}_f)_A.$$

This follows directly from 4.3 (where g is the identity mapping from A into M) and from 4.2 (where M and N should be replaced by A and M, respectively).

4.5. *If $W_1, \ldots, W_n$ is a vector basis of the $\mathscr{D}$-module $\mathscr{W}$, then $W_1 \circ f, \ldots$ $\ldots, W_n \circ f$ is a vector basis of the $\mathscr{C}$-module $\mathscr{W}_f$. Thus $V \in \mathscr{W}_f$ if and only if $V = a^i \cdot W_i \circ f$ for certain functions $a^i \in \mathscr{C}$.*

Let $\mathscr{V}$ denote, as on p. 56-57, the $\mathscr{C}$-module of all linear combinations

$$V = a^i \cdot \overline{W}_i \circ f, \quad \text{where } a^i \in \mathscr{C} \text{ and } \overline{W}_i \in \mathscr{W}.$$

Since $\overline{W}_i = \beta_i^j W_j$ for certain functions $\beta_i^j \in \mathscr{D}$, we have

$$V = \gamma^j \cdot W_j f, \tag{4}$$

where $\gamma^j = a^i \cdot (\beta_i^j \circ f) \in \mathscr{C}$. Conversely, every linear field (4), where $\gamma^j \in \mathscr{C}$, belongs to $\mathscr{V}$. Thus $\mathscr{V}$ is the set of all linear combinations (4), where $\gamma^j \in \mathscr{C}$. Since elements $W_1 \circ f(p), \ldots, W_n \circ f(p)$ form a basis of the linear space $\Phi \circ f(p)$, the sequence $W_1 \circ f, \ldots, W_n \circ f$ is a vector basis of $\mathscr{V}$. On the other hand, $\mathscr{W}_f = \mathscr{V}_M = \mathscr{V}$ by (1) and 2.6, which completes the proof of 4.5.

Theorem 4.5 is a particular case of the following theorem:

4.6. *If $W_1, \ldots, W_n$ is a vector basis of the $\mathscr{D}$-module $\mathscr{W}$ on a set $B \subset N$, and if $A \subset M$, $f(A) \subset B$, then*

$$W_1 \circ f|A, \ldots, W_n \circ f|A \tag{5}$$

is a vector basis of the $\mathscr{C}$-module $\mathscr{W}_f$ on the set A.

Replacing in 4.5 f by $f|A$, N by B, $\mathscr{W}$ by $\mathscr{W}_B$ and M by A, we infer that (5) is a vector basis of the $\mathscr{C}_A$-module $(\mathscr{W}_B)_{f|A}$. On the other hand, denoting by g the identity mapping of B into N we have, by 4.2, $\mathscr{W}_B = \mathscr{W}_g$ and, consequently, by 4.3 and 4.4,

$$(\mathscr{W}_B)_{f|A} = (\mathscr{W}_g)_{f|A} = \mathscr{W}_{g \circ f|A} = \mathscr{W}_{f|A} = (\mathscr{W}_f)_A.$$

This proves that (5) is a vector basis of $(\mathscr{W}_f)_A$, i.e., of $\mathscr{W}_f$ on the set A.

4.7. *If* $\dim \mathscr{W} = n$, *then* $\dim \mathscr{W}_f = n$. *Thus if* $\mathscr{W}$ *is a differential modulus, so is* $\mathscr{W}_f$.

If $\dim \mathscr{W} = n$, there exists an open covering $\mathscr{B}$ of the space N such that for every $B \in \mathscr{B}$ there is a vector basis $W_1, \ldots, W_n$ of the $\mathscr{D}$-module $\mathscr{W}$ on the set B. The sets $A = f^{-1}(B)$, where $B \in \mathscr{B}$, form an open covering $\mathscr{A}$ of the space M. By 4.6, for every set $A \in \mathscr{A}$, the sequence (5) is a vector basis of the $\mathscr{C}$-module $\mathscr{W}_f$ on the set A. This proves that $\dim \mathscr{W}_f = n$.

4.8. *If the differential space* $(N, \mathscr{D})$ *is of finite dimension, and* $\mathscr{W} = \mathfrak{W}(N)$, *then* $\mathscr{W}_f = \mathfrak{W}_f(M, N)$.

In other words, under the hypotheses of 4.8,

$$\mathfrak{W}(N)_f = \mathfrak{W}_f(M, N). \tag{6}$$

Let $\mathscr{V}$ has the same meaning as on p. 56-57 or in the proof of 4.5.

If $V \in \mathscr{V}$, i.e., if

$$V = a^i \cdot W_i \circ f, \quad \text{where } a^i \in \mathscr{C} \text{ and } W_i \in \mathscr{W},$$

then for every $\gamma \in \mathscr{D}$

$$\partial_V \gamma = a^i \cdot (\partial_{W_i} \gamma) \circ f, \tag{7}$$

because for every point $p \in M$

$$\partial_V \gamma(p) = a^i(p) \partial_{W_i(f(p))} \gamma = a^i(p) \partial_{W_i} \gamma\big(f(p)\big).$$

It follows from (7) that $\partial_V \gamma \in \mathscr{C}$ for every $\gamma \in \mathscr{D}$, i.e., that the vector f-field V is smooth. In other words, $\mathscr{V} \subset \mathfrak{W}_f(M, N)$. Hence $\mathscr{W}_f = \mathscr{V}_M \subset \big(\mathfrak{W}_f(M, N)\big)_M = \mathfrak{W}_f(M, N)$ (see the second part of 3.5). Consequently,

$$\mathscr{W}_f \subset \mathfrak{W}_f(M, N).$$

Suppose now that $V \in \mathfrak{W}_f(M, N)$. For every fixed point $p_0 \in M$ there exists an neighbourhood A of $q_0 = f(p_0)$ such that there is a vector basis $W_1, \ldots, W_n$ of $\mathfrak{W}(N)$ on A. The set $A' = f^{-1}(A)$ is a neighbourhood of p_0 and

$$V|A' = a^i \cdot W_i \circ (f|A')$$

for certain functions $a^i \in \mathscr{C}_{A'}$ (see 3.8). Let a function $\beta \in \mathscr{D}$ separate the point q_0 in the set A, i.e.

$$\beta|B = 1 \quad \text{for a neighbourhood } B \text{ of } q_0,\ B \subset A,$$

$$\beta|A_0 \quad \text{for an open set } A_0,\ A \cup A_0 = N.$$

Let

$$\overline{W}_i(q) = \begin{cases} \beta(q) W_i(q) & \text{for } q \in A, \\ 0 & \text{for } q \in A_0. \end{cases}$$

By the definition, $\overline{W}_i \in \mathfrak{W}(N)$. The set $B' = f^{-1}(B)$ is a neighbourhood of p_0, $B' \subset A'$, and

$$V|B' = \gamma^i \cdot \overline{W}_i \circ (f|B'), \tag{8}$$

where $\gamma^i = a^i|B'$. So we have proved that every point $p_0 \in M$ has a neighbourhood B' such that $V|B'$ is of the form (8), where $\gamma^i \in \mathscr{C}_{B'}$ and $\overline{W}_i \in \mathscr{W}$. Hence $V \in \mathscr{W}_f$. This proves that

$$\mathfrak{W}_f(M, N) \subset \mathscr{W}_f.$$

4.9. *If $(M, \mathscr{C})$ is a differential subspace of the differential space $(N, \mathscr{D})$ of finite dimension, then*

$$\mathfrak{W}(M) \subset \mathfrak{W}(M, N) = \mathfrak{W}(N)_M. \tag{9}$$

For every $W \in \mathfrak{W}(M)$ and for every point $p \in M$ there exist a neighbourhood A of p in the space M and a $V \in \mathfrak{W}(N)$ such that

$$V|M \in \mathfrak{W}(M), \qquad V|A = W|A. \tag{10}$$

It is obvious that $\mathfrak{W}(M) \subset \mathfrak{W}(M, N)$. The identity $\mathfrak{W}(M, N) = \mathfrak{W}(N)_M$ follows from 4.8, where f is the identity mapping on M (see also 4.2).

Let $W \in \mathfrak{W}(M)$ and $p \in M$. Since $W \in \mathfrak{W}(M, N)$, it follows from (9) that there exists a $V_1 \in \mathfrak{W}(N)$ such that $W|A_1 = V_1|A_1$ for a neighbourhood A_1 of p in the space M. Let A_2 be an open subset of the space N, such that $M \cap A_2 = A_1$, and let $\beta \in \mathscr{D}$ be a function separating p in A_2. The field $V = \beta V_1$ and the set $A = M \cap B$, where B is a neighbourhood of p such that $\beta|B = 1$, have the properties required in the second part of 4.9.

5. Modules of tensor fields. Let $(M, \mathscr{C})$ be a differential space.

In this section we shall consider $n+1$ fixed functions Φ_j $(j = 1, \ldots, n+1)$ which assign linear spaces $\Phi_j(p)$ to any $p \in M$. We shall also examine $n+1$ fixed $\mathscr{C}$-modules $\mathscr{W}_j$ $(j = 1, \ldots, n+1)$. Elements of $\mathscr{W}_j$ are Φ_j-fields.

The letter Φ will now denote the function which assigns, to every $p \in M$, the linear space

$$\Phi(p) = \mathfrak{L}_E\big(\Phi_1(p), \ldots, \Phi_n(p); \Phi_{n+1}(p)\big) \tag{1}$$

of all E-n-linear mappings of the product $\Phi_1(p) \times \ldots \times \Phi_n(p)$ into $\Phi_{n+1}(p)$. Therefore a Φ-field on M is now a function which assigns, to every point $p \in M$, an E-n-linear mapping (an E-tensor)

$$L(p)\colon \Phi_1(p) \times \ldots \times \Phi_n(p) \to \Phi_{n+1}(p). \tag{2}$$

Linear fields of this kind will be called *tensor fields*. If L is a Φ-field (2), then the symbol L' will denote the function which assigns, to any $W_1 \epsilon \mathscr{W}_1, \ldots, W_n \epsilon \mathscr{W}_n$, the function $L'(W_1, \ldots, W_n)$ defined on M by

(3) $$L'(W_1, \ldots, W_n)(p) = L(p)\big(W_1(p), \ldots, W_n(p)\big) \quad \text{for } p \epsilon M.$$

By the definition, $L'(W_1, \ldots, W_n)$ is a Φ_{n+1}-field. Let $\mathscr{W}$ denote the set of all Φ-fields L such that

(4) $$L'(W_1, \ldots, W_n) \epsilon \mathscr{W}_{n+1} \quad \text{for all } W_1 \epsilon \mathscr{W}_1, \ldots, W_n \epsilon \mathscr{W}_n.$$

In other words,

(5) $$L \epsilon \mathscr{W} \text{ if and only if } L' \epsilon \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1}),$$

i.e., if L' is a $\mathscr{C}$-tensor, L': $\mathscr{W}_1 \times \ldots \times \mathscr{W}_n \to \mathscr{W}_{n+1}$.

5.1. *If the $\mathscr{C}$-module $\mathscr{W}_{n+1}$ is closed with respect to localization and A is an open subset of M, then $\mathscr{W}_A$ is the set of all $(\Phi|A)$-fields L such that*

(6) $$L'(W_1, \ldots, W_n) \epsilon (\mathscr{W}_{n+1})_A$$

for arbitrary linear fields

(7) $$W_1 \epsilon (\mathscr{W}_1)_A, \ldots, W_n \epsilon (\mathscr{W}_n)_A.$$

If $L \epsilon \mathscr{W}_A$ and (7) holds, then for every point $p \epsilon A$ there exist a neighbourhood B of p in the space A, a Φ-field $\bar{L} \epsilon \mathscr{W}$, and fields

(8) $$\overline{W}_1 \epsilon \mathscr{W}_1, \ldots, \overline{W}_n \epsilon \mathscr{W}_n$$

such that $\bar{L}|B = L|B$, $\overline{W}_1|B = W_1|B, \ldots, \overline{W}_n|B = W_n|B$. Thus, by (4),

$$L'(W_1, \ldots, W_n)|B = \bar{L}'(\overline{W}_1, \ldots, \overline{W}_n)|B \epsilon (\mathscr{W}_{n+1})_B.$$

This proves (6).

Conversely, if L has the property (6) and p is any point in A, let $\beta \epsilon \mathscr{C}$ be a function separating the point p in the set A, that is,

(9) $$\beta|B = 1 \quad \text{for a neighbourhood } B \text{ of } p,$$

(10) $$\beta|A_0 = 0 \quad \text{for an open set } A_0,\ A \cup A_0 = M.$$

Let

$$\bar{L}(q) = \begin{cases} \beta(q)L(q) & \text{for } q \epsilon A, \\ 0 & \text{for } q \epsilon A_0. \end{cases}$$

Since $\bar{L}|A = \beta L$ and $\bar{L}|A_0 = 0 \epsilon \mathscr{W}_{A_0}$, we have

$$\bar{L}'(W_1, \ldots, W_n)|A = \beta L'(W_1|A, \ldots, W_n|A \epsilon (\mathscr{W}_{n+1})_A,$$
$$\bar{L}'(W_1, \ldots, W_n)|A_0 = 0 \epsilon (\mathscr{W}_{n+1})_{A_0}$$

for $W_1 \epsilon \mathscr{W}_1, \ldots, W_n \epsilon \mathscr{W}_n$. This proves that, by 1.2,

(11) $$\bar{L}'(W_1, \ldots, W_n) \epsilon (\mathscr{W}_{n+1})_M = \mathscr{W}_{n+1} \quad \text{for } W_1 \epsilon \mathscr{W}_1, \ldots, W_n \epsilon \mathscr{W}_n.$$

It follows from (11) that $\bar{L} \in \mathscr{W}$, and from (9) that $L|B = \bar{L}|B$. Since p is an arbitrary point in A, we infer that $L \in \mathscr{W}_A$.

5.2. *If the $\mathscr{C}$-module $\mathscr{W}_{n+1}$ is closed with respect to localization, then the $\mathscr{C}$-module $\mathscr{W}$ is also closed with respect to localization.*

This follows directly from 5.1 where $A = M$.

5.3. *If $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ are differential modules, then $\mathscr{W}$ is also a differential module. If*

$$\dim \mathscr{W}_1 = m_1, \quad \ldots, \quad \dim \mathscr{W}_{n+1} = m_{n+1},$$

then

$$\dim \mathscr{W} = m_1 \ldots m_{n+1}.$$

We shall prove 5.3 only in the case $n = 2$ to simplify the notation. It will be evident how to generalize the proof for an arbitrary n.

By 5.2, it suffices to prove that if, for an open set $A \subset M$, there exist

a vector basis $W'_1, \ldots, W'_{m_1}$ of $\mathscr{W}_1$ on the set A,

a vector basis $W''_1, \ldots, W''_{m_2}$ of $\mathscr{W}_2$ on the set A,

and a vector basis $W'''_1, \ldots, W'''_{m_3}$ of $\mathscr{W}_3$ on the set A,

then there exists a vector basis of $\mathscr{W}$ on A, which is composed of $m_1 m_2 m_3$ fields.

Let $L_k^{l,r}$ be a $(\Phi|A)$-field such that for any $p \in A$

$$L_k^{l,r}(p)\colon \ \Phi_1(p) \times \Phi_2(p) \to \Phi_3(p)$$

is a bilinear mapping and

$$L_k^{l,r}(p)\big(W'_i(p),\ W''_j(p)\big) = \delta_i^l \delta_j^r W'''_k(p),$$

where δ_i^l and δ_j^r are the Kronecker symbols.

Such E-tensors $L_k^{l,r}(p)$ $(l = 1, \ldots, m_1,\ r = 1, \ldots, m_2,\ k = 1, \ldots, m_3)$ exist and form a basis of the linear space $\Phi(p) = \mathfrak{L}_E\big(\Phi_1(p), \Phi_2(p); \Phi_3(p)\big)$, i.e., the fields $L_k^{l,r}$ satisfy condition (a) in the definition of a vector basis.

If $W' \in (\mathscr{W}_1)_A$ and $W'' \in (\mathscr{W}_2)_A$, then $W' = a^i W'_i$ and $W'' = \beta^j W''_j$ for certain functions $a^i, \beta^j \in \mathscr{C}_A$.

Thus

$$L_k^{l,r}(W', W'') = a^l \beta^r W'''_k \in (\mathscr{W}_3)_A.$$

This implies, by 5.1, that $L_k^{l,r} \in \mathscr{W}_A$.

It follows from the property (a) that any $(\Phi|A)$-field L can be uniquely represented in the form

$$L = a_{l,r}^k L_k^{l,r},$$

where $a^k_{l,r}$ are real functions on A. If $L \in \mathscr{W}_A$, then $a^k_{l,r} \in \mathscr{C}_A$, since $L(W'_i, W''_j) = a^k_{i,j} W'''_k$, $L(W'_i, W''_j) \in (\mathscr{W}_3)_A$ and the W'''_k $(k = 1, \ldots, m_3)$ form a vector basis in $(\mathscr{W}_3)_A$. This proves that the fields $L^{l,r}_k$ have the property (b) in the definition of a vector basis. Thus they form a vector basis for $\mathscr{W}_A$.

We shall now examine properties of the $\mathscr{C}$-module

$$\mathscr{W}' = \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1}) \tag{12}$$

of all $\mathscr{C}$-n-linear mappings ($\mathscr{C}$-tensors)

$$\mathbf{L}\colon \mathscr{W}_1 \times \ldots \times \mathscr{W}_n \to \mathscr{W}_{n+1}. \tag{13}$$

5.4. *If $\mathbf{L} \in \mathscr{W}'$, $W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n$, if, for a positive integer $j_0 \leqslant n$, there exists a vector basis $V_1, \ldots, V_m$ of the $\mathscr{C}$-module $\mathscr{W}_{j_0}$ on a neighbourhood A of a point p, if the $\mathscr{C}$-module $\mathscr{W}_{j_0}$ is closed with respect to the localization and if $W_{j_0}(p) = 0$, then*

$$\mathbf{L}(W_1, \ldots, W_n)(p) = 0.$$

Let us assume, for simplicity, that $j_0 = 1$. We have $W_1|A = a^i V_i$, $a^i \in \mathscr{C}_A$. Let $\beta \in \mathscr{C}$ separate the point p in the set A, i.e., β has properties (9) and (10). The formula

$$\overline{V}_i(q) = \begin{cases} \beta(q) V_i(q) & \text{for } q \in A, \\ 0 & \text{for } q \in A_0 \end{cases}$$

defines a Φ_1-field $\overline{V}_i$ which is in $\mathscr{W}_1$, because $\overline{V}_i|A = \beta V_i \in (\mathscr{W}_1)_A$, $\overline{V}_i|A_0 = 0 \in (\mathscr{W}_1)_{A_0}$, the sets A, A_0 form an open covering of M, and the module $\mathscr{W}_1$ is closed with respect to the localization (see 1.2). The formula

$$\bar{a}^i(q) = \begin{cases} \beta(q) a^i(q) & \text{for } q \in A, \\ 0 & \text{for } q \in A_0 \end{cases}$$

defines a function $\bar{a}^i \in \mathscr{C}$. Let $\overline{W}_1 = \beta^2 W_1 = \bar{a}^i \overline{V}_i$. Since $W_1(p) = 0$, we have $a^i(p) = 0$ and, consequently, $\bar{a}^i(p) = 0$ for every i. Since $\beta(p) = 1$, we get

$$\begin{aligned} \mathbf{L}(W_1, \ldots, W_n)(p) &= \beta(p)^2 \mathbf{L}(W_1, \ldots, W_n)(p) \\ &= \mathbf{L}(\overline{W}_1, W_2, \ldots, W_n)(p) = \mathbf{L}(\bar{a}^i \overline{V}_i, W_2, \ldots, W_n)(p) \\ &= \bar{a}^i(p) \mathbf{L}(\overline{V}_i, W_2, \ldots, W_n) = 0. \end{aligned}$$

5.5. *If $\mathscr{W}_1, \ldots, \mathscr{W}_n$ are differential modules, $\mathbf{L} \in \mathscr{W}'$, $p \in M$, $W_j, \overline{W}_j \in \mathscr{W}_j$ and $W_j(p) = \overline{W}_j(p)$ for $j = 1, \ldots, n$, then*

$$\mathbf{L}(W_1, \ldots, W_n)(p) = \mathbf{L}(\overline{W}_1, \ldots, \overline{W}_n)(p).$$

This follows from the identity

$$\begin{aligned} &\mathbf{L}(W_1, \ldots, W_n)(p) - \mathbf{L}(\overline{W}_1, \ldots, \overline{W}_n)(p) \\ &\quad = \mathbf{L}(W_1 - \overline{W}_1, W_2, \ldots, W_n)(p) + \mathbf{L}(\overline{W}_1, W_2 - \overline{W}_2, \ldots, W_n)(p) + \\ &\qquad + \ldots + \mathbf{L}(\overline{W}_1, \ldots, \overline{W}_{n-1}, W_n - \overline{W}_n)(p), \end{aligned}$$

because all terms on the right-hand side of the equation are equal to 0 by 5.4.

It follows directly from the definition that the *canonical mapping* from $\mathscr{W}$ into $\mathscr{W}'$, which assigns $L' \in \mathscr{W}'$ to $L \in \mathscr{W}$, is $\mathscr{C}$-linear. We shall prove that

5.6. *If $\mathscr{W}_1, \ldots, \mathscr{W}_n$ are differential modules, then the canonical mapping is one-to-one and onto $\mathscr{W}'$, i.e., it is a module-isomorphism from $\mathscr{W}$ onto $\mathscr{W}'$.*

Suppose that $\mathbf{L} \in \mathscr{W}'$, $L \in \mathscr{W}$ and $\mathbf{L} = L'$. If $w_j \in \Phi_j(p)$ $(j = 1, \ldots, n$, $p \in M)$, let $W_j \in \mathscr{W}_j$ be a Φ_j-field such that $W_j(p) = w_j$ (see 2.9). By (3),

$$L(p)(w_1, \ldots, w_n) = \mathbf{L}(W_1, \ldots, W_n)(p). \tag{14}$$

Equation (14) determines $L(p)$ uniquely for every $p \in M$. Thus for every $\mathbf{L} \in \mathscr{W}'$ there exists at most one $L \in \mathscr{W}$ such that $\mathbf{L} = L'$.

On the other hand, if $\mathbf{L} \in \mathscr{W}'$ is given, equation (14) defines an E-n-linear mapping

$$L(p)\colon \Phi_1(p) \times \ldots \times \Phi_n(p) \to \Phi_{n+1}(p)$$

for every $p \in M$ (the right-hand side of (14) does not depend on the choice of the fields W_j, $j = 1, \ldots, n$, on account of 5.5). Thus (14) defines a Φ-field L. It follows from (14) that $L' = \mathbf{L}$, which proves at the same time that $L \in \mathscr{W}$.

The canonical isomorphism mentioned in 5.6 allows to identify (under the hypothesis that $\mathscr{W}_1, \ldots, \mathscr{W}_n$ are differential modules) any $L \in \mathscr{W}$ with $L' \in \mathscr{W}'$. The identification is very convenient in practice. We shall treat L' as another interpretation of the Φ-field L and, as a rule, we shall omit the sign $'$ at the symbol L. In other words, every Φ-field $L \in \mathscr{W}$ on M will be often interpreted as a mapping ($\mathscr{C}$-tensor)

$$L \in \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})$$

which assigns to any linear fields $W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n$ the linear fields

$$L(W_1, \ldots, W_n) \in \mathscr{W}_{n+1} \tag{15}$$

defined by

$$L(W_1, \ldots, W_n)(p) = L(p)\big(W_1(p), \ldots, W_n(p)\big) \quad \text{for } p \in M. \tag{16}$$

We see that, after the identification, the notion of a tensor field $L \in \mathscr{W}$ has two interpretations: pointwise and global (similarly to the notion of vector field — see p. 53). In the *pointwise interpretation* L is a function which assigns the E-n-tensor (2) to any $p \in M$. In the global interpretation L is a $\mathscr{C}$-n-tensor with properties (15) and (16).

In consequence, we shall assume that

$$\mathscr{W} = \mathfrak{L}_{\mathscr{C}}(W_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1}). \tag{17}$$

Using the convention (17) we can formulate Theorem 5.1 as follows

$$(18)\qquad \mathfrak{L}_{\mathscr{C}}\big((\mathscr{W}_1)_A, \ldots, (\mathscr{W}_n)_A; (\mathscr{W}_{n+1})_A\big) = \big(\mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})\big)_A.$$

6. A commutative theorem. In Sections 4 and 5 we have defined two methods of construction of new differential modulus from the given differential modulus (see 4.7 and 5.3). In this section we shall prove that they are commutative — see Theorem 6.1 below.

Let $(M, \mathscr{C})$, Φ_j, $\mathscr{W}_j$ $(j = 1, \ldots, n+1)$, Φ, $\mathscr{W}$ have the same meaning as in Section 5 (see p. 60-61). Suppose that $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ are differential modules and that f is a smooth mapping of a differential space $(M', \mathscr{C}')$ into $(M, \mathscr{C})$.

Applying the construction described in Section 5 to the differential C-modules $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ we get the differential $\mathscr{C}$-module

$$(1)\qquad \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})$$

whose elements in the pointwise interpretations are linear Φ-fields L on M such that

$$(2)\qquad L(W_1, \ldots, W_n) \in \mathscr{W}_{n+1} \quad \text{for } W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n.$$

Applying the construction described in Section 4 to the differential module (1) we get the differential $\mathscr{C}'$-module

$$(3)\qquad \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})_f$$

whose elements are linear $(\Phi \circ f)$-fields on M'.

On the other hand, applying the construction from Section 4 to the $\mathscr{C}$-modules $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ we get the differential $\mathscr{C}'$-modules

$$(4)\qquad \mathscr{W}_{1,f}, \ldots, \mathscr{W}_{n+1,f};$$

the elements of $\mathscr{W}_{j,f}$ are linear $\Phi_j \circ f$-fields on M'. Applying the construction from Section 5 to the differential $\mathscr{C}'$-modules (4) we get the differential $\mathscr{C}'$-module

$$(5)\qquad \mathfrak{L}_{\mathscr{C}'}(\mathscr{W}_{1,f}, \ldots, \mathscr{W}_{n,f}; \mathscr{W}_{n+1,f})$$

whose elements, in the pointwise interpretation, are linear $(\Phi \circ f)$-fields L on M' such that

$$L(W_1, \ldots, W_n) \in \mathscr{W}_{n+1,f} \quad \text{for } W_1 \in \mathscr{W}_{1,f}, \ldots, W_n \in \mathscr{W}_{n,f}.$$

6.1. *The following identity is true*

$$(6)\qquad \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})_f = \mathfrak{L}_{\mathscr{C}'}(\mathscr{W}_{1,f}, \ldots, \mathscr{W}_{n,f}; \mathscr{W}_{n+1,f}).$$

To simplify notation, we shall prove 6.1 only for $n = 2$. It will be evident how to generalize the proof for an arbitrary positive integer n.

We have to prove that

$$(6') \qquad \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \mathscr{W}_2; \mathscr{W}_3)_f = \mathfrak{L}_{\mathscr{C}'}(\mathscr{W}_{1,f}, \mathscr{W}_{2,f}; \mathscr{W}_{3,f}).$$

Let A be an open subset of M such that there exist a vector basis

$$(7) \qquad W'_1, \ldots, W'_{m_1}$$

of the $\mathscr{C}$-module $\mathscr{W}_1$ on the set A, a vector basis

$$(8) \qquad W''_1, \ldots, W''_{m_2}$$

of the $\mathscr{C}$-module $\mathscr{W}_2$ on the set A, and a vector basis

$$(9) \qquad W'''_1, \ldots, W'''_{m_3}$$

of the $\mathscr{C}$-module $\mathscr{W}_3$ on the set A.

We shall prove the identity (6') first under the additional hypothesis that

$$(10) \qquad f(M') \subset A.$$

In this case (see 4.5)

$$(11) \qquad W'_1 \circ f, \ldots, W'_{m_1} \circ f$$

is a vector basis of the $\mathscr{C}'$-module $\mathscr{W}_{1,f}$,

$$(12) \qquad W''_1 \circ f, \ldots, W''_{m_2} \circ f$$

is a vector basis of the $\mathscr{C}'$-module $\mathscr{W}_{2,f}$, and

$$(13) \qquad W'''_1 \circ f, \ldots, W'''_{m_3} \circ f$$

is a vector basis of the $\mathscr{C}'$-module $\mathscr{W}_{3,f}$.

In the proof of 5.3 we have shown that the vector bases (7), (8), (9) determine a vector basis

$$(14) \qquad L_k^{l,r} \quad (l = 1, \ldots, m_1, r = 1, \ldots, m_2, k = 1, \ldots, m_3)$$

of the $\mathscr{C}$-module $\mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \mathscr{W}_2; \mathscr{W}_3)$ on the set A. Hence it follows that

$$(15) \qquad L_k^{l,r} \circ f$$

is a vector basis of the $\mathscr{C}'$-module $\mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \mathscr{W}_2; \mathscr{W}_3)_f$. The construction of (14) from (7), (8), (9) is of such a kind that if it is applied to the bases (11)-(13), then it yields the vector basis (15) for the $\mathscr{C}'$-modulus $\mathfrak{L}_{\mathscr{C}'}(\mathscr{W}_{1,f}, \mathscr{W}_{2,f}; \mathscr{W}_{3,f})$. Thus (15) is simultaneously a vector basis of both modules appearing in (6'). Modules having a common vector basis are identical. Thus (6') holds.

Now we shall prove (6') without the hypothesis (10).

Let $\mathscr{A}$ be an open covering of the space M by means of open sets A, on which there exist simultaneously vector bases (7), (8), (9). The class $\mathscr{B}$ of all sets $B = f^{-1}(A)$, where $A \in \mathscr{A}$, is an open covering of the space M'.

Replacing in the part of 6.1 just proved the mapping f by $f|B$ $(B \in \mathscr{B})$ we get

$$\mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \mathscr{W}_2; \mathscr{W}_3)_{f|B} = \mathfrak{L}_{\mathscr{C}'_B}(\mathscr{W}_{1,f|B}, \mathscr{W}_{2,f|B}; \mathscr{W}_{3,f|B}),$$

that is, by 4.4 and (18) from Section 5,

$$(16) \qquad \big(\mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \mathscr{W}_2; \mathscr{W}_3)_f\big)_B = \big(\mathfrak{L}_{\mathscr{C}'}(\mathscr{W}_{1,f}, \mathscr{W}_{2,f}; \mathscr{W}_{3,f})\big)_B.$$

A $(\Phi \circ f)$-field L on M' belongs to the module on the left-hand side of (6′) if and only if for every $B \in \mathscr{B}$ the field $L|B$ belongs to the module on the left-hand side of (16). Similarly, a $(\Phi \circ f)$-field L on M' belongs to the module on the right-hand side of (6′) if and only if for every $B \in \mathscr{B}$ the field $L|B$ belongs to the module on the right-hand side of (16). Thus (6′) follows directly from (16).

7. The differential space of a differential module. Let $(M, \mathscr{C})$ be a differential space and let Φ be, as usual, a mapping which assigns to every $p \in M$ a linear space $\Phi(p)$. Let $\mathscr{W}$ be a differential $\mathscr{C}$-modulus of Φ-fields on M, and let $n = \dim \mathscr{W}$.

Let Q be the set of all pairs (p, w), where $p \in M$ and $w \in \Phi(p)$. By the *projection* of Q onto M we shall mean the mapping $\pi \colon Q \to M$ defined by

$$(1) \qquad \pi(p, w) = p \quad \text{for } (p, w) \in Q.$$

If A is an open subset of M and $W_1, \ldots, W_n$ is a vector basis of $\mathscr{W}$ on the set A, then the formula

$$(2) \qquad f(p, q) = \big(p, x^i W_i(p)\big) \quad \text{for } p \in A \text{ and } q = (x^1, \ldots, x^n) \in E^n$$

defines a mapping $f \colon A \times E^n \to Q$ (E^n denotes here the n-dimensional Euclidean space). Each mapping f of the form (2) will be called *fundamental.*

Let $\mathscr{F}$ be the set of all real functions $\alpha \colon Q \to E$ such that, for every fundamental mapping (2), the composition $\alpha \circ f \colon A \times E^n \to E$ is a smooth function on the Cartesian product of the differential spaces $(A, \mathscr{C}_A)$ and $(E^n, \mathscr{E})$.

It can be proved that the set $\mathscr{F}$ is a differential structure on the set Q. The differential space $(Q, \mathscr{F})$ is said to be the *differential space of the differential modulus* $\mathscr{W}$. By definition Q is a fibre boundle with the basis M and projection π. The fibre is the n-dimensional linear space. If $(M, \mathscr{C})$ is an m-dimensional manifold, then $(Q, \mathscr{F})$ is an $(m+n)$-dimensional manifold.

For every Φ-field W on M let $\overline{W} \colon M \to Q$ be the mapping defined by

$$\overline{W}(p) = \big(p, W(p)\big) \quad \text{for } p \in M.$$

It can be proved that $W \in \mathscr{W}$ if and only if $\overline{W}$ is a smooth mapping of the differential space $(M, \mathscr{C})$ into the differential space $(Q, \mathscr{F})$.

8. Covariant derivative. Let $(M, \mathscr{C})$ be a differential space, let Φ be a mapping which assigns to every point $p \in M$ a linear space $\Phi(p)$, and let $\mathscr{W}$ be a $\mathscr{C}$-module of Φ-fields.

Suppose that for every $p \in M$, for every $v \in M_p$, and for every $W \in \mathscr{W}$ there is defined an element

$$\nabla_v W \in \Phi(p). \tag{1}$$

If $V \in \mathfrak{W}(M)$ is a smooth tangent vector field on M, then the symbol $\nabla_V W$ will always denote the linear Φ-field on M (called the *global interpretation* of the function (1)) defined by

$$\nabla_V W(p) = \nabla_{V(p)} W \quad \text{for } p \in M. \tag{2}$$

A function $\nabla_v W$ of two variables

$$v \in \bigcup_{p \in M} M_p \quad \text{and} \quad W \in \mathscr{W}$$

is said to be a *covariant derivative* in the $\mathscr{C}$-module $\mathscr{W}$ if it satisfies condition (1) and condition

$$\nabla_V W \in \mathscr{W} \quad \text{for every } V \in \mathfrak{W}(M) \text{ and every } W \in \mathscr{W}, \tag{3}$$

if it is an E-linear function of v, i.e.,

$$\nabla_{av} W = a \nabla_v W \quad \text{and} \quad \nabla_{v+v_1} W = \nabla_v W + \nabla_{v_1} W \quad \text{for } a \in E,\ v, v_1 \in M_p,\ W \in \mathscr{W}, \tag{4}$$

if it is an E-linear function of W, i.e.,

$$\nabla_v aW = a \nabla_v W \quad \text{and} \quad \nabla_v (W + W_1) = \nabla_v W + \nabla_v W_1 \quad \text{for } a \in W,\ v \in M_p,\ W_1 \in \mathscr{W}, \tag{5}$$

and if, moreover, it satisfies the condition

$$\nabla_v aW = \partial_v a \cdot W(p) + a(p) \cdot \nabla_v W \quad \text{for } a \in \mathscr{C},\ v \in M_p,\ W \in \mathscr{W}. \tag{6}$$

By a *global covariant derivative* in the $\mathscr{C}$-module $\mathscr{W}$ we shall mean a function which assigns to every $V \in \mathfrak{W}(M)$ and to every $W \in \mathscr{W}$ a Φ-field $\nabla_V W$ on M in such a way that (3) holds, $\nabla_V W$ is a $\mathscr{C}$-linear function of the variable V, i.e.,

$$\nabla_{aV} W = a \nabla_V W \quad \text{and} \quad \nabla_{V+V_1} W = \nabla_V W + \nabla_{V_1} W \quad \text{for } a \in \mathscr{C},\ V, V_1 \in \mathfrak{W}(M),\ W \in \mathscr{W}, \tag{7}$$

$\nabla_V W$ is an E-linear function of the variable W, i.e.,

$$\nabla_V aW = a \nabla_V W \quad \text{and} \quad \nabla_V (W + W_1) = \nabla_V W + \nabla_V W_1 \quad \text{for } a \in E,\ V \in \mathfrak{W}(M),\ W, W_1 \in \mathscr{W}, \tag{8}$$

and, moreover, that it satisfies the following condition

$$\nabla_V aW = \partial_V a \cdot W + a \cdot \nabla_V W \quad \text{for } a \in \mathscr{C},\ V \in \mathfrak{W}(M),\ W \in \mathscr{W}. \tag{9}$$

It is easy to verify that

8.1. *The global interpretation of a covariant derivative in a $\mathscr{C}$-module $\mathscr{W}$ is a global covariant derivative in $\mathscr{W}$.*

The following theorem is a partial converse of 8.1.

8.2. *If the differential space $(M, \mathscr{C})$ is of a finite dimension, then every global covariant derivative in $\mathscr{W}$ is the global interpretation of exactly one covariant derivative in $\mathscr{W}$.*

If $\nabla_V W$ is defined for every $V \in \mathfrak{W}(M)$ and $W \in \mathscr{W}$, we define $\nabla_v W$ for a $v \in M_p$ $(p \in M)$ and $W \in \mathscr{W}$ by the formula

$$\nabla_v W = (\nabla_V W)(p), \tag{10}$$

where $V \in \mathfrak{W}(M)$ is a vector field such that $v = V(p)$. The vector field V exists by 2.9. It is easy to verify that properties (7), (8) and (9) imply properties (4), (5), (6), respectively, under the hypothesis that the definition (10) is correct, i.e. that the right-hand side of (10) does not depend on the choice of V.

To show it, observe that for any fixed $W \in \mathscr{W}$ the formula

$$\mathcal{L}(V) = \nabla_V W \qquad (V \in \mathfrak{W}(M))$$

defines a $\mathscr{C}$-tensor $\mathcal{L} \in \mathfrak{L}_{\mathscr{C}}(\mathfrak{W}(M); \mathscr{W})$. By 5.6 there exists exactly one linear field L such that $\mathcal{L} = L'$, i.e. that L assigns to every $p \in M$ an E-linear mapping $L(p)\colon M_p \to \Phi(p)$ in a way such that for every $V \in \mathfrak{W}(M)$

$$L(p)\big(V(p)\big) = \mathcal{L}(V)(p).$$

In other words, if $p \in M$, $v \in M(p)$, $V \in \mathfrak{W}(M)$ and $V(p) = v$, then

$$\nabla_v W = \nabla_V W(p) = L(p)(v).$$

This proves the correctness of the definition (10).

In the sequel we shall always denote by the same symbol ∇ a covariant derivative in $\mathscr{W}$ and its global interpretation.

9. Covariant derivative on open subsets. As in Section 8, let $(M, \mathscr{C})$ be a differential space, let Φ be a mapping which assigns to every point $p \in M$ a linear space $\Phi(p)$, and let $\mathscr{W}$ be a $\mathscr{C}$-module of Φ-fields.

9.1. *If ∇ is a covariant derivative in $\mathscr{W}$, A is an open subset of M and $W \in \mathscr{W}$ is a Φ-field such that $W|A = 0$, then $\nabla_v W = 0$ for every $v \in M_p$, $p \in A$.*

Let $p \in A$ and let $\beta \in \mathscr{C}$ separate p in A, i.e.,

$$\beta|B = 1 \quad \text{for a neighbourhood } B \text{ of } p,$$

$$\beta|A_0 = 0 \quad \text{for an open set } A_0,\ A \cup A_0 = M.$$

Since $\beta W = 0$, we have, for any $v \in M_p$,

$$0 = \partial_v(\beta W) = \partial_v\beta \cdot W(p) + \beta(p) \cdot \nabla_v W.$$

Since $\beta|B = 1$, we have $\partial_v\beta = 0$ which proves that $\nabla_v W = 0$.

9.2. *If ∇ is a covariant derivative in $\mathscr{W}$, $W_1, W_2 \in \mathscr{W}$, and $W_1|B = W_2|B$ for a neighbourhood B of a point $p \in M$, then $\nabla_v W_1 = \nabla_v W_2$ for every $v \in M_p$.*

This follows directly from 9.1.

9.3. *Let ∇ be a covariant derivative in $\mathscr{W}$ and let A be an open subset of M. There exists exactly one covariant derivative in $\mathscr{W}_A$, denoted by ∇/A, such that*

$$(1) \qquad (\nabla/A)_v(W|A) = \nabla_v W$$

for every $p \in A$, $v \in A_p = M_p$ and $W \in \mathscr{W}$.

Consequently, if $p \in A$, $v \in M_p$, $W \in \mathscr{W}_A$, $W_1 \in \mathscr{W}$ and $W|B = W_1|B$ for a neighbourhood B of p, then

$$(2) \qquad (\nabla/A)_v W = \nabla_v W_1.$$

If $V \in \mathfrak{W}(A)$, $V_1 \in \mathfrak{W}(M)$, $W \in \mathscr{W}_A$, $W_1 \in \mathscr{W}$, $V|B = V_1|B$ and $W|B = W_1|B$ for an open set $B \subset A$, then

$$(3) \qquad \big((\nabla/A)_V W\big)|B = (\nabla_{V_1} W_1)|B.$$

In particular,

$$(4) \qquad (\nabla/A)_{V|A}(W|A) = (\nabla_V W)|A$$

for every $V \in \mathfrak{W}(M)$ and $W \in \mathscr{W}$.

If a covariant derivative ∇/A in $\mathscr{W}_A$ has property (1), then it has also property (2). In fact, under the hypotheses formulated in (2) we have, by Lemma 9.2 applied to ∇/A,

$$(\nabla/A)_v W = (\nabla/A)_v(W_1|A).$$

On the other hand, it follows from (1) that

$$(\nabla/A)_v(W_1|A) = \nabla_v W_1,$$

which implies (2).

Identity (2) defines $(\nabla/A)_v W$ uniquely (the right-hand side of (2) does not depend on the choice of W_1 on account of Lemma 9.2 applied to ∇). Thus if a covariant derivative ∇/A with the property (1) exists, it is unique, viz. it is defined by (2).

On the other hand, it is evident that if the expression $(\nabla/A)_v W$ has property (2), then it has property (1) (replace in (2) W_1 by W, and W by $W|A$). Then it has also property (3) since under the hypotheses formulated in (3) we have, for every $p \in B$,

$$(\nabla/A)_V W(p) = (\nabla/A)_{V(p)} W = \nabla_{V(p)} W_1 = \nabla_{V_1(p)} W_1 = \nabla_{V_1} W_1(p).$$

Assuming $B = A$, $V_1 = V$ and $W_1 = W$ in (3) we get also (4).

Thus, in order to prove 9.3, it suffices to show that the expression $(V/A)_v W$ defined by means of (2) is a covariant derivative in $\mathscr{W}_A$. The properties (1), (4), (5) and (6) from Section 8 of V/A follow from the same properties of V and from (2). To verify the property (3) from Section 8 (where, of course, M should be replaced by A, and $\mathscr{W}$ by $\mathscr{W}_A$) let us observe that for any $V \in \mathfrak{W}(A)$ and $W \in \mathscr{W}_A$ there exists an open covering $\mathscr{B}$ of A such that for every $B \in \mathscr{B}$ there are $V_B \in \mathfrak{W}(M)$ and $W_B \in \mathscr{W}$ with $V|B = V_B|B$ and $W|B = W_B|B$. By (3),

$$\big((V/A)_A W\big)|B = (V_{V_B} W_B)|B \in \mathscr{W}_B \quad \text{for every } B \in \mathscr{B}.$$

This implies, by 1.2, that $(V/A)_V W \in \mathscr{W}$.

The only covariant derivative V/A determined in $\mathscr{W}_A$ by V in $\mathscr{W}$ will be called the *restriction* of the covariant derivative V to the open subset A. For simplicity, we write V instead of V/A. In other words, if V is a covariant derivative in $\mathscr{W}$, we extend the meaning of the symbol $V_v W$ to the case where $W \in \mathscr{W}_A$, A is an open subset of M, $p \in A$ and $v \in M_p$. By the definition,

$$V_v W = V_v W_1, \tag{5}$$

where $W_1 \in \mathscr{W}$ is any field such that $W|B = W_1|B$ for a neighbourhood B of p.

9.4. *If A and B are open subsets of M, $B \subset A$, and V is a covariant derivative in $\mathscr{W}$, then*

$$(V/A)/B = V/B.$$

Let us observe that the following theorem can be proved in an analogous way.

9.5. *Let V be a global covariant derivative in $\mathscr{W}$ and let A be an open subset of M. There exists exactly one global covariant derivative in $\mathscr{W}_A$, denoted by V/A, such that*

$$(V/A)_{V|A}(W|A) = (V_V W)|A$$

for every $V \in \mathfrak{W}(M)$ and $W \in \mathscr{W}$. If $V \in \mathfrak{W}(A)$, $V_1 \in \mathfrak{W}(M)$, $W \in \mathscr{W}$, $W_1 \in \mathscr{W}$, $V|B = V_1|B$ and $W|B = W_1|B$ for an open set $B \subset A$, then

$$\big((V/A)_V W\big)|B = (V_{V_1} W_1)|B.$$

The only V/A in $\mathscr{W}_A$ will be called the *restriction* of the global covariant derivative V to the open subset A.

If V is a covariant derivative in $\mathscr{W}$, then the restriction of the global interpretation of V to A is equal to the global interpretation of the restriction of V to A. This follows directly from the properties (3) and (4) of the restriction of V.

10. Covariant derivative induced by a smooth mapping. As in the two previous sections, let $(M, \mathscr{C})$ be a differential space, Φ a function which assigns to every $p \in M$ a linear space $\Phi(p)$, and $\mathscr{W}$ a $\mathscr{C}$-module of Φ-fields. Suppose additionally that f is a smooth mapping from a differential space $(M', \mathscr{C}')$ into $(M, \mathscr{C})$.

Generalizing definition (2) from Section 8 we introduce the following notation: If $V \in \mathfrak{W}_f(M', M)$ (i.e., if V is a smooth vector f-field on M', tangent to M), $W \in \mathscr{W}$, and ∇ is a covariant derivative in $\mathscr{W}$, then $\nabla_V W$ denotes the linear $\Phi \circ f$-field on M' defined by

$$(1) \qquad (\nabla_V W)(p) = \nabla_{V(p)} W \quad \text{for } p \in M'.$$

10.1. *If ∇ is a covariant derivative in $\mathscr{W}$, the differential space $(M, \mathscr{C})$ is of finite dimension, $V \in \mathfrak{W}_f(M', M)$ and $W \in \mathscr{W}$, then $\nabla_V W \in \mathscr{W}_f$.*

Let $\mathscr{B}$ be an open covering of M such that for every $B \in \mathscr{B}$ there exists a vector basis $V_1, \ldots, V_m$ of $\mathfrak{W}(M)$ on the set B. The sets $A = f^{-1}(B)$, where $B \in \mathscr{B}$, form an open covering $\mathscr{A}$ of the space M'. By 3.8 we have $V|A = a^i \cdot V_i \circ f|A$ for certain functions $a^i \in \mathscr{C}'_A$. Hence

$$(\nabla_V W)|A = a^i \cdot \nabla_{V_i \circ f|A} W = a^i \cdot (\nabla_{V_i} W) \circ f|A.$$

Since $\nabla_{V_i} W \in \mathscr{W}_B$, we infer that $(\nabla_V W)|A \in (\mathscr{W}_B)_{f|A} = (\mathscr{W}_f)_A$. Consequently, $\nabla_V W \in \mathscr{W}_f$ by 1.2.

10.2. *Let $(M, \mathscr{C})$ be a differential space of finite dimension, let $\mathscr{W}$ be a differential module of Φ-fields on M and let ∇ be a covariant derivative in $\mathscr{W}$. There exists exactly one covariant derivative in $\mathscr{W}_f$, denoted by ∇^f, such that for every $p \in M'$, $v \in M'_p$, and $W \in \mathscr{W}$*

$$(2) \qquad \nabla^f_v (W \circ f) = \nabla_{\mathbf{d}f(v)} W.$$

Consequently, for every $V \in \mathfrak{W}(M')$ and $W \in \mathscr{W}$,

$$(3) \qquad \nabla^f_V (W \circ f) = \nabla_{\mathbf{d}f \circ V} W.$$

Let $\mathscr{B}$ be an open covering of M such that for every $B \in \mathscr{B}$ there exists a vector basis of $\mathscr{W}$ on B. The class $\mathscr{A}$ of all sets

$$(4) \qquad A = f^{-1}(B), \quad \text{where } B \in \mathscr{B},$$

is an open covering of M'.

Let $p \in M'$ and $v \in M'_p$. There exists a set (4) such that $p \in A$. Let $W_1, \ldots, W_n$ be a vector basis of $\mathscr{W}$ on the set B. By 4.6 the $\Phi \circ f$-fields $W_i \circ f|A$ $(i = 1, \ldots, n)$ form a vector basis of $\mathscr{W}_f$ on A.

Suppose that ∇^f is a covariant derivative in $\mathscr{W}_f$ such that (2) holds. It follows from the considerations of Section 9 that the symbol $\nabla^f_v W$ is well defined for every $W \in \mathscr{W}_{f|A} = (\mathscr{W}_f)_A$ and that the symbol $\nabla_{\mathbf{d}f(v)} \overline{W}$

is well defined for every $\overline{W} \in \mathscr{W}_B$ (more precisely, we should write here ∇^f/A and ∇/B instead of ∇^f and ∇). By (2),

$$\nabla^f_v(\overline{W} \circ f|A) = \nabla_{\mathbf{d}f(v)} \overline{W} \quad \text{for } \overline{W} \in \mathscr{W}_B. \tag{5}$$

If $W \in \mathscr{W}_f$, then $W|A = a^i \cdot W_i \circ f|A$ for certain functions $a^i \in \mathscr{C}'_A$. Hence

$$\begin{aligned} \nabla^f_v W = \nabla^f_v(W|A) &= \nabla^f_v(a^i \cdot W_i \circ f|A) \\ &= \partial_v a^i \cdot W_i(f(p)) + a^i(p) \cdot \nabla^f_v(W_i \circ f|A), \end{aligned}$$

that is, by (5),

$$\nabla^f_v W = \partial_v a^i \cdot W_i(f(p)) + a^i(p) \cdot \nabla_{\mathbf{d}f(v)} W_i. \tag{6}$$

Hence it follows that if the required covariant derivative does exist, it is only one, viz., it is defined by (6).

Assume now (6) as the definition of the symbol $\nabla^f_v W$. By a simple calculation we verify that the definition is correct, i.e., that the right-hand side of (6) does not depend on the choice of the set $A \in \mathscr{A}$ and of the basis $W_1, \ldots, W_n$ of $\mathscr{W}$ on B. It follows directly from (6) that the expression $\nabla^f_v W$ has properties (1), (4), (5) and (6) from Section 8 (where M should be replaced by M'). It has also the property (3) from Section 8 since, by 10.1, for $V \in \mathfrak{W}(M')$ and $A \in \mathscr{A}$,

$$(\nabla^f_V W)|A = \partial_V a^i \cdot (W_i \circ f|A) + a^i \cdot \nabla_{\mathbf{d}f \circ V|A} W_i \in (\mathscr{W}_f)_A$$

which proves, by 1.2, that $\nabla^f_V W \in \mathscr{W}_f$. Thus (6) defines a covariant derivative ∇^f in $\mathscr{W}_f$.

If $W \in \mathscr{W}$ and $p \in A \in \mathscr{A}$, let $B \in \mathscr{B}$ be such that (4) holds. Since $W|B = \beta^i \cdot W_i$ for certain $\beta^i \in \mathscr{C}_B$, we have

$$W \circ f|A = (\beta^i \circ f|A) \cdot (W_i \circ f|A).$$

Assuming in (6) $a^i = \beta^i \circ f|A$ we get

$$\begin{aligned} \nabla^f_v(W \circ f) &= \partial_v(\beta^i \circ f) \cdot W_i(f(p)) + \beta^i(f(p))\, \nabla_{\mathbf{d}f(v)} W_i \\ &= \partial_{\mathbf{d}f(v)} \beta^i \cdot W_i(f(p)) + \beta^i(f(p))\, \nabla_{\mathbf{d}f(v)} W_i = \nabla_{\mathbf{d}f(v)} W. \end{aligned}$$

This proves (2). And (3) is a direct consequence of (2).

The only covariant derivative ∇^f in $\mathscr{W}_f$ satisfying (2) will be called the *covariant derivative induced by* ∇ *and* f.

Suppose now additionally that $(M', \mathscr{C}')$ is a differential space of finite dimension and that g is a smooth mapping from a differential space $(M'', \mathscr{C}'')$ into $(M', \mathscr{C}')$. We shall prove that

10.3. *The following identity holds*

$$(\nabla^f)^g = \nabla^{f \circ g}. \tag{7}$$

By the definition $\nabla^{f\circ g}$ is the only covariant derivative in the $\mathscr{C}''$-module $\mathscr{W}_{f\circ g}$ (i.e., in $(\mathscr{W}_f)_g$ — see 4.3) such that

$$\nabla_v^{f\circ g}\big(W\circ(f\circ g)\big) = \nabla_{\mathbf{d}(f\circ g)(v)}W \tag{8}$$

for $v\in M''_p$, $p\in M''$, and $W\in\mathscr{W}$. Using property (2) for ∇^f and the analogous property for $(\nabla^f)^g$ we verify that the covariant derivative $(\nabla^f)^g$ satisfies (8) and therefore is identical with $\nabla^{f\circ g}$. In fact,

$$(\nabla^f)_v^g\big(W\circ(f\circ g)\big) = (\nabla^f)_v^g\big((W\circ f)\circ g\big) = \nabla^f_{\mathbf{d}g(v)}(W\circ f)$$
$$= \nabla_{\mathbf{d}f(\mathbf{d}g(v))}W = \nabla_{\mathbf{d}(f\circ g)(v)}W.$$

Consider now the case where $(M',\mathscr{C}')$ is a differential subspace of $(M,\mathscr{C})$.

10.4. *If $(M,\mathscr{C})$ is a differential space of a finite dimension, $\mathscr{W}$ is a differential $\mathscr{C}$-module of Φ-fields on M, ∇ is a covariant derivative in $\mathscr{W}$ and $A\subset M$, then there exists exactly one covariant derivative in $\mathscr{W}_A$, denoted by ∇^A, such that for every $p\in A$, $v\in A_p$, and for every $W\in\mathscr{W}$*

$$\nabla_v^A(W|A) = \nabla_v W. \tag{9}$$

Consequently, for every $V\in\mathfrak{W}(M)$ and $W\in\mathscr{W}$

$$\nabla^A_{V|A}(W|A) = (\nabla_A W)|A. \tag{10}$$

If A is an open subset of M, then

$$\nabla^A = \nabla/A. \tag{11}$$

To get the first part of 10.4 it suffices to assume in 10.2 $M' = A$, $\mathscr{C}' = \mathscr{C}_A$ and $f =$ the identity mapping on A. We recall that $\mathscr{W}_f = \mathscr{W}_A$ on account of 4.2.

If A is open, then ∇/A has property (9) by 9.3 (1). This implies (11).

Generalizing the definition on p. 71 we shall call ∇^A the *restriction of ∇ to A*.

10.5. *If $(M,\mathscr{C})$ is a differential space of a finite dimension, $B\subset A\subset M$, $(A,\mathscr{C}_A)$ is of finite dimension, $\mathscr{W}$ is a differential module of Φ-fields on M, and ∇ is a covariant derivative in $\mathscr{W}$, then*

$$\nabla^B = (\nabla^A)^B. \tag{12}$$

This follows directly from 10.3 where f is the identity mapping on A and g is the identity mapping on B.

To simplify notation, we often write ∇ instead of ∇^f or ∇^A.

To illustrate the notion of the covariant derivative ∇^f, let us assume that $(M,\mathscr{C})$ is a differential space of a finite dimension (in particular, a smooth manifold M with the class $\mathscr{C}$ of all smooth real functions on M) and that f is a *smooth curve*, i.e., a smooth mapping $f\colon (M',\mathscr{C}')\to(M,\mathscr{C})$, where M' is a bounded or unbounded interval of real numbers, and $\mathscr{C}'$ is the class of all smooth (= infinitely derivable) real functions on M'. Let ∇ be a covariant derivative on $(M, \mathscr{C})$, i.e., in the differential $\mathscr{C}$-module $\mathscr{W} = \mathfrak{W}(M)$ of all smooth tangent vector fields on M. By 10.2, ∇ determines uniquely a covariant derivative ∇^f in $\mathscr{W}_f = \mathfrak{W}_f(M', M)$ (see 4.8).

For every $x \in M'$ let $e(x)$ be the *unit vector* tangent to M' at the point x, i.e.,

$$e(x)(a) = \partial_{e(x)} a = a'(x) \quad \text{for } a \in \mathscr{C}',$$

where a' denotes the ordinary derivative of the function a. Thus e denotes the tangent unit-vector field on M' that assigns to every $x \in M'$ the tangent vector $e(x) \in M'_x$. If $W \in \mathscr{W}_f$, i.e., if W is a smooth vector f-field on M' tangent to M, then the *derivative* of W (more precisely: the *∇-derivative* of W) is the vector f-field $\mathbf{D}W$ defined as follows:

$$(13) \qquad \mathbf{D}W = \nabla^f_e W, \quad \text{i.e.,} \quad \mathbf{D}W(x) = \nabla^f_{e(x)} W \text{ for } x \in M'.$$

The following properties of W are equivalent:

1° W is a parallel translation on the curve f;

2° $\nabla^f W = 0$, i.e., $\nabla^f_v W = 0$ for every $v \in M'_x$, $x \in M'$;

3° $\mathbf{D}W = 0$.

The property 2° or 3° can be used as a definition of 1°.

11. Covariant derivative in a module of tensor fields. Similarly as in Section 5 we shall consider in this section a differential space $(M, \mathscr{C})$, $n+1$ functions Φ_j $(j = 1, \ldots, n+1)$ each of which assigns a linear space $\Phi_j(p)$ to any $p \in M$, and $n+1$ $\mathscr{C}$-modules $\mathscr{W}_j$ of Φ_j-fields $(j = 1, \ldots, n+1)$. The letter Φ will denote, as in Section 5, the function which assigns, to every $p \in M$, the linear space

$$\Phi(p) = \mathfrak{L}_E(\Phi_1(p), \ldots, \Phi_n(p); \Phi_{n+1}(p)),$$

and the letter $\mathscr{W}$ will denote the $\mathscr{C}$-module of all Φ-fields L satisfying the condition (4) from Section 5.

11.1. *Let ∇^i be a global covariant derivative in $\mathscr{W}_j$, $j = 1, \ldots, n+1$. There exists exactly one global covariant derivative ∇ in the $\mathscr{C}$-module*

$$(1) \qquad \mathscr{W}' = \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})$$

such that for every $V \in \mathfrak{W}(M)$, $W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n$, and $L \in \mathscr{W}'$

$$(2) \quad (\nabla_V L)(W_1, \ldots, W_n) = \nabla^{n+1}_V\big(L(W_1, \ldots, W_n)\big) - \\ -\big(L(\nabla^1_V W_1, W_2, \ldots, W_n) + L(W_1, \nabla^2_V W_2, W_3, \ldots, W_n) + \ldots + \\ + L(W_1, \ldots, W_{n-1}, \nabla^n_V W_n)\big).$$

The formula (2) defines uniquely a mapping

$$(3) \qquad \nabla_V L \colon \mathscr{W}_1 \times \ldots \times \mathscr{W}_n \to \mathscr{W}_{n+1}.$$

Thus if a global covariant derivative ∇ with the required property exists, it is unique, viz., it is defined by (2). On the other hand, it is easy to verify that the mapping (3) defined by (2) satisfies the conditions mentioned in the definition of a global derivative (see p. 68). Thus it is the required covariant derivative.

11.2. *Let $(M, \mathscr{C})$ be a differential space of a finite dimension, let $\mathscr{W}_1, \ldots, \mathscr{W}_n$ be differential modules, and let $\nabla^1, \ldots, \nabla^{n+1}$ be covariant derivatives in $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$, respectively. There exists exactly one covariant derivative ∇ in the $\mathscr{C}$-module $\mathscr{W}$ such that*

$$(4)\qquad (\nabla_v L)\big(W_1(p), \ldots, W_n(p)\big) = \nabla_v^{n+1}\big(L(W_1, \ldots, W_n)\big) - \\ -\big(L(p)\,(\nabla_v^1 W_1, W_2(p), \ldots, W_n(p)) + \ldots + \\ + L(p)\,(W_1(p), \ldots, W_{n-1}(p), \nabla_v^n W_n)\big)$$

for every $W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n$, $v \in M_p$, $p \in M$ and $L \in \mathscr{W}$.

Applying Theorem 11.1 to the global interpretations of the covariant derivatives $\nabla^1, \ldots, \nabla^{n+1}$, we get a global covariant derivative in the modulus (1) which is identified with $\mathscr{W}$ by the convention accepted in Section 5, p. 64. The global covariant derivative in $\mathscr{W}$ is the global interpretation of exactly one covariant derivative ∇ in $\mathscr{W}$, see 8.2. The covariant derivative ∇ in $\mathscr{W}$ is the only covariant derivative in $\mathscr{W}$ having property (4).

To formulate the next theorem let us suppose that $(M, \mathscr{C})$ is of a finite dimension, $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ are differential modules,

$$(5)\qquad \nabla^1, \ldots, \nabla^{n+1}$$

are covariant derivatives in $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ respectively, and f is a smooth mapping from a differential space $(M', \mathscr{C}')$ of a finite dimension into $(M, \mathscr{C})$. The covariant derivatives (5) uniquely determine, by 11.2, a covariant derivative ∇ in the differential $\mathscr{C}$-module,

$$(6)\qquad \mathscr{W} = \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1}).$$

By 10.2 the covariant derivative ∇ in (6) uniquely determines a covariant derivative ∇^f in the differential $\mathscr{C}$-module

$$(7)\qquad \mathscr{W}_f = \mathfrak{L}_{\mathscr{C}}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})_f$$

such that

$$(8)\qquad \nabla_v^f(L \circ f) = \nabla_{df(v)} L \quad \text{for } L \in \mathscr{W},\ v \in M'_p,\ p \in M'.$$

It follows from 6.1 that

$$(8')\qquad \mathscr{W}_f = \mathfrak{L}_{\mathscr{C}'}(\mathscr{W}_{1,f}, \ldots, \mathscr{W}_{n,f}; \mathscr{W}_{n+1,f}).$$

The covariant derivatives (5) uniquely determine, by 10.2, covariant derivatives $\nabla^{1,f}, \ldots, \nabla^{n+1,f}$ in the differential $\mathscr{C}'$-modules $\mathscr{W}_{1,f}, \ldots, \mathscr{W}_{n+1,f}$, respectively, such that

$$(9) \quad \nabla_v^{j,f}(W \circ f) = \nabla^j_{df(v)} W \quad \text{for } W \in \mathscr{W}_j,\ v \in M'_p,\ p \in M',\ j = 1, \ldots, n+1.$$

The covariant derivatives uniquely determine, by 11.2, a covariant derivative $\overline{\nabla}$ in the $\mathscr{C}'$-module (7) such that

$$(10) \quad \begin{aligned}(\overline{\nabla}_v L)\big(W_1(p), \ldots, W_n(p)\big) = {} & \nabla_v^{n+1,f}\big(L(W_1, \ldots, W_n)\big) - \\ & -\Big(L(p)\big(\nabla_v^{1,f} W_1, W_2(p), \ldots, W_n(p)\big) + \ldots + \\ & + L(p)\big(W_1(p), \ldots, W_{n-1}(p), \nabla_v^{n,f} W_n\big)\Big)\end{aligned}$$

for $W_1 \in \mathscr{W}_{1,f}, \ldots, W_n \in \mathscr{W}_{n,f}$, $v \in M'_p$, $p \in M'$ and $L \in \mathscr{W}_f$. We shall prove that

$$(11) \qquad \nabla^f = \overline{\nabla}.$$

In other words, we shall prove:

11.3. *Under the hypotheses mentioned, the following identity holds*

$$(12) \quad \begin{aligned}(\nabla_v^f L)\big(W_1(p), \ldots, W_n(p)\big) = {} & \nabla_v^{n+1,f}\big(L(W_1, \ldots, W_n)\big) - \\ & -\Big(L(p)\big(\nabla_v^{1,f} W_1, W_2(p), \ldots, W_n(p)\big) + \ldots + \\ & + L(p)\big(W_1(p), \ldots, W_{n-1}(p), \nabla_v^{n,f} W_n\big)\Big)\end{aligned}$$

for $W_1 \in \mathscr{W}_{1,f}, \ldots, W_n \in \mathscr{W}_{n,f}$, $v \in M'_p$, $p \in M'$ and $L \in \mathscr{W}_f$. Consequently,

$$(13) \quad \begin{aligned}\nabla_v^{n+1,f}\big((\mathfrak{L} \circ f)(W_1, \ldots, W_n)\big) = {} & (\nabla_{df(v)} \mathfrak{L})\big(W_1(p), \ldots, W_n(p)\big) + \\ & + \mathfrak{L}\big(f(p)\big)\big(\nabla_v^{1,f} W_1, W_2(p), \ldots, W_n(p)\big) + \\ & + \ldots + \mathfrak{L}\big(f(p)\big)\big(W_1(p), \ldots, W_{n-1}(p), \nabla_v^{n,f} W_n\big)\end{aligned}$$

for $W_1 \in \mathscr{W}_{1,f}, \ldots, W_n \in \mathscr{W}_{n,f}$, $v \in M'_p$, $p \in M'$ and $\mathfrak{L} \in \mathscr{W}$.

We shall prove the identity

$$(14) \qquad \nabla_v^f L = \overline{\nabla}_v L \quad (v \in M'_p,\ p \in M')$$

first in the case where $L = \mathfrak{L} \circ f$, $\mathfrak{L} \in \mathscr{W}$. Let $w_1 \in \Phi_1\big(f(p)\big), \ldots, w_n \in \Phi_n\big(f(p)\big)$ be arbitrary but fixed, and let $V_1 \in \mathscr{W}_1, \ldots, V_n \in \mathscr{W}_n$ be such that

$$V_1\big(f(p)\big) = w_1, \quad \ldots, \quad V_n\big(f(p)\big) = w_n.$$

The linear fields $W_1 = V_1 \circ f \in \mathscr{W}_{1,f}, \ldots, W_n = V_n \circ f \in \mathscr{W}_{n,f}$ satisfy the equations

$$W_1(p) = w_1, \quad \ldots, \quad W_n(p) = w_n.$$

Consequently, by (7), (9), (4) and (10),

$$
\begin{aligned}
(\nabla_v^f L)(w_1, \ldots, w_n) &= (\nabla_v^f \mathbf{L} \circ f)(w_1, \ldots, w_n) = (\nabla_{\mathrm{d}f(v)} \mathbf{L})(w_1, \ldots, w_n) \\
&= \nabla_{\mathrm{d}f(v)}^{n+1}\big(\mathbf{L}(V_1, \ldots, V_n)\big) - \big(\mathbf{L}(f(p))\big)(\nabla_{\mathrm{d}f(v)}^1 V_1, w_2, \ldots, w_n) + \ldots + \\
&\quad + \mathbf{L}(f(p))(w_1, \ldots, w_{n-1}, \nabla_{\mathrm{d}f(v)}^n V_n) = \nabla_v^{n+1,f} L(W_1, \ldots, W_n) - \\
&\quad - \big(L(p)(\nabla_v^{1,f} W_1, w_2, \ldots, w_n) + \ldots + L(p)(w_1, \ldots, w_{n-1}, \nabla_v^{n,f} W_n)\big) \\
&= (\overline{\nabla}_v L)(w_1, \ldots, w_n).
\end{aligned}
$$

This proves (14) in the particular case under consideration.

For every $L \in \mathscr{W}_f$ and for a fixed point $p \in M'$ there exist a neighbourhood A of p, functions $a^i \in \mathscr{C}'$ and $\mathbf{L}_i \in \mathscr{W}$ such that

$$L|A = (a^i \cdot \mathbf{L}_i \circ f)|A.$$

Consequently, for $v \in M'_p$,

$$
\begin{aligned}
\nabla_v^f L &= \nabla_v^f(a^i \cdot \mathbf{L}_i \circ f) = \partial_v a^i \cdot \mathbf{L}_i \circ f(p) + a^i(p) \cdot \nabla_v^f(\mathbf{L}_i \circ f) \\
&= \partial_v a^i \cdot \mathbf{L} \circ f(p) + a^i(p) \cdot \overline{\nabla}_v(\mathbf{L}_i \circ f) = \overline{\nabla}_v(a^i \cdot \mathbf{L}_i \circ f) = \overline{\nabla}_v L.
\end{aligned}
$$

This proves (14). Since L is arbitrary in (14), (11) is true. (12) is another formulation of (11), and (13) is a particular case of (12) (if $L = \mathbf{L} \circ f$, $\mathbf{L} \in \mathscr{W}$).

To illustrate Theorem 11.2 let us suppose that $(M, \mathscr{C})$ is a differential space of a finite dimension. Let T be the smallest set such that

1° the numbers 0 and 1 are in T,

2° if $t_1, \ldots, t_{n+1} \in T$ $(n > 0)$, then the sequence $t = (t_1, \ldots, t_{n+1})$ is in T.

We shall define, by induction, functions Φ_t on M and modules $\mathscr{W}_t$ of Φ_t-fields as follows:

1) $\Phi_0(p) = E$ for every $p \in M$ and $\mathscr{W}_0 = \mathscr{C}$,

2) $\Phi_1(p) = M_p$ for every $p \in M$ and $\mathscr{W}_1 = \mathfrak{W}(M)$,

3) if $t = (t_1, \ldots, t_{n+1}) \in T$, then

$$\Phi_t(p) = L_E\big(\Phi_{t_1}(p), \ldots, \Phi_{t_n}(p); \Phi_{t_{n+1}}(p)\big) \quad \text{for } p \in M,$$

and $\mathscr{W}_t$ is the $\mathscr{C}$-module of all Φ_t-fields L on M such that $L'_t(W_1, \ldots, W_n) \in \mathscr{W}_{t_{n+1}}$ for $W_1 \in \mathscr{W}_{t_1}, \ldots, W_n \in \mathscr{W}_{t_n}$ (see Section 5 (3)).

It follows from 5.3 (by induction with respect to t) that, for every $t \in T$, $\mathscr{W}_t$ is a differential module.

Suppose that ∇ is a covariant derivative on $(M, \mathscr{C})$, i.e., in the differential module $\mathscr{W}_1 = \mathfrak{W}(M)$. There exists exactly one function which assigns, to every $t \in T$, a covariant derivative ∇^t in the differential module $\mathscr{W}_t$ in a way such that

a) $\nabla^0 = \partial$, i.e., $\nabla_v^0 a = \partial_v a$ for $v \in M_p$, $p \in M$ and $a \in \mathscr{C}$;

b) $\nabla^1 = \nabla$;

c) if $t = (t_1, \ldots, t_{n+1}) \in T$, then ∇^t is the only covariant derivative obtained from $\nabla^{t_1}, \ldots, \nabla^{t_{n+1}}$ by means of Theorem 11.2.

In the pointwise interpretation, $\mathscr{W}_t$ is the $\mathscr{C}$-module of all *smooth tangent tensor fields on M of type $t \in T$.*

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