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ON COMPACTIFICATION OF T_0-SPACES

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Wallman proved in [4] that the set of ultrafilters consisting of closed subsets of a T_1-space, with a topology introduced in the usual way, is a compact T_1-extension of this space. This method was used by Banaschewski in [1] to obtain compact T_2-extensions of spaces having the so called Wallman basis (the ultrafilters of the Wallman basis are the points of this compactifications). By a suitable choice of the Wallman basis, Banaschewski obtained compactifications for several classes of spaces such as the Čech-Stone compactification of normal spaces, the Alexandroff one-point compactification of locally compact T_2-spaces, the Banaschewski compactification of null-dimensional T_0-spaces and the Freudenthal compactification of rim-compact T_2-spaces. In this paper, a similar method is used to construct some compact extensions of T_0-spaces.

1. The space Ω. Let $\mathfrak{A}$ be a finitely multiplicative family of subsets of a set X, Ω' — the set of all ultrafilters of the family $\mathfrak{A}$, Ω'' — the set of some filters of this family (not necessarilly all of them; filters are well defined because of the multiplicativity of $\mathfrak{A}$).

Let $\Omega = \Omega' \cup \Omega''$ and $\Omega_A = \{\xi \in \Omega : A \in \xi\}$. The points of Ω are filters, hence (for each $\xi \in \Omega$) $A \cap B \in \xi$ iff $A \in \xi$ and $B \in \xi$, i.e.

(1.1) $\Omega_{A\cap B} = \Omega_A \cap \Omega_B$, $A, B \in \mathfrak{A}$.

Moreover,

(1.2) $\Omega_A = \emptyset$ *iff* $A = \emptyset$.

In fact, if $A \neq \emptyset$, the filter $\{A\}$ may be extended to a maximal filter ξ, i.e. to an ultrafilter $\xi \in \Omega$; clearly $\xi \in \Omega_A$. The converse implication is obvious, because $\emptyset \notin \xi$.

Taking the family $\{\Omega_A\}_{A\in\mathfrak{A}}$ as a subbasis of the family of closed subsets, we introduce a topology in Ω. The sets

$$\Omega_{A_1} \cup \ldots \cup \Omega_{A_n}, \quad A_1, \ldots, A_n \in \mathfrak{A},$$

form a basis $\mathfrak{B}$ for the family of all closed sets in Ω.

(1.3) *The basis $\mathfrak{B}$ is finitely multiplicative.*

Proof. Let $B_1, \ldots, B_n \in \mathfrak{B}$, i.e. $B_k = \Omega_{A_1^k} \cup \ldots \cup \Omega_{A_{m_k}^k}$, $k = 1, 2, \ldots, n$. According to (1.1) we have

$$\bigcap_{k=1}^{n} B_k = \bigcap_{k=1}^{n} \bigcup_{j=1}^{m_k} \Omega_{A_j^k} = \bigcup_{\substack{(j_1,\ldots,j_n)\\ 1\leqslant j_k\leqslant m_k}} \bigcap_{k=1}^{n} \Omega_{A_{j_k}^k} = \bigcup_{\substack{(j_1,\ldots,j_n)\\ 1\leqslant j_k\leqslant m_k}} \Omega_{\bigcap\limits_{k=1}^{n} A_{j_k}^k} = \bigcup_{t\in T} \Omega_{A_t},$$

where T is a finite set and $A_t \in \mathfrak{A}$. Thus $\bigcap\limits_{k=1}^{n} B_k \in \mathfrak{B}$.

THEOREM 1. *Ω is a compact T_0-space.*

Proof. To show the compactness, we must prove that each maximal centered family $\boldsymbol{B} = \{B_t\}_{t\in T}$ of closed subsets from the basis $\mathfrak{B}$ has a non-void intersection.

First we prove that

(1) for each $B_t = \Omega_{A_1^t} \cup \ldots \cup \Omega_{A_{n_t}^t}$ there exists an integer $k = k(t)$, $1 \leqslant k \leqslant n_t$, such that $\Omega_{A_k^t} \in \boldsymbol{B}$.

Suppose, for some $B_{t_0} \in \boldsymbol{B}$ and each k, $1 \leqslant k \leqslant n_{t_0}$, $\Omega_{A_k^{t_0}} \notin \boldsymbol{B}$. The set $\Omega_{A_k^{t_0}}$ belongs to the basis $\mathfrak{B}$ and $\boldsymbol{B} \subset \mathfrak{B}$ is a maximal centered family, thus for each k there exists a $B_{t_k} \in \boldsymbol{B}$ such that $\Omega_{A_k^{t_0}} \cap B_{t_k} = \emptyset$. Hence, for each k, $1 \leqslant k \leqslant n_{t_0}$, $\Omega_{A_k^{t_0}} \cap \bigcap\limits_{k=1}^{n_{t_0}} B_{t_k} = \emptyset$. Thus, by $\bigcup\limits_{k=1}^{n_{t_0}} \Omega_{A_k^{t_0}} = B_{t_0}$, we have $B_{t_0} \cap \bigcap\limits_{k=1}^{n_{t_0}} B_{t_k} = \emptyset$. By (1.3), $\bigcap\limits_{k=1}^{n_{t_0}} B_{t_k}$ belongs to $\mathfrak{B}$. But $B_{t_k} \in \boldsymbol{B}$, and $\boldsymbol{B} \subset \mathfrak{B}$ is a maximal centered family, therefore $\bigcap\limits_{k=1}^{n_{t_0}} B_{t_k} \in \boldsymbol{B}$. By $B_{t_0} \cap \bigcap\limits_{k=1}^{n_{t_0}} B_{t_k} = \emptyset$, we have $B_{t_0} \notin \boldsymbol{B}$, contrary to the assumption.

Consider the set $\xi = \{A \in \mathfrak{A}\colon \Omega_A \in \boldsymbol{B}\}$. We shall prove, that

(2) ξ is an ultrafilter of the family $\mathfrak{A}$.

First, let us see that

(a) ξ is a filter of the family $\mathfrak{A}$.

Since $\Omega_A \neq \emptyset$ for each $\Omega_A \in B$, we have $A \neq \emptyset$ by (1.2). Thus, by definition of ξ, $\emptyset \notin \xi$. Let $A, B \in \xi$ (i.e. Ω_A, $\Omega_B \in \boldsymbol{B}$); then $\Omega_{A\cap B} = \Omega_A \cap \Omega_B \in \boldsymbol{B}$ (for $\boldsymbol{B}$ is a maximal centered family of sets belonging to the multiplicative basis $\mathfrak{B}$). Thus $A \cap B \in \xi$. Finally, let $A \in \xi$ (i.e. $\Omega_A \in \mathfrak{B}$) and $A \subset B$, $B \in \mathfrak{A}$. The $\Omega_A \subset \Omega_B$ and hence, by the maximality of $\boldsymbol{B}$, $\Omega_B \in \boldsymbol{B}$, i.e. $B \in \xi$. Thus ξ is a filter.

(b) ξ is maximal.

In fact, let $B \in \mathfrak{A}$ and $B \cap A \neq \emptyset$ for each $A \in \xi$. Then $\Omega_B \cap \Omega_A = \Omega_{B\cap A} \neq \emptyset$ for each $\Omega_A \in \boldsymbol{B}$. Thus, by (1), Ω_B has a non-void intersection with all $B_t \in \mathbf{B}$. Since $\mathbf{B}$ is maximal, $\Omega_B \in \mathbf{B}$, i.e. $B \in \xi$. Thus ξ is a maximal filter, belonging to Ω.

The definition of ξ implies $\xi \in \Omega_A$ for each $\Omega_A \in \mathbf{B}$. Thus, from (1) it follows that $\xi \in B_t$ for each $B_t \in \mathbf{B}$ and hence $\bigcap_{B_t \in B} B_t \supset \{\xi\}$. The family $\mathbf{B}$ has a non-void intersection, which is nothing else as the compactness of Ω.

Ω is a T_0-space, for if $\xi', \xi'' \in \Omega$ and $\xi' \neq \xi''$, then there exists an $A \in \mathfrak{A}$ such that $A \in \xi'$ and $A \notin \xi''$, i.e. $\xi' \in \Omega_A$ and $\xi'' \notin \Omega_A$; thus the points ξ', ξ'' may be T_0-separeted.

(1.4) *Ω is a T_1-space iff $\Omega = \Omega'$.*

Proof. A filter $\xi \in \Omega \setminus \Omega'$ may be extended to an ultrafilter $\xi^* \in \Omega$. If $\xi \in \Omega_A$, i.e. $A \in \xi$, then $A \in \xi^*$ and $\xi^* \in \Omega_A$. Thus the point ξ cannot be T_1-separeted from ξ^*, hence Ω is not a T_1-space.

Let ξ', ξ'' be two-different ultrafilters from $\Omega = \Omega'$. Then there exist $\xi'_t \in \xi'$ and $\xi''_\tau \in \xi''$ such that $\xi'_t \cap \xi''_\tau = \emptyset$, i.e. $\xi''_\tau \notin \xi'$ and $\xi'_t \notin \xi''$. Hence $\xi' \in \Omega_{\xi'_t}$ and $\xi'' \notin \Omega_{\xi'_t}$, $\xi' \notin \Omega_{\xi''_\tau}$ and $\xi'' \in \Omega_{\xi''_\tau}$. Thus Ω is a T_1-space.

In the sequel we shall assume that the (finitely multiplicative) family $\mathfrak{A}$ covers the set X. Then, for each point $x \in X$, the non-void family $\xi_x = \{A \in \mathfrak{A}: x \in A\}$ (consisting of all $\mathfrak{A}$-neighbourhoods of x) is a filter from Ω.

Assume that $\Omega'' = \{\xi_x\}_{x \in X}$ (i.e. $\Omega = \Omega' \cup \Omega''$ consists only of all ultrafilters of $\mathfrak{A}$ and filters of the form ξ_x, $x \in X$).

(1.5) $A_1 \cup \ldots \cup A_n = X$ *iff* $\Omega_{A_1} \cup \ldots \cup \Omega_{A_n} = \Omega$.

Proof. Let $A_1 \cup \ldots \cup A_n = X$ and $\xi \in \Omega$.

Let $\xi = \xi_x$ (i.e. ξ is the filter of neighbourhoods of x). There exists an integer k such, that $x \in A_k$, thus $\xi_x \in \Omega_{A_k} \subset \Omega_{A_1} \cup \ldots \cup \Omega_{A_n}$.

Let ξ be an ultrafilter. Then there exists an integer k such that $A_k \in \xi$ (for if we have $A_k \notin \xi$ for each k, then taking $\xi_{t_k} \in \xi$ with the property $A_k \cap \xi_{t_k} = \emptyset$, we have $\bigcup_{k=1}^{n} A_k \cap \bigcap_{k=1}^{n} \xi_{t_k} = \emptyset$, i.e. $X \cap \bigcap_{k=1}^{n} \xi_{t_k} = \emptyset$, contrary to $\bigcap_{k=1}^{n} \xi_{t_k} \neq \emptyset$). Thus $\xi \in \Omega_{A_k}$, and, in consequence, $\Omega = \Omega_{A_1} \cup \ldots \cup \Omega_{A_n}$. Conversely, let $x \in X$. Then $\xi_x \in \Omega = \Omega_{A_1} \cup \ldots \cup \Omega_{A_n}$. Thus $\xi_x \in \Omega_{A_k}$, i.e. $x \in A_k$. This implies $X = A_1 \cup \ldots \cup A_n$.

(1.6) *Ω is a T_1-space iff*

(i) *for each $x \in X$ and each $A \in \mathfrak{A}$, if $x \notin A$, then there exists a $B \in \mathfrak{A}$ such that $x \in B$ and $A \cap B = \emptyset$.*

Proof. Due to (1.4), Ω is a T_1-space iff ξ_x is an ultrafilter for each each $x \in X$.

Now, let ξ_x be an ultrafilter. If $x \notin A$, then $A \notin \xi_x$. Hence there exists a $B \in \xi_x$ (thus $x \in B$) such that $A \cap B = \emptyset$.

Conversely, we must prove that ξ_x is an ultrafilter for each $x \in X$. Let $A \cap \xi_x^t \neq \emptyset$ for each $\mathfrak{A}$-neighbourhood ξ_x^t of x. Now, in virtue of (i), $x \in A$, i.e. $A \in \xi_x$, whence ξ_x is an ultrafilter.

(1.7) *Ω is a T_2-space iff Ω is a T_1-space and*

(ii) *for each $A, B \in \mathfrak{A}$, if $A \cap B = \emptyset$, then there exists a collection $\{A_1, \ldots, A_m, B_1, \ldots, B_n\}$, $A_1, \ldots, A_m, B_1, \ldots, B_n \in \mathfrak{A}$ such that*

$$A \cap \bigcup_{k=1}^{m} A_k = \emptyset, \qquad B \cap \bigcup_{l=1}^{n} B_l = \emptyset \qquad \text{and} \qquad \bigcup_{k=1}^{m} A_k \cup \bigcup_{l=1}^{n} B_l = X$$

(i.e. Ω is a T_2-space iff Ω is a T_1-space and $\mathfrak{A}$ is "structurally" normal).

Proof. Let $A, B \in \mathfrak{A}$ be disjoint sets. Then $\Omega_A \cap \Omega_B = \Omega_{A \cap B} = \emptyset$, whence, in consequence of normality of the compact T_2-space Ω, there exist disjoint open sets

$$U = \bigcup_{t \in T} (\Omega \setminus B_t), \qquad V = \bigcup_{\tau \in T} (\Omega \setminus B_\tau).$$

(B_t, B_τ are closed sets from the basis in Ω) covering the sets Ω_A and Ω_B respectively, i.e. $\Omega_A \subset U$ and $\Omega_B \subset V$. But the sets Ω_A and Ω_B, being closed subsets of a compact space, are compact, and therefore we can choose finite subcoverings of them:

$$\Omega_A \subset \bigcup_{i=1}^{m} (\Omega \setminus B_{t_i}) = U_0, \qquad \Omega_B \subset \bigcup_{j=1}^{n} (\Omega \setminus B_{\tau_j}) = V_0 \qquad (U_0 \cap V_0 = \emptyset),$$

of course). The first inclusion implies, in virtue of (1.3), that

$$\emptyset = \Omega_A \cap \bigcap_{i=1}^{m} B_{t_i} = \Omega_A \cap \Big(\bigcup_{k \in K} \Omega_{A_k}\Big) = \bigcup_{k \in K} \Omega_A \cap \Omega_{A_k},$$

where K is a finite set. Hence $\Omega_A \cap \Omega_{A_k} = \emptyset$ for each $k \in K$, thus $\Omega_{A \cap A_k} = \emptyset$ and therefore $A \cap A_k = \emptyset$. Finally, $A \cap (\bigcup_{k \in K} A_k) = \emptyset$. Similarly, the second inclusion implies $B \cap \Big(\bigcup_{l \in L} B_l\Big) = \emptyset$, where L is a finite set.

Moreover,

$$\emptyset = U_0 \cap V_0 = \bigcup_{i=1}^{m} (\Omega \setminus B_{t_i}) \cap \bigcup_{j=1}^{n} (\Omega \setminus B_{\tau_j}) = (\Omega \setminus \bigcap_{i=1}^{m} B_{t_i}) \cap (\Omega \setminus \bigcap_{j=1}^{n} B_{\tau_j})$$
$$= \Omega \setminus (\bigcap_{i=1}^{m} B_{t_i} \cup \bigcap_{j=1}^{n} B_{\tau_j}).$$

Hence

$$\Omega = \bigcap_{i=1}^{m} B_{t_i} \cup \bigcap_{j=1}^{n} B_{\tau_j} = \bigcup_{k \in K} \Omega_{A_k} \cup \bigcup_{l \in L} \Omega_{B_l}.$$

Thus, according to (1.5), $\bigcup_{k \in K} A_k \cup \bigcup_{l \in L} B_l = X$.

Conversely, let $\xi', \xi'' \in \Omega$ and $\xi' \neq \xi''$; thus there exists a $\xi'_t \in \xi'$ and a $\xi''_\tau \in \xi''$ such that $\xi'_t \cap \xi''_\tau = \emptyset$ (ξ', ξ'' being ultrafilters). The last equality implies the existence of a covering consisting of the sets $A_1, \ldots, A_m, B_1, \ldots, B_n$ such that $\xi'_t \cap A_k \subset \xi'_t \cap (\bigcup_{k \in K} A_k) = \emptyset$, $k = 1, 2, \ldots, m$ and $\xi''_\tau \cap B_l = \emptyset$, $l = 1, 2, \ldots, n$ which means that $A_k \notin \xi'$, $B_l \notin \xi''$, $k = 1, 2, \ldots, m$, $l = 1, 2, \ldots, n$. Now, $\xi' \in \Omega \setminus (\Omega_{A_1} \cup \ldots \cup \Omega_{A_m})$ and $\xi'' \in \Omega \setminus (\Omega_{B_1} \cup \ldots \cup \Omega_{B_n})$. But $\{A_1, \ldots, A_m, B_1, \ldots, B_n\}$ covers the set X, thus, by (1.5), $\Omega_{A_1} \cup \ldots \cup \Omega_{A_m} \cup \Omega_{B_1} \cup \ldots \cup \Omega_{B_n} = \Omega$.

Since $[\Omega \setminus (\Omega_{A_1} \cup \ldots \cup \Omega_{A_m})] \cap [\Omega \setminus (\Omega_{B_1} \cup \ldots \cup \Omega_{B_n})] = \Omega \setminus (\Omega_{A_1} \cup \ldots \cup \Omega_{A_m} \cup \Omega_{B_1} \cup \ldots \cup \Omega_{B_n}) = \emptyset$, the points ξ', ξ'' may be separated by open disjoint sets. Thus Ω is a T_2-space.

In [1] there was given a similar necessary and sufficient condition for the space Ω to be a T_2-space.

2. The mapping Φ and the extension $\mathfrak{A}_X$. Let us define a mapping $\Phi: X \to \Omega$ by $\Phi(x) = \xi_x$.

We introduce in X a topology taking as a subbasis of the family of closed sets the family $\mathfrak{A}$.

It is easy to check that

(2.1) *Φ is one-to-one iff X is a T_0-space.*

(2.2) *$\Phi^{-1}(\Omega_A) = A$ for each $A \in \mathfrak{A}$.*

Proof. $\Phi^{-1}(\Omega_A) \subset A$, for if $x \in \Phi^{-1}(\Omega_A)$, then $\Phi(x) = \xi_x \in \Omega_A$, hence $A \in \xi_x$ ane, finally, $x \in A$.

Now, let $x \in A$. Then $A \in \xi_x = \Phi(x)$, which means that $\Phi(x) \in \Omega_A$. Thus $x \in \Phi^{-1}(\Omega_A)$ — the converse inclusion.

For each B belonging to the basis of X, i.e. for each set of the form $B = A_1 \cup \ldots \cup A_n$, where $A_1, \ldots, A_n \in \mathfrak{A}$ let $B^\Omega = \Omega_{A_1} \cup \ldots \cup \Omega_{A_n}$.

(2.3) $X_0 \subset B = A_1 \cup \ldots \cup A_n$ *iff* $\Phi(X_0) \subset B^\Omega$.

Proof. Let $X_0 \subset B$ and let $\xi \in \Phi(X_0)$. Thus $\xi = \xi_x$ for some $x \in X_0$. Then $x \in A_k$, for some integer k, hence $\xi_x \in \Omega_{A_k} \subset B^\Omega$.

Conversely, let $\Phi(X_0) \subset B^\Omega$ and $x \in X_0$. Thus $\Phi(x) = \xi_x \in B^\Omega$, i.e. $\xi_x \in \Omega_{A_k}$ for some k. Hence $x \in A_k \subset B$.

(2.4) *If $B_t = A_1^t \cup \ldots \cup A_{n_t}^t$, where $A_k^t \in \mathfrak{A}$, $k = 1, 2, \ldots, n_t$, then*

$$\Phi(\bigcap_{t \in T} B_t) = \bigcap_{t \in T} B_t^\Omega \cap \Phi(X).$$

Proof. According to (2.3) we have

$$\Phi(\bigcap_{t \in T} B_t) = \Phi(\bigcap_{t \in T} B_t) \cap \Phi(X) \subset \bigcap_{t \in T} \Phi(B_t) \cap \Phi(X) \subset \bigcap_{t \in T} B_t^\Omega \cap \Phi(X).$$

Conversely, let $\xi \in \bigcap_{t \in T} B_t^\Omega \cap \Phi(X)$, i.e. $\xi = \xi_x$ and $\xi \in B_t^\Omega$ for each t. Thus for each t there exists an integer $k = k(t)$ such that $\xi = \xi_x \in \Omega_{A_k^t}$.

This implies that $x \in A_k^t \subset B_t$. Hence $x \in \bigcap_{t \in T} B_t$, and, finally, $\xi \in \Phi(\bigcap_{t \in T} B_t)$.

(2.5) *For any closed sets B_1, B_2 from the basis, if $B_1 \subset B_2$, then $B_1^\Omega \subset B_2^\Omega$.*

Proof. If $\xi \in B_1^\Omega$, then $\xi \in \Omega_{A_k^1}$ for some k.

In the case $\xi = \xi_x$ we have $x \in A_k^1 \subset B_1 \subset B_2$. Hence there exists an integer l such that $x \in A_l^2$. Then $\xi = \xi_x \in \Omega_{A_l^2} \subset B_2^\Omega$.

In the case when ξ is an ultrafilter, taking k such that $\xi \in \Omega_{A_k^1}$, we have $A_k^1 \in \xi$. Now there exists an A_l^2 such that $A_l^2 \in \xi$. To show this, suppose on the contrary that for each l there exists an element $\xi_{t_l} \in \xi$ disjoint with A_l^2. We get

$$\bigcup_{l=1}^{m} A_l^2 \cap \bigcap_{l=1}^{m} \xi_{t_l} = \emptyset,$$

whence $B_2 \cap \xi_{t_0} = \emptyset$, where $\xi_{t_0} = \bigcap_{l=1}^{m} \xi_{t_l} \in \xi$. This leads to a contradiction, for $A_k^1 \cap \xi_{t_0} \subset B_1 \cap \xi_{t_0} \subset B_2 \cap \xi_{t_0} = \emptyset$ and simultaneously $A_k^1 \cap \xi_{t_0} \neq \emptyset$, because of $A_k^1 \in \xi$ and $\xi_{t_0} \in \xi$. We conclude that $\xi \in B_1^\Omega$ implies $\xi \in B_2^\Omega$.

(2.6) $\overline{\Phi(B)}^\Omega = B^\Omega$ *for each closed set B from the basis.*

Proof. We have $\Phi(B) \subset B^\Omega$, by (2.3).

On the other hand, B^Ω is the minimal closed basic set in Ω containing the set $\Phi(B)$. To show this, suppose $\Phi(B) \subset B_*^\Omega$. Then (2.3) implies $B \subset B_*$ and hence, according to (2.5), $B^\Omega \subset B_*^\Omega$. Thus B is contained in each closed set containing $\Phi(B)$. Hence $\overline{\Phi(B)}^\Omega = B^\Omega$.

(2.7) $\overline{\Phi(B_1 \cap \ldots \cap B_n)}^\Omega = \overline{\Phi(B_1)}^\Omega \cap \ldots \cap \overline{\Phi(B_n)}^\Omega$ *for any basic closed sets $B_1, \ldots, B_n$.*

Proof. Clearly $\Phi(B_1 \cap \ldots \cap B_n) \subset \Phi(B_1) \cap \ldots \cap \Phi(B_n) \subset \overline{\Phi(B_1)}^\Omega \cap \ldots \cap \overline{\Phi(B_n)}^\Omega$.

Besides, $\overline{\Phi(B_1)}^\Omega \cap \ldots \cap \overline{\Omega(B_n)}^\Omega$ is the minimal closed set F in Ω with the property $\Phi(B_1 \ldots B_n) \subset F$. In fact, if $\Phi(B_1 \cap \ldots \cap B_n) \subset \Omega_{A_1} \cup \ldots \cup \Omega_{A_m}$, then, in view of (2.3), $B_1 \cap \ldots \cap B_n \subset A_1 \cup \ldots \cup A_m$. Hence (see also (1.3))

$$\bigcap_{k=1}^{n} A_{j_k}^k \subset \bigcup_{\substack{(j_1,\ldots,j_n)\\ 1 \leqslant j_k \leqslant n_k}} \bigcap_{k=1}^{n} A_{j_k}^k = B_1 \cap \ldots \cap B_n \subset A_1 \cup \ldots \cup A_m$$

for each collection $(j_1, \ldots, j_n)$. Then, by (2.5), we have $\Omega_{\bigcap_{k=1}^{n} A_{j_k}^k} \subset \Omega_{A_1} \cup \ldots \cup \Omega_{A_n}$ for each collection $(j_1, \ldots, j_n)$. Thus, according to (2.6),

$$\overline{\Phi(B_1)}^\Omega \cap \ldots \cap \overline{\Phi(B_n)}^\Omega = B_1^\Omega \cap \ldots \cap B_n^\Omega = \bigcap_{k1=}^{n} \bigcup_{j=1}^{n_k} \Omega_{A_j^k}$$

$$= \bigcup_{\substack{(j_1,\ldots,j_n)\\ 1\leqslant j_k\leqslant n_k}} \bigcap_{k=1}^{n} \Omega_{A^k_{j_k}} \subset \bigcup_{\substack{(j_1,\ldots,j_n)\\ 1\leqslant j_k\leqslant n_k}} \Omega_{\bigcap_{k=1}^{n} A^k_{j_k}} \subset \Omega_{A_1} \cup \ldots \cup \Omega_{A_m}.$$

This proves formula (2.7).

(2.8) $\overline{\Phi(X)}^{\Omega} = \Omega$.

Proof. It is sufficient to show that any non-void open basic set $U = \Omega \setminus (\Omega_{A_1} \cup \ldots \cup \Omega_{A_n})$ has a non-void intersection with $\Phi(X)$. From $U \neq \emptyset$, i.e. from $\Omega_{A_1} \cup \ldots \cup \Omega_{A_n} \neq \Omega$, we obtain by (1.5) $A_1 \cup \ldots \cup A_n \neq X$. Thus there exists a point $x \in X$ such that $x \notin A_k$ for each k. Then $A_k \notin \xi_x = \Phi(x)$ for each k, which means that $\Phi(x) \notin \Omega_{A_k}$. Thus $\Phi(x) \in U$, i.e. $\Phi(X) \cap U \neq \emptyset$.

Formula (2.2) means that Φ is continuous. Formula (2.4) means that $\Phi \colon X \to \Phi(X)$ is closed. Thus, if the topology generated by the family $\mathfrak{A}$ in X is T_0, then, by (2.1), the mapping Φ imbeds the space X into the space Ω. By (2.8), Φ is an imbedding onto a dense subset.

In other words, (Ω, Φ) is a compact T_0-extension of the space X. We denote it by $\mathfrak{A}X$. This compactification has the properties (2.6) and (2.7).

Remarks. 1. The family $\mathfrak{A}$ of all closed subsets of a T_1-space is multiplicative, covers X and has the property (i) from (1.5) (because $\{x\} = \overline{\{x\}}$ for each $x \in X$). The extension $\mathfrak{A}X$, corresponding to $\mathfrak{A}$, is simply the Wallman extension, described in [3].

2. Let X be a locally compact (but not compact) T_0-space, $\mathfrak{A}'$ — the family of all closed compact sets, $\mathfrak{A}''$ — the family of complements of open sets with compact closure. The family $\mathfrak{A} = \mathfrak{A}' \cup \mathfrak{A}''$ is multiplicative. Indeed, if $A \in \mathfrak{A}'$, $B \in \mathfrak{A}$, then $A \cap B$ is a closed subset of the compact set A, thus $A \cap B \in \mathfrak{A}'$; if $A \in \mathfrak{A}''$, $B \in \mathfrak{A}''$ (i.e. the sets $\overline{X \setminus A}$, $\overline{X \setminus B}$ are compact), then $\overline{X \setminus A \cap B} = \overline{(X \setminus A) \cup (X \setminus B)} = \overline{X \setminus A} \cup \overline{X \setminus B}$ is a compact set, thus $A \cap B \in \mathfrak{A}''$. On the other hand, the family of open sets with compact closure is a basis for the open sets in X, thus $\mathfrak{A}''$ forms a basis for the closed sets. $\mathfrak{A}X$ is the Alexandroff one-point compactification ([2], p. 150) of the space X. The proof is analogous to that of [1].

3. The question of uniqueness of $\mathfrak{A}X$. The extension $\mathfrak{A}X$ is minimal with respect to the properties (2.6) and (2.7) in the following sense:

THEOREM 2. *If (K, ψ) is a compact T_0-extension of the space X such that*

1. $\{\overline{\psi(A)}^K\}_{A \in \mathfrak{A}}$ forms a subbasis for the closed subsets in K,

2. $\overline{\psi(A_1 \cap \ldots \cap A_n)}^K = \overline{\psi(A_1)}^K \cap \ldots \cap \overline{\psi(A_n)}^K$, $A_1, \ldots, A_n \in \mathfrak{A}$,

then there exists an imbedding $i_K \colon \Omega \to K$ for which $i_K \circ \Phi = \psi$.

Proof. 1° The family $\{\overline{\psi(\xi_t)}^K \colon \xi_t \in \xi\}$, $\xi \in \Omega$, is the filter of closed neighbourhoods from the subbasis $\{\overline{\psi(A)}^K\}_{A \in \mathfrak{A}}$ of a point $k \in K$.

In fact, if $\xi = \xi_x$, then $x \in \xi_x^t$ for each $\xi_x^t \in \xi_x$. Therefore $\psi(x) \in \psi(\xi_x^t) \subset \overline{\psi(\xi_x^t)}^K$ and, in addition, each neighbourhood from the closed subbasis of the point $\psi(x)$ is of the form $\overline{\psi(\xi_x^t)}^K$. For if $\psi(x) \in \overline{\psi(A)}^K$, $A \in \mathfrak{A}$, then $\psi^{-1}(\psi(x)) \in \psi^{-1}(\overline{\psi(A)}^K) = A$ (because ψ is an imbedding map and A is closed in X), whence $x \in A$, i.e. $A \in \xi_x$. Thus, for $\xi = \xi_x \in \Omega$, the family $\{\overline{\psi(\xi_t)}^K\}_{\xi_t \in \xi}$ is the filter of neighbourhoods of the point $k = \psi(x)$.

If ξ is an ultrafilter with void intersection, i.e. if $\xi \in \Omega \setminus \Phi(X)$, then $\{\overline{\psi(\xi_t)}^K\}_{\xi_t \in \xi}$ is a centered family of closed sets in K. K being a compact space, there exists a point $k \in \bigcap_{\xi_t \in \xi} \overline{\psi(\xi_t)}^K$.

Note that each neighbourhood of k from $\{\overline{\psi(A)}^K\}_{A \in \mathfrak{A}}$ is of the form $\overline{\psi(\xi_t)}^K$. Indeed, let $k \in \overline{\psi(A)}^K$. Then, according to property 2 of (K, ψ), $k \in \overline{\psi(A)}^K \cap \overline{\psi(\xi_t)}^K = \overline{\psi(A \cap \xi_t)}^K$, whence $A \cap \xi_t \neq \emptyset$ for each $\xi_t \in \xi$. Since ξ is an ultrafilter, the last inequality implies $A \in \xi$, i.e. $A = \xi_{t_0}$.

2° Since K is a T_0-space, the point $k \in K$, whose existence for each $\xi \in \Omega$ was proved in 1°, is uniquely determined by ξ. This enables us to define a mapping $i_K\colon \Omega \to K$ by $i_K(\xi) = k$.

3° The mapping i_K is one-to-one. Let $\xi' \neq \xi''$. Thus there exists $A \in \mathfrak{A}$ such that $A \in \xi'$ and $A \notin \xi''$, which implies $A \neq \xi_t''$ for each $\xi_t'' \in \xi''$. Hence $\overline{\psi(A)}^K \neq \overline{\psi(\xi_t'')}^K$ because the sets A and ξ_t'' are closed in X. Hence $\overline{\psi(A)}^K \notin \{\overline{\psi(\xi_t'')}^K\}_{\xi_t'' \in \xi''}$, thus the neighbourhood filters $\{\overline{\psi(\xi_t')}^K\}_{\xi_t' \in \xi'}$, $\{\overline{\psi(\xi_t'')}^K\}_{\xi_t'' \in \xi''}$ are different and therefore determine different points $i_K(\xi')$ and $i_K(\xi'')$.

4° Finally, for each $A \in \mathfrak{A}$, $i_K(\xi) \in \overline{\psi(A)}^K \cap i_K(\Omega)$ iff $\xi \in \Omega_A$. For if $k = i_K(\xi) \in \overline{\psi(A)}^K \cap i_K(\Omega)$, then $\overline{\psi(A)}^K$ is a neighbourhood of k, hence $A \in \xi$, i.e. $\xi \in \Omega_A$.

Conversely, let $\xi \in \overline{\Omega_K}$. Then $A \in \xi$, and, by definition of i_K, $i_K(\xi) \in \overline{\psi(A)}^K$, A being one of the $\xi_t \in \xi$. Evidently $i_K(\xi) \in i_K(\Omega)$. This is the same as the continuity of i_K. Moreover, the mapping $i_K\colon \Omega \to i_K(\Omega)$ is closed $(i_K(\Omega_A) = \overline{\psi(A)}^K \cap i_K(\Omega))$. By 3°, it is one-to-one. Therefore it is an imbedding. The commutativity $(i_K \circ \Phi)(x) = i_K(\xi_x) = \psi(x)$, $x \in X$, follows immediately from the construction of k for ξ_x in 1°.

Remarks. 1. There exist compact T_0-extensions (K, ψ) of the space X such that $i_K(\Omega) \underset{\neq}{\subset} K$.

Flachsmeyer constructed in [2] a compactification of T_0-spaces (denoted in [2] by $(\gamma X, \Phi)$) having the properties 1 and 2 with respect to the family $\mathfrak{A}$ of all closed subsets of X. Let $\alpha = \{A_1, \ldots, A_n\}$ be a finite open covering of X, and let $f = A_1^* \cap \ldots \cap A_n^*$, where $A_k^* = A_k$ or $A_k^* = X \setminus A_k$. Let A be the family of all finite open coverings α, ordered by set-theoretical inclusion. The points of K are collections $\{f_\alpha\}_{\alpha \in A}$ for which $f_a \subset f_\beta$ for $a \supset \beta$. The points of X are represented in K by collections $\{f_a\}_{a\in A}$ with non-void intersection.

Now, take a countable set $X = \{x_1, x_2, \ldots\}$. Let $A_n = \{x_i \in X: i \leqslant n\}$ be the closed subsets in X and $\mathfrak{A} = \{A_n\}_{n\in N}$. X is a compact T_0-space, in just defined topology, and thus the spaces X and Ω are homeomorphic.

The extension (K, ψ) has the properties 1 and 2 from Theorem 2, thus there exists an imbedding $i_K: \Omega \to K$. But $K \setminus \psi(X) \neq \emptyset$, because the collection $\{\bigcap_{u\in a} U\}_{a\in A}$ is a point of K and has a void intersection. Finally, by commutativity $i_K \circ \Phi = \psi$, and since Φ is a homeomorphism, we have $(i_K \circ \Phi)(X) = i_K(\Phi(X)) = i_K(\Omega) = \psi(X)$, i.e. Ω is imbedded as a proper subset of K.

2. The imbedding i_K uniquely determined. For let $j_K: \Omega \to K$ be an imbedding with the property $j_K \circ \Phi = \psi$. Then it suffices to prove that $j_K(\xi) = i_K(\xi)$ for each $\xi \in \Omega \setminus \Phi(X)$. Observe that

$$j_K(\Omega_A) = j_K(\overline{\Phi(A)}^{\Omega}) = \overline{j_K(\Phi(A))}^{j_K(\Omega)} = \overline{\psi(A)}^K \cap j_K(\Omega).$$

But the subspace $\Omega \setminus \Phi(X)$, consisting of ultrafilters of $\mathfrak{A}$, is a T_1-space. Thus $\xi = \bigcap_{\xi_t\in\xi} \Omega_{\xi_t}$ for each $\xi \in \Omega \setminus \Phi(X)$. Finally,

$$j_K(\xi) = j_K\Big(\bigcap_{\xi_t\in\xi} \Omega_{\xi_t}\Big) = \bigcap_{\xi_t\in\xi} j_K(\Omega_{\xi_t}) = \bigcap_{\xi_t\in\xi} \overline{\psi(\xi_t)}^K \cap j_K(\Omega) \subset \bigcap_{\xi_t\in\xi} \overline{\psi(\xi_t)}^K = i_K(\xi)$$

(the last equality follows from the definition of i_K for $\xi \in \Omega \setminus \Phi(X)$). Thus $j_K(\xi) = i_K(\xi)$ for each $\xi \in \Omega \setminus \Phi(X)$.

3. The extension (Ω, Φ) is characterized by the property described in Theorem 2. To prove the assertion, let (Ω', Φ') have also this property. Then there exists an imbedding $i'_\Omega: \Omega' \to \Omega$ such that $i'_\Omega \circ \Phi' = \Phi$. Since $i'_\Omega(\Omega')$ is a subspace of Ω, the family $\{\overline{\Phi'(A)}^{\Omega'}\}_{A\in\mathfrak{A}}$ forms a subbasis for the closed subsets in Ω' and the equality $\overline{\Phi'(A_1 \ldots A_n)}^{\Omega'} = \overline{\Phi'(A_1)}^{\Omega'} \cap \ldots \cap \overline{\Phi'(A_n)}^{\Omega'}$ holds for each $A_1, \ldots, A_n \in \mathfrak{A}$. But $\Omega' \setminus \Phi'(X)$ is a subspace of the T_1-space $\Omega \setminus \Phi(X)$, hence, if (K, ψ) is an extension of X, with the properties 1 and 2 from Theorem 2, then the corresponding imbedding $i'_K: \Omega' \to K$ is uniquely determined (the proof is analogous to that of Remark 2). Finally, by Theorem 2, there exists an imbedding $i_{\Omega'}: \Omega \to \Omega'$ with the property $i_{\Omega'} \circ \Phi = \Phi'$. But $i_\Omega \circ \Phi' = \Phi$, thus $i_{\Omega'} \circ i_\Omega \circ \Phi' = \Phi'$ and $i'_\Omega \circ i_{\Omega'} \circ \Phi = \Phi$, where $i_{\Omega'} \circ i'_\Omega$ is the imbedding $i'_{\Omega'}$ (corresponding to the extension (Ω', Φ)) and $i'_\Omega \circ i_{\Omega'}$, is the imbedding i_Ω (corresponding to the extension (Ω, Φ)). In consequence of the uniqueness of these imbeddings, $i'_{\Omega'} = e_{\Omega'}$ and $i_\Omega = e_\Omega$ ($e_{\Omega'}$ and e_Ω being the identity mappings of Ω' and Ω). Thus $i_{\Omega'}$ and i'_Ω are homeomorphisms and the spaces Ω and Ω' are homeomorphic.

4. If X is a T_1-space and $\mathfrak{A}$ satisfies the condition (i) from (1.5), then $\mathfrak{A}X$ is the unique compact T_1-extension of X, with properties (2.6) and (2.7).

First we prove a

LEMMA. *For each* $k \in K \setminus \psi(X)$, $\{k\} = \overline{\{k\}}^K$ *iff* $k \in i_K(\Omega) \setminus \psi(X)$.

Proof. Let k be a closed point, belonging to $K \setminus \psi(X)$. It is easy to check that the non-void family $\xi = \{A \in \mathfrak{A}: k \in \overline{\psi(A)}^K\}$ is a filter of the subbasis $\{\overline{\psi(A^K)}\}_{A \in \mathfrak{A}}$. Furthermore, ξ is a maximal filter. To prove this, take any $B \in \mathfrak{A}$ such that $B \cap A \neq \emptyset$ for each $A \in \xi$. Then for each $A_1, \ldots, A_n \in \xi$ we have

$$\overline{\psi(B)}^K \cap \overline{\psi(A_1)}^K \cap \ldots \cap \overline{\psi(A_n)}^K = \overline{\psi(B \cap A_1 \cap \ldots \cap A_n)}^K = \overline{\psi(B \cap A)}^K \neq \emptyset,$$

where $A = A_1 \cap \ldots \cap A_n \in \xi$. It means that the family $\{\overline{\psi(B)}^K\} \cup \cup \{\overline{\psi(A)}^K\}_{A \in \xi}$ is a centered family of closed subsets of the compact space K. Hence

$$\emptyset \neq \overline{\psi(B)}^K \cap \bigcap_{A \in \xi} \overline{\psi(A)}^K \subset \bigcap_{A \in \xi} \overline{\psi(A)}^K = \{k\},$$

for k is a closed point. This means that $\{k\} = \overline{\psi(B)}^K \cap \bigcap_{A \in \xi} \overline{\psi(A)}^K$, whence $k \in \overline{\psi(B)}^K$ and, in consequence, $B \in \xi$. This proves the maximality of ξ. The ultrafilter ξ is a point of Ω. By definition of i_K, $i_K(\xi) = k$. Then, by the assumption, $k \in i_K(\Omega) \setminus \psi(X)$.

To prove the converse implication, observe that $i_K(\Omega) \setminus \psi(X) = i_K(\Omega \setminus \Phi(X))$, where $\Omega \setminus \Phi(X)$ is a T_1-space and i_K is an imbedding. Thus $\{k\} = \overline{\{k\}}^K$ for $k \in i_K(\Omega) \setminus \psi(X)$.

Now, if K is any compact T_1-extension of X, with the properties 1 and 2 from Theorem 2, then, by the Lemma, $i_K(\Omega) = K$. Thus the spaces Ω and K are homeomorphic.

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