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ON CONFORMAL MAPPING PROBLEM OF STOÏLOW AND OF WOLIBNER

BY

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The object of this paper is to answer in the affirmative the following question of Wolibner (see [6]):

Does there exist an analytic single-valued function fulfilling the following conditions:

1. it is a function of Pompeiu-Urysohn type (cf. [5]), i.e. the set S of its essential singular points is non-void, zero-dimensional and perfect, and at every point of S the function is defined and finite (hence it is a continuous function on the whole plane);

2. it possesses a simple pole at infinity;

3. it is univalent (schlicht) in the whole plane (hence it establishes an automorphism of the whole complex plane).

This problem is also mentioned explicitly in Stoïlow's book [4] (p. 124-125). The first reference to this question, however, goes back to Denjoy [1], as far as I know.

We are going to construct a function satisfying these conditions and mapping a set of singularities of positive measure onto a set of zero measure (plane Lebesgue measure), both these sets being zero-dimensional.

In § 2 we sketch the construction intuitively. §§ 3, 4 and 5 give the technical details. These include a certain principle of "Gebietsstetigkeit" (§ 4, Lemma 5) and a principle of "Randstetigkeit" (§ 4, Lemma 6). Whether these lemmas, in the particular form I give them, are new, I do not know; but I am convinced that very similar theorems have been proved a long time ago.

1. Consider a (multiply connected) domain containing the point at infinity and bounded by a finite number of smooth (1) Jordan curves, so that its complement consists of a finite number of "islands".

(1) "Smooth" is not really necessary. The reader may take it to mean, say, twice differentiable, or piecewise analytic, etc.

It is well known that such a domain can be mapped univalently onto a slit-domain (Schlitzbereich) leaving the point at infinity fixed. We may also specify, which is permissible, that the slits should be parallel to the imaginary axis and that the function should be of the form

$$w = z + \frac{a}{z} + \frac{b}{z^2} + \dots \tag{1}$$

(with constant term missing) in the neighbourhood of infinity.

We note that we have uniqueness: for each domain D there is just one function $f(z)$ satisfying these conditions. It can also be shown that if all the islands are inside the disc $|z| \leqslant 1$, the slits will all be inside the disc $|w| \leqslant 2$, and 2 is best possible (see § 3, Lemma 2).

We shall consider such mappings and shall construct, by a certain limit process, a function which

(a) maps the z-plane topologically onto the w-plane,

(b) maps a certain 0-dimensional set of positive measure (plane Lebesgue measure) onto a 0-dimensional set of 0 measure,

(c) is regular every where else and satisfies (1).

2. The process consists of constructing a sequence of functions $\varphi_n(z)$ mapping n-tuply connected domains ($n = 1, 2, \dots$) onto slit-domains (as specified in § 1), and then taking the limit. Roughly speaking, take a domain bounded by n islands, something like the water on a map of Venice (with an ocean around it). Then we dig more and more thin canals, until, in the limit, we get a zero-dimensional perfect set. We want it to be of positive measure and $\lim \varphi_n(z)$ to satisfy the conditions (a), (b) and (c).

We shall first explain it intuitively, taking all limit processes for granted, and then give the details to justify it, in the next paragraphs.

Let $\varphi_1(z) = z - 1/z$, which maps the domain $|z| > 1$ onto a slit domain whose slit is $x = 0$, $-2 \leqslant y \leqslant +2$.

Now, suppose $\varphi_n(z)$ already defined, so that it maps a domain with n islands (contained in the unit circle) onto a slit domain (fig. 1, very schematic) and satisfies the conditions of the last paragraph, in particular condition (1). Also, let U_n be an already defined ε-neighbourhood of these slits (see fig. 2). For $n = 1$, we may take $\varepsilon = 1$, otherwise the ε should be sufficiently small, say $\varepsilon < \lambda_n/2^n$, where λ_n is the length of the shortest one of the slits; also such as to satisfy (2) below.

Next, cut one of the islands in two by a very thin "canal", and let $\varphi_{n+1}(z)$ be the corresponding function, mapping the new, $(n+1)$-tuply connected domain onto a slit domain (fig. 3a and fig. 3b).

In this way, one of the slits will be replaced by a couple of slits nearby, the other slits being slightly displaced and slightly modified (to justify

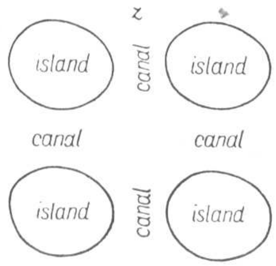

Fig. 1

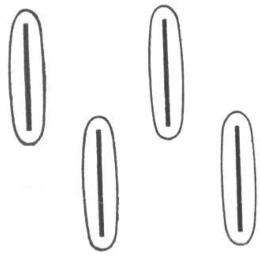

Fig. 2

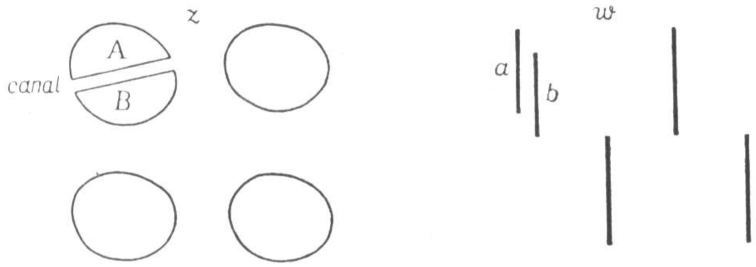

Fig. 3a

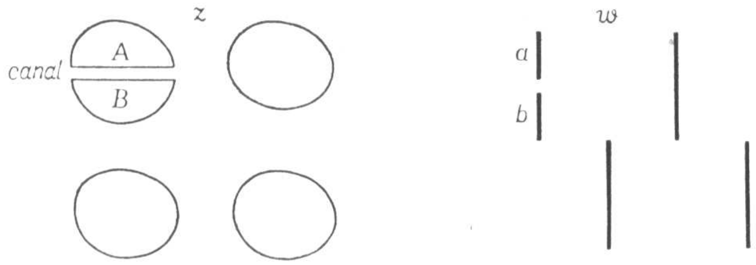

Fig. 3b

this statement, we shall need a certain principle of "Gebietsstetigkeit", see § 4). We shall make the canal so thin that all the new slits are inside U_n. Now let U_{n+1} be an ε-neighbourhood of the new set of slits, such that

(2) $$U_{n+1} \subset U_n \text{ and the measure } \mu(U_{n+1}) < \tfrac{1}{2}\mu(U_n).$$

This condition assures that, when $n \to \infty$, the slit sets (²) will tend to a limit set (³) of measure 0. Also, if the canals are sufficiently thin, the island sets will tend to a limit set of positive measure. It remains to make the limit sets zero-dimensional.

Let δ_n be the largest of the diameters of the islands of φ_n and, similarly, let δ'_n be the maximum length of the slits. It is then sufficient to place the canals successively in such a way that $\delta_n \to 0$ and $\delta'_n \to 0$. The

(²) Here "slit set" means "boundary of the slit domain", i.e., strictly speaking "the set of points on the set of slits".

(³) "Limit set" in the sense of definition 1, § 4.

former is obviously possible, and the latter follows from the fact (to be shown in § 5) that, by means of a suitable canal, a slit can be replaced by two shorter ones (of approximately half the length; see fig. 3b). In order to do both simultaneously, we might proceed thus: first make a number of canals to reduce the size of the islands, then others to reduce the lengths of the slits, and so on alternately; then, for a certain sequence $n_1, n_2, n_3, \ldots$ we get $\delta_{n_1}, \delta'_{n_2}, \delta_{n_3}, \delta'_{n_4}, \delta_{n_5}, \ldots$ with both the δ_n and the δ'_n tending to 0.

Now let $f(z) = \lim \varphi_n(z)$. To justify this construction and to prove that $f(z)$ satisfies the conditions of the theorem, viz. (a), (b), (c) of § 1, we need to prove three things:

1. The $\varphi_n(z)$'s form a normal family and their inverses $\varphi_n^{-1}(w)$ also.

2. A certain principle of "Gebietsstetigkeit".

3. The possibility of reducing the lengths of the slits ("halving principle").

The regularity and schlichtness outside the zero-dimensional set follows at once, and the continuity on this set is easy to show; the same goes for its inverse, $f^{-1}(w)$.

This set of singularities is a perfect set because it is closed and bounded, and cannot have any isolated points; neither can it be the empty set, since no regular function can map a null set onto a set of positive measure, or vice-versa.

In the sequel we shall need the following well known (cf. [7], Theorem 6.7)

TOPOLOGICAL LEMMA. *In the plane, or in n-space with $n \geqslant 2$, if a domain which is complementary to a bounded zero-dimensional (closed) set is mapped topologically* (4) *onto another such domain, the mapping can be extended continuously to the zero-dimensional set, giving thus a topological mapping of the whole plane.*

3. The normal family $\mathscr{F}$. Let $\mathscr{D}$ be the family of all (multiply connected) domains which contain the point at infinity and whose boundary lies entirely within the unit disc $|z| \leqslant 1$.

Let $\mathscr{F}$ be the family of all functions $w = f(z)$ which are schlicht in *some* — not always the same — domain $D \in \mathscr{D}$ and which are of the form (1), viz.,

$$w = z + \frac{a}{z} + \frac{b}{z^2} + \ldots,$$

in the neighbourhood of infinity.

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(4) For our purpose, we may assume the mapping to be conformal.

We note that (1) can be put in a symmetric form, as follows:

(3) if $z \to \infty$ and $w \to \infty$, then

$$\lim \frac{w}{z} = 1 \quad \text{and} \quad \lim (w - z) = 0.$$

We want to show that $\mathcal{F}$ is a normal family.

LEMMA 1. *If $w = z + 0 + c_3 z^3 + c_4 z^4 + \ldots$ is schlicht in the unit circle, but not necessarily holomorphic (it may have one pole), then the "Koebe constant" becomes equal to $\frac{1}{2}$, i.e., if ξ_1 is a boundary point of the (unit circle's) image domain in the w-plane, then $|\xi_1| \geqslant \frac{1}{2}$.*

Remark. Here the condition of holomorphism in Koebe's distortion theorem is replaced by the condition that the second derivative vanishes at the origin.

Proof (cf. [3], p. 214). The function $g(z) = w/(1 - w/\xi_1)$ is schlicht and holomorphic inside the unit circle if $w \neq \xi_1$ throughout. We find that $g(0) = 0$, $g'(0) = 1$, $g''(0) = 2/\xi_1$, so that $g(z) = z + z^2/\xi_1 + \ldots$ Thus, by a well-known theorem, $|1/\xi_1| \leqslant 2$ and $|\xi_1| \geqslant \frac{1}{2}$.

LEMMA 2. *If $f(z)$ belongs to the family $\mathcal{F}$ defined above, and if ξ is a boundary point of the image-domain $f(D)$ in the w-plane, then $|\xi| \leqslant 2$, and 2 is the best possible. In particular, if the image-domain is a slit domain, then all the slits lie in the disc $|w| \leqslant 2$.*

Proof. Let $z_1 = 1/z$ and $w_1 = 1/w$. Then, since $f(z)$ satisfies (1), the function $w_1 = f_1(z_1)$ satisfies the conditions of Lemma 1, in particular, $f_1(z_1)$ is schlicht inside the unit circle because (by hypothesis on the domain D) $f(z)$ is schlicht outside. Now, by Lemma 1, $|\xi_1| \geqslant \frac{1}{2}$, and we may put $\xi = 1/\xi_1$, and so $|\xi| \leqslant 2$ and the boundary of $f(D)$ lies in the disc $|w| \leqslant 2$. That 2 is the best possible follows from the example $w = z - 1/z$ (cf. § 2).

Remark. Restricting this function $f(z)$ to the domain outside the unit circle, we find that $|z| = 1$ implies $|w| \leqslant 2$, and, similarly, the circle $|z| = r$ $(1 < r < \infty)$ is mapped onto some curve with $|w| \leqslant 2r$. Hence, because of schlichtness,

(4) $$|z| \leqslant r \text{ implies } |w| \leqslant 2r \qquad (r > 1).$$

Also

(4') $$|w/z| \leqslant |2z/z| = 2 \qquad (|z| > 1).$$

Incidentally, letting $r \to 1$ in (4), we find that if $z = 0$ belongs to D, then $|f(0)| \leqslant 2$.

LEMMA 3. *$\mathcal{F}$ is a compact normal family. More precisely, any sequence $f_n \in \mathcal{F}$ is normal in the common part of the respective domains D_n (for $n > n_0$) and the limit function belongs to $\mathcal{F}$ also.*

Proof. It follows from (4) that $\mathcal{F}$ is bounded, and therefore normal, in the circle of any radius r $(1 < r < \infty)$. As to the neighbourhood of infinity, the normality of $\mathcal{F}$ follows from (4'). Also, from (4'), if $f_n \in \mathcal{F}$ (and convergent), $\lim f_n$ cannot be a constant, and it easily follows that $\lim f_n$ satisfies (1) (or (3)).

Let us now consider the family, say $\mathcal{F}^{-1}$, of all inverse functions $z = f^{-1}(w)$ where $f(z) \in \mathcal{F}$. Because of (3), they have the same properties as the functions in $\mathcal{F}$, except that, by Lemma 2, the unit circle is to be replaced by a circle of radius 2. Hence:

LEMMA 4. *The family $\mathcal{F}^{-1}$ of all inverse functions $f^{-1}(w)$, $f \in \mathcal{F}$, is normal in the same sense as in Lemma* 3, *and contains all limit functions.*

It follows that

The functions $\varphi_n(z)$ of § 1 *form a normal family, and their inverse functions $\varphi_n^{-1}(w)$ likewise, and, since they are schlicht and satisfy* (1), *the same holds for the limit functions.*

4. Gebietsstetigkeit. There is a well-known definition of distance between closed sets, which annuls only when the two sets coincide. All we need here, is the corresponding definition of limit of a sequence.

Definition 1. If $\{F_n\}$ is a sequence of closed sets, we define $\lim F_n = F$ if every neighbourhood of the closed set F contains almost all F_n's ("almost all" means "all but a finite number of") and if no smaller closed set has the property that every neighbourhood of it contains infinitely many F_n's. The sequence is said to be *convergent* if such a limit set F exists.

If S_n is the boundary of a slit domain ("slit set" for short) in the w-plane, we shall talk about $\lim S_n$ in this sense.

Definition 2. If D_n is a sequence of domains, we say that $\lim D_n = D$ if:

(a) the complements $\complement D_n$ of these domains converge, $\lim \complement D_n = F$, say;

(b) $\complement D = F$;

(c) D is a domain, i.e., an open and connected set.

It will be better to use the following equivalent definition:

Definition 3. We shall say that $\lim D_n = D$ (the D_n's being domains, as above) if:

(a) $z \in D$ if and only if there exists a neighbourhood U_z contained in almost all D_n's,

(b) if, for any given sub-sequence of the D_n's, D' is defined as in (a), then $D' = D$,

(c) D is a domain, i.e., open and connected.

LEMMA 5. *Let $f_n(z)$, $n = 1, 2, \ldots$, be a convergent sequence of schlicht functions mapping respectively domains D_n onto E_n, $f_n(D_n) = E_n$, and suppose the D_n's converge:* $\lim D_n = D$. *Then, if* $\lim f_n(z) = f(z) \neq$ *constant and if the $f_n(z)$ belong to a normal family (in the sense of* § 3), *e.g., $f_n \in \mathscr{F}$ (and the inverse functions $f_n^{-1}(w)$ likewise* ([5])), *we have:*

(I) *$f(z)$ is schlicht in D,*

(II) *if $f(D) = E$, then $E = \lim E_n$, i.e., $f(D) = \lim f_n(D_n)$.*

Proof. Statement (I) follows at once from the normality and the definition of $\lim D_n$. As to (II), let $w_0 \in E$ and let U_{w_0} be a sufficiently small neighbourhood of w_0. Then, because of normality, we shall have $U_{w_0} \subset E_n$ for almost all n.

On the other hand, if $U_{w_0} \subset E_n$ (for all $n > n_0$), then $g_n(w) = f_n^{-1}(w)$ is normal in U_{w_0}. But $\lim g_n(w)$ is not a constant, thus $\lim g_n'(w_0) = a \neq 0$. Now let $g_n(U_{w_0}) = U_n$ (in the z-plane); then, by Koebe's theorem, each U_n contains a circle C_n with center at $g_n(w_0)$ and the same radius, independent of n. These circles converge to a circle C with center at $\lim g_n(w_0)$ and the same positive radius. Thus $C \subset D$ and, by definition, $f(C) \subset E$, so that $U_{w_0} \subset E_n$ (for almost all n) implies that $w_0 \in E$.

This settles condition (a) of Definition 3. As to (b), it is sufficient to replace the sequences $\{f_n\}$, $\{D_n\}$ by corresponding subsequences. Condition (c) is obviously satisfied, since $E = f(D)$ and f is schlicht, q.e.d.

We shall now apply this lemma, which says, very roughly speaking, that small modifications of a domain imply small modifications of the counter-domain, to the situation in § 2.

Let D be a domain with a finite number of islands within the unit circle, and let D_ε be the domain obtained from D by making a thin canal, of width ε through one of the islands (see figs. 3a or 3b). Let $f_\varepsilon(z)$ and $f(z)$ (both in $\mathscr{F}$) be the (unique) functions mapping D_ε and D respectively, onto slit domains the slit sets of which shall be called S_ε and S respectively. Then, if $\varepsilon \to 0$, we have $\lim D_\varepsilon = D$ and, at least for a certain sequence of ε's, $\lim f_\varepsilon(z) = f(z)$ and so, by Lemma 5, $\lim S_\varepsilon = S$. This shows that, for any given ε, we can make this canal so thin that the slits will only be slightly displaced (less than ε), except for one of them which will be replaced by two slits in its ε-neighbourhood.

As to the last part of the construction in § 2, viz., the passage to the limit, we note that the domains D_n corresponding respectively to $\varphi_n(z)$, $n = 1, 2, \ldots$, obviously converge and, by Lemma 5 and the Topological Lemma, the zero-dimensional set $\lim CD_n$ will be mapped by $\lim \varphi_n(z) = f(z)$ onto $\lim S_n$, where S_n is the slit set belonging to D_n, provided the sequence $\{\varphi_n(z)\}$ converges. Actually, it does, but it is suf-

([5]) This hypothesis is redundant, but it saves trouble to assume it.

ficient for our purposes to replace it by a convergent subsequence, which is always possible because of the normality.

There still remains the problem of making a canal in such a way as to cut a slit approximately in half (because we have to reduce the lengths of the slits also). This will be done in the next section; for this purpose, a certain principle of "Randstetigkeit" is useful. In the following lemma the reader may suppose the arc a to lie on the boundary of an island; that part of a neighbourhood of a which lies in the domain and outside the island will be called a *half-neighbourhood*; more generally, a half-neighbourhood of a is a region bounded by a and by another Jordan arc.

LEMMA 6. *Let $f_n(z)$ be a convergent sequence of functions which are schlicht and holomorphic in a certain (the same) half-neighbourhood of a Jordan arc a (see fig. 4). If $f_n(z)$ maps a onto a straight segment s_n and if, for some interior point z_0, $\lim f'_n(z_0) = f'(z_0) \neq 0$ $\left(\text{where } f(z) = \lim f_n(z)\right)$, then $f(z)$ will map a onto $\lim s_n$ and, if t is a point on a, not an end-point, $\lim f_n(t) = f(t)$. We may add the hypothesis that the counter-domains of the half-neighbourhood are uniformly bounded.* (This simplifies the proof, and is sufficient for our purpose).

Sketch of proof. If a is a segment, we may apply the reflection principle, which puts a into the interior. Then, since normality is assured, there is no difficulty. Otherwise, we can transform a into a segment $a' = g(a)$, where $z' = g(z)$, and consider $f_n\left(g^{-1}(z')\right)$.

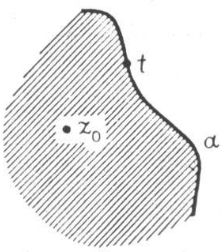

Fig. 4

5. Halving a slit. Let $f(z)$ be one of the functions $\varphi_n(z)$ of § 2 and let t_1, t_2 be two points on the circumference of one of the islands. Then $f(t_1), f(t_2)$ will be points on the corresponding slit. Now let us make an ε-canal connecting neighbourhoods of t_1 and t_2, dividing the island into two new ones, A and B. Thus the slit will be replaced by two slits a and b (see fig. 3a or 3b). (We only consider mappings belonging to $\mathscr{F}$.) Our object is to prove the following

LEMMA 7. *An island can be cut in two by a thin canal in such a way that the two new slits a and b will be approximately half the length of the original slit, such as in fig.* 3b.

The proof will follow from the following lemmas 8 and 9.

Definition. We shall say that two slits a and b *approximately overlap* if their projections onto the original slit overlap, i.e., if, when the width ε of the canal tends to 0, $\lim a$ and $\lim b$ overlap, or, which is the same thing, if, for small ε, both a and b contain points in the neighbourhood of $f(t_1)$ and of $f(t_2)$; that is, such as in fig. 3a, but not as in fig. 3b.

LEMMA 8. *If* $f(t_1) \neq f(t_2)$, *then* a *and* b *approximately overlap.*

This follows easily from Lemma 6; only, to avoid difficulties with the end-points of the segment $[f(t_1), f(t_2)]$, we replace it by a slightly shorter subsegment, say $[\zeta_1, \zeta_2]$. We find then that $[\zeta_1, \zeta_2]$ is contained both in $\lim a$ and in $\lim b$.

LEMMA 9. *If* $f(t_1) = f(t_2)$, $t_1 \neq t_2$, *there is no approximate overlapping* (*that is, the two slits* a *and* b *will be "edge on"*).

Proof. Let t be a point moving along the circumference of the island from t_1 to t_2 (counterclockwise, say) and let t' move simultaneously from t_2 to t_1, in the same manner. We note that $f(t) \neq f(t')$, except at the end-points t_1, t_2. Now consider a canal between t and t'. Then, if the couple (t, t') varies continuously from (t_1, t_2) to (t_2, t_1), the canal will "rotate" around 180°, and the islands A and B will permute. Therefore, the slits a and b will also permute, so that, for some couple (t_0, t_0'), they will be "edge on", that is, no approximate overlapping. But, by Lemma 8, there is always approx. overlapping when $f(t) \neq f(t')$, that is, except when t_0 and t_0' coincide (in some order) with t_1 and t_2, q.e.d.

Proof of Lemma 7. Let w_0 be the mid-point of the original slit and let t_1 and t_2 be the corresponding points: $f(t_1) = f(t_2) = w_0$. Then, again by Lemma 6, for sufficiently small ε, both a and b contain points in the neighbourhood of w_0. But they are "edge on"; also, when $\varepsilon \to 0$, the union of a and b tends to the original slit. Thus, both a and b are approximately half its length, q.e.d.

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