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THERE IS NO UNIVERSAL-PROJECTING HOMEOMORPHISM OF THE CANTOR SET

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Suppose X and Y are topological spaces, and f and g are homeomorphisms of X and Y, respectively, onto themselves. Then f is said *to project onto* g, and g is said *to be raised to* f, if there exists a mapping φ of X onto Y such that for each $x \epsilon X$,

$$\varphi f(x) = g\varphi(x).$$

In [2] and [3] R. D. Anderson asks the following question: Is there a homeomorphism h of the Cantor set C onto itself such that if g is any homeomorphism of C onto itself, h projects onto g? If so, then h would project onto any homeomorphism of a compact metric space onto itself; see [1].

It is the purpose of this note to observe that no such homeomorphism exists.

THEOREM. *Suppose h is a homeomorphism of C onto itself; then there is a homeomorphism f of C onto itself such that h does not project onto f.*

Proof. Select $x_0 \epsilon C$ and consider the sequence $\{h^n(x_0)\}_{n=1}^{\infty}$. There exists a convergent subsequence $\{h^{n_i}(x_0)\}_{i=1}^{\infty} \to y$. By theorem 21 of [4] there is a number $q \epsilon [0, 1]$ such that the set $\{n_i q (\text{mod } 1)\}_{i=1}^{\infty}$ is dense in $[0, 1]$. Let K be the circle group realized as $[0, 1]$ with the endpoints identified. Then $g: K \to K$ defined by $g(x) = q+x$ (mod 1) is a rotation such that the set $\{g^{n_i}(x)\}_{i=1}^{\infty}$ is dense for each $x \epsilon K$. By theorem 4 in [1] g can be raised to a homeomorphism f of C onto itself. That is, there is a mapping $\pi: C \to K$ such that $\pi f = g\pi$. Now suppose h projected onto f. Then a mapping φ exists from C onto C with $\varphi h = f\varphi$. Since $\{h^{n_i}(x_0)\}_{i=1}^{\infty} \to y$, $\{\varphi h^{n_i}(x_0)\}_{i=1}^{\infty} \to \varphi(y)$. And since $\varphi h^{n_i}(x_0) = f^{n_i}\varphi(x_0)$, $\{f^{n_i}\varphi(x_0)\}_{i=1}^{\infty} \to \varphi(y)$. Similarly, it follows that $\{g^{n_i}\pi\varphi(x_0)\}_{i=1}^{\infty} \to \pi\varphi(y)$; but this is impossible since $\{g^{n_i}\pi\varphi(x_0)\}_{i=1}^{\infty}$ is dense.

REFERENCES

[1] R. D. Anderson, *On raising flows and mappings*, Bulletin of the American Mathematical Society 69 (1963), p. 259-264.

[2] — *Problème P 460*, Colloquium Mathematicum 12 (1964), p. 148.

[3] — *Quasi-universal flows*, Fundamenta Mathematicae (to appear).

[4] Hermann Weyl, *Über die Gleichverteilung von Zahlen mod Eins*, Mathematische Annalen 77 (1916), p. 313-352.

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