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# MAPPINGS ON MANIFOLDS

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**0. Introduction.** All spaces under consideration are supposed to be separable and metrizable. A subset $X$ of a space $M$ is said to be *densely connected in* $M$ if for any open connected subset $U$ of $M$, $U \cap X$ is connected. This notion is due to A. Lelek (see [4]). In [4], A. Lelek restricted himself to the case where $M = S^n$, the $n$-sphere, and proved the following interesting theorem:

THEOREM 0. *Let $X$ be a non-degenerate densely connected subset of $S^n$ and $h$ a homeomorphism of $X$ into a compact metric space $Y$ such that*

$$\dim[Y - h(X)] \leqslant 0.$$

*Then* $\dim Y \geqslant n$.

In the same paper, A. Lelek posed the following question:

(Q) Given the hypothesis of Theorem 0, and $h(X)$ is dense in $Y$, is it true that every closed separator of $Y$ has dimension $\geqslant n-1$?

In a series of papers, E. G. Sklyarenko studied the notions of perfect mappings and perfect extensions (see [7] and [8]). As consequences of Theorem 1 of [7] and Theorem 1 of [8], Theorem 0 and question (Q) can be rephrased, respectively, as follows:

THEOREM 0′. *Let $X$ be a non-degenerate densely connected subset of $S^n$ and $f$ a mapping of $S^n$ onto $Y$ such that* (i) *$f$ restricted to $X$ is a homeomorphism, and* (ii) $\dim[Y - f(X)] \leqslant 0$.

*Then* $\dim Y \geqslant n$.

(Q′) Given the hypothesis of Theorem 0′, is it true that every closed separator of $Y$ has dimension $\geqslant n-1$?

It is in the form (Q′) that we shall give an affirmative answer to A. Lelek's question (see Remarks following Theorem 4). Also Theorem 0′ follows easily from Theorem 2, and in fact, a strengthened form of Theorem 0′ holds (see Remark of Section 3).

In another paper, A. Lelek also posed the following question (see [5]):

(P) If $f$ is a non-constant map of $S^n$ onto $Y$ and if $\dim Y = k \leqslant n-1$, is it true that the set $F = \{y \in Y \mid n-k \leqslant \dim f^{-1}(y)\}$ has dimension $> 0$?

An affirmative answer was given by E. G. Sklyarenko (see [9]) with the aid of sheaf theory. In this paper we show that this result may be obtained by quite different methods (see Remark following Theorem 5).

**1. Mappings on Cantor-manifolds.** The following theorem is a generalization of a theorem of A. Lelek (see [5], Theorem 1). However, it is interesting to note that the proof we shall give is quite different from that of Theorem 1 in [5].

The following lemma can easily be proved:

LEMMA 1. *Suppose $f$ is a closed mapping of $M$ onto $N$. Let $N'$ be a subset of $N$, $M' = f^{-1}[N']$ and $g = f|_{M'}$. Then $g$ is a closed mapping of $M'$ onto $N'$.*

COROLLARY. *Suppose $f$ is a closed mapping of $M$ onto $N$. Let*

$$D_f = \{x \in M \mid f^{-1}f(x) = x\}.$$

*Then $g = f \mid D_f$ is a homeomorphism of $D_f$ onto $f(D_f)$.*

THEOREM 1. *Let $S = \bigcup_{i=1}^{\infty} S_i$ be a countable union of $n_i$-dimensional Cantor-manifolds, $\infty > n_i \geqslant n \geqslant 1$ for $i = 1, 2, 3, \ldots$, and $f$ a mapping of $S$ such that*

$$\dim f(S - D_f) \leqslant 0.$$

*Then* $\dim f(S) \geqslant n$ *or $f(S)$ is countable.*

Proof. If $f(S)$ is not countable, then $f$ is not constant on $S_i$, for some $i = 1, 2, \ldots$ Let $g = f \mid_{S_i}$ be non-constant. Then it suffices to show that $\dim g(S_i) \geqslant n$.

Suppose $\dim g(S_i) \leqslant n-1$. Since $g$ is not constant, let $p$ and $q$ be distinct points of $g(S_i)$. Since $D_g$ is $G_\delta$ (see [10], p. 162), $S_i - D_g$ is $F_\sigma$. But $g$ is a closed map; therefore, $g(S_i - D_g)$ is $F_\sigma$. Moreover, $D_g \supset D_f \cap S_i$ and so $\dim g(S_i - D_g) \leqslant 0$. By a theorem of T. Nishiura (see [6], p. 922), there is a closed set $C$ in $g(S_i)$ separating $p$ and $q$ in $g(S_i)$ and such that $\dim C \leqslant n-2$, and $C \subset g(D_g)$.

By the corollary to Lemma 1, $g$ is a homeomorphism on $D_g$ and since dimension is a topological invariant, $\dim g^{-1}[C] = \dim C \leqslant n-2 \leqslant n_i-2$. Thus $g^{-1}[C]$ cannot separate the $n_i$-dimensional Cantor-manifold $S_i$. This contradicts the fact that $g^{-1}[C]$ separates $S_i$ between $g^{-1}(p)$ and $g^{-1}(q)$, since $C$ separates $g(S_i)$ between $p$ and $q$. Thus $\dim g(S_i) \geqslant n$.

COROLLARY. *Let $S$ be an $n$-dimensional Cantor-manifold $(n \geqslant 1)$ and $f$ a non-constant map of $S$ such that*

$$\dim f(S - D_f) \leqslant 0.$$

*Then* $\dim f(S) \geqslant n$.

**2. Mappings on manifolds.** A natural question arises as to whether or not the conclusion of Theorem 1 can be strengthened to that of equality. The following example shows that the answer is negative. However, in case $S$ is a manifold, the strengthened conclusion holds true, as is given by Theorem 2.

Example. There exists a non-constant mapping $f$ of a 1-dimensional Cantor-manifold $S$, namely the Sierpiński curve, onto a closed 2-cell such that $f(S-D_f)$ is countable.

Let $I^2$ be a closed 2-cell and $S$ a Sierpiński curve in $I^2$, as constructed in [3], 5, p. 202. Since $S$ is a 1-dimensional compact connected subset of $I^2$, $S$ is a 1-dimensional Cantor-manifold. Let

$$P = \{\overline{U} \mid U \text{ is a component of } I^2-S \text{ in } I^2\}.$$

Then $P$ is a null sequence (see [10], p. 67) of disjoint continua in int $(I^2)$, the interior of $I^2$. The set

$$Q = (I^2-\bigcup P) \cup \{\overline{U} \mid \overline{U} \in P\}$$

is an upper semi-continuous decomposition of $I^2$, which contains no separator of $I^2$. We note that Proposition (2.1), of [10], p. 171, implies "the hyperspace of any upper semi-continuous decomposition of a closed 2-cell $I^2$ into continua, not separating $I^2$ and whose non-degenerate continua are contained in int $(I^2)$, is topologically a closed 2-cell". Hence $Q$ is a closed 2-cell. Let $g$ be the mapping of $I^2$ onto $Q$ induced by the decomposition $Q$ and $f = g \mid_S$. Then $g$ is clearly a non-constant map of $S$ onto $Q$. Since $I^2-S$ has only a countable number of components in $I^2$, $f(S-D_f)$ is countable.

THEOREM 2. *Let $M$ be an $n$-manifold and $f$ a non-constant mapping of $M$ such that*

$$\dim[f(M-D_f)] \leqslant 0.$$

*Then* $\dim f(M) = n$.

Proof. The theorem is obviously true for $n = 0$; so assume $n \geqslant 1$.

Since an $n$-manifold is a countable union of closed $n$-cells, $f$ is non-constant and $M$ is connected, the inequality $\dim f(M) \geqslant n$ follows from Theorem 1. To show that the reverse inequality holds, let $D$ be a countable dense subset of the interior of $D_f$. Then the set $(M-D_f) \cup D$ is dense in $M$ and its complement $D_f-D$ contains no non-void open subset of $M$. Since $M$ is an $n$-manifold, $\dim(D_f-D) \leqslant n-1$ (see [1], Corollary 1, p. 47). $D$, being a countable set (or $\emptyset$), is at most a 0-dimensional $F_\sigma$ subset of $M$. As in the proof of Theorem 1, $M-D_f$ is $F_\sigma$ in $M$. Since $M$ is a countable union of compact sets, namely closed $n$-cells, and since a closed subset of a compact set is compact, a $F_\sigma$ subset of $M$ is actually a countable union of compact subsets of $M$. It follows that $f(D)$ and

$f(M-D_f)$ are both $F_\sigma$ in $f(M)$ and both are 0-dimensional. Applying the Sum Theorem (see [2], Corollary 1, p. 172), $\dim[f(D \cup (M-D_f))] \leqslant 0$. Hence (see [1], Property B, p. 28),

$$\begin{aligned}\dim f(M) &\leqslant 1+\dim f(D_f-D)+\dim[f(D \cup (M-D_f))]\\ &\leqslant 1+n-1+0\\ &= n;\end{aligned}$$

i.e., $\dim f(M) \leqslant n$.

### 3. Some Lemmas.

Lemma 2. *Let $X$ be a dense subset of $S$, $f$ a mapping of $S$ and $g = f|_X$ a homeomorphism on $X$. Then $X \subset D_f$.*

Proof. Let $x$ be in $X$ and $C = f^{-1}f(x)$. Let $y$ be an element of $C$. Since $X$ is dense in $S$, there is a sequence $y_k$ in $X$ converging to $y$. By the continuity of $f, f(y_k)$ converges to $f(y) = f(x)$. By hypothesis, $g^{-1}$ is continuous on $f(X)$ to $X$; therefore, $y_k = g^{-1}f(y_k)$ converges to $x = g^{-1}f(x)$. Since $S$ is Hausdorff, $f^{-1}f(x) = x$ and $x$ is in $D_f$. Thus $X \subset D_f$.

Lemma 3. *Let $X$ be a dense and connected subset of an $n$-dimensional Cantor-manifold $S$ $(n \geqslant 2)$ and $f$ a mapping of $S$ such that $f$ restricted to $X$ is a homeomorphism. Then $f^{-1}(y)$ does not separate $S$ for any $y$ in $f(S)$.*

Proof. Let $y$ be a point of $f(S)$ and suppose $f^{-1}(y)$ separates $S$. Since $X$ is dense in $S$, $f^{-1}(y) \cap X$ separates $X$. Because $X \subset D_f$ and $f^{-1}(y)$ is non-degenerate, $f^{-1}(y) \cap X = \emptyset$. But $X$ is connected; we have a contradiction. Hence $f^{-1}(y)$ does not separate $S$.

Lemma 4. *Let $X$ be a dense and connected subset of an $n$-dimensional Cantor-manifold $S$ $(n \geqslant 2)$, $f$ a mapping of $S$ onto $Y$ such that*

(i) *$f$ restricted to $X$ is a homeomorphism,*

*and*

(ii) $\dim[Y-f(X)] \leqslant 0$.

*If $C$ is an $(n-1)$-dimensional Cantor-manifold in $S$ which separates $S$, then $\dim f(C) \geqslant n-1$.*

Proof. Let $g$ be $f$ restricted to $C$. Then, by Lemma 3, $g$ is non-constant. Since $D_g \supset D_f \cap C$, we have

$$\dim[g(C-D_g)] \leqslant \dim[f(C-D_f)] \leqslant 0,$$

where the last inequality follows from hypothesis (ii) and Lemma 2. Hence $g$ satisfies the hypothesis of the corollary of Theorem 1 and, therefore, $\dim f(C) \geqslant n-1$.

The following lemma is well known (see [3], 3, p. 348):

Lemma 5. *If $C$ is an irreducible closed separator of $S^n$ $(n \geqslant 2)$ between $a$ and $b$, then $C$ is an $(n-1)$-dimensional Cantor-manifold.*

Lemma 6 can be easily proved by using Lemma 5.

LEMMA 6. *If $C$ is an irreducible closed separator of $I^n$ ($n \geqslant 2$) between $a$ and $b$, then $C$ is an $(n-1)$-dimensional Cantor-manifold.*

LEMMA 7. *Let $X$ be a non-degenerate densely connected subset of an $n$-manifold $M$ (with or without boundary), $n \geqslant 2$. Then $X$ is connected and, in case $n \geqslant 2$, $X$ is dense in $M$.*

Proof. Since $M$ is connected, $X = X \cap M$ is connected.

Suppose $n \geqslant 2$ and $X$ is not dense in $M$. Then $M-\bar{X}$ is a non-void open subset of $M$. Let $U$ be a component of $M-\bar{X}$. Then $U$ is open and connected in $M$. Since $X$ is non-degenerate, there are non-void open and connected subsets $V$ and $W$ of $M$ which are separated and both of which intersect $U$ and $X$. Then $U \cup V \cup W$ is a non-void open connected subset of $M$ whose intersection with $X$ is $(X \cap V) \cup (X \cap W)$. The latter set, however, is the union of two non-void separated sets, which contradicts the densely connectedness of $X$. Hence, $X$ is dense in $M$.

Remark. The inequality in the conclusion of Theorem 0′ can be replaced, even under the weaker hypothesis that the set $X$ is dense and connected only (not necessarily densely connected), by an equality, since $X \subset D_f$ by virtue of Lemma 2, and, therefore, the condition that $\dim[Y-f(X)] \leqslant 0$ implies $\dim[f(S^n-D_f)] \leqslant 0$. The strengthened conclusion of Theorem 0′ now follows from Theorem 2, even in a more general case, that is for arbitrary $n$-dimensional manifold instead of a sphere $S^n$.

## 4. Mappings on spheres.

THEOREM 3. *Let $X$ be a dense and connected subset of $S^n$ ($n \geqslant 1$), $f$ a mapping of $S^n$ onto $Y$ such that*

(i) *$f$ restricted to $X$ is a homeomorphism*

*and*

(ii) $\dim[Y-f(X)] \leqslant 0$.

*Then every closed separator of $Y$ has dimension $\geqslant n-1$.*

Thus, by the Remark following Lemma 7, $Y$ is an $n$-dimensional Cantor-manifold.

Proof. The theorem is obviously true for $n = 1$; therefore, assume $n \geqslant 2$.

Let $K$ be a closed separator of $Y$. It follows from the hypotheses on $X$ that we may suppose there are points $a$ and $b$ in $X$ such that $K$ separates $Y$ between $f(a)$ and $f(b)$. Then $f^{-1}[K]$ is a closed separator of $S^n$ between $a$ and $b$. By a theorem of Mazurkiewicz (see [3], p. 176), there is an irreducible closed separator $C$ of $S^n$ between $a$ and $b$ and $C \subset f^{-1}[K]$. By Lemma 5, $C$ is an $(n-1)$-dimensional Cantor-manifold and the theorem now follows from Lemma 4.

**5. Mappings on manifolds** (continued).

THEOREM 4. *Let $X$ be a non-degenerate densely connected subset of an $n$-manifold $M$ (with or without boundary), $n \geqslant 1$, $f$ a map of $M$ onto $Y$ such that*

(i) *$f$ restricted to $X$ is a homeomorphism*

*and*

(ii) $\dim[Y - f(X)] \leqslant 0$.

*Then every closed separator of $Y$ has dimension $\geqslant n-1$.*

Proof. The case when $n = 1$ is trivial. Assume $n \geqslant 2$.

Let $K$ be a closed separator of $Y$. As in the proof of Theorem 3, we may assume $C$ separates $Y$ between $f(a)$ and $f(b)$ for some points $a, b \in X$. Then $f^{-1}[K]$ is a closed separator of $M$ between $a$ and $b$. Since $M$ is locally connected, applying a theorem of Mazurkiewicz again, there is an irreducible closed separator $C$ of $M$ between $a$ and $b$ and $C \subset f^{-1}[K]$. Since the boundary of $M$ cannot separate $M$, there is a point $p$ in $C$ which is an interior point of $M$. Let $I$ be a closed $n$-cell in $M$ whose interior is an open neighborhood of $p$. Then $C \cap I$ is a closed separator of $I$ between some points $a'$ and $b'$, $a', b' \in I$. Applying the same theorem of Mazurkiewicz once more, let $C'$ be an irreducible closed separator of $I$ between $a'$ and $b'$ and $C' \subset C \cap I \subset f^{-1}[K]$. By Lemma 6, $C'$ is an $(n-1)$-dimensional Cantor-manifold.

By Lemma 7, $X$ is dense in $M$ and, hence, $X' = X \cap I$ is dense in $I$. By densely connectedness, $X \cap \operatorname{int}(I)$ is connected, where $\operatorname{int}(I)$ denotes the interior of $I$. Since $X \cap \operatorname{int}(I) \subset X' \subset \overline{X \cap \operatorname{int}(I)} = \overline{X} \cap \overline{\operatorname{int}(I)} = I$, $X'$ is connected. Hence $g = f/_I$ satisfies the hypothesis of Lemma 4 and so $n-1 \leqslant \dim g(C') = \dim f(C')$. Thus $\dim K \geqslant n-1$.

Remarks. 1. An affirmative answer to A. Lelek's question (Q′) for $n \geqslant 1$ follows by Theorem 3 and Lemma 7 as well as by Theorem 4, and in both cases it is apparent for $n = 0$.

2. We do not know whether Theorem 4 still holds true if "densely connectedness" is replaced by "non-degenerate, dense and connected" **(P 580)**.

**6. Mappings that lower dimensions of Cantor-manifolds*.** Let $X$ be a subset of an arbitrary space $M$. Then the $n$-dimensional degree of $X$ with respect to $M$ will be denoted by $d_n(X)$ and is defined as the infimum of the numbers $\varepsilon > 0$ for which there is a finite family of open subsets $G_0, \ldots, G_m$ such that (cf. [3], p. 60)

* The author would like to express his thanks to Professor A. Lelek for his help in proving Lemmas 8 and 9.

(i) $X \subset G_0 \cup G \cup \ldots \cup G_m$,

(ii) $\delta(G_i) < \varepsilon$ for $i = 0, 1, 2, \ldots, m$

and

(iii) $G_{i_0} \cap G_{i_1} \cap \ldots \cap G_{i_n} = \emptyset$ for any $i_0 < i_1 < \ldots < i_n \leqslant m$.

LEMMA 8. *Let $K$ be compact and $f$ a map of $K$ onto $Y$. For any $\varepsilon > 0$, and for any integer $m \geqslant 1$, $F_\varepsilon = \{y \in Y \mid d_m f^{-1}(y) \geqslant \varepsilon\}$ is closed in $Y$.*

Proof. Let $\{y_\mu\}_{\mu=1}^{\infty}$ be a sequence in $F_\varepsilon$ converging to a point $p \notin F_\varepsilon$. Then $d_m f^{-1}(p) < \varepsilon$ and, therefore, there is a finite family of open subsets $G_0, G_1, \ldots, G_l$ such that

(i) $f^{-1}(p) \subset G_0 \cup G_1 \cup \ldots \cup G_l$,

(ii) $\delta(G_i) < \varepsilon$,

and

(iii) $G_{i_0} \cap G_{i_1} \cap \ldots \cap G_{i_m} = \emptyset$ for $i_0 < i_1 < \ldots < i_m \leqslant l$.

As the collection $\{f^{-1}(y) \mid y \in Y\}$ forms an upper semi-continuous decomposition of $K$, there is at least one $\mu_0$ with $f^{-1}(y_{\mu_0}) \subset \bigcup_{i=0}^{l} G_i$. This implies that $d_m f^{-1}(y_{\mu_0}) < \varepsilon$ which is a contradiction. Hence $p \in F_\varepsilon$ and $F_\varepsilon$ is closed in $Y$.

LEMMA 9. *Let $K$ be compact and $f$ a map of $K$ onto $Y$. For any integer $m \geqslant -1$, the set $F = \{y \in Y \mid m \leqslant \dim f^{-1}(y)\}$ is $F_\sigma$ in $Y$.*

Proof. In both cases $m = -1$ and $m = 0$, then $F = Y$ is $F_\sigma$ in $Y$. Assume that $m \geqslant 1$. For each $y \in Y, f^{-1}(y)$ is compact; so that $\dim f^{-1}(y) \geqslant m$ is equivalent to $d_m f^{-1}(y) > 0$ (see [3], 1, p. 60). For each integer $j \geqslant 1$, let

$$F_j = \left\{y \in Y \mid d_m f^{-1}(y) \geqslant \frac{1}{j}\right\}.$$

It follows from Lemma 8 that each $F_j$ is a closed subset of $Y$. As $F = \bigcup_{j=1}^{\infty} F_j$, $F$ is $F_\sigma$ in $Y$.

THEOREM 5. *Let $S$ be an $n$-dimensional Cantor-manifold ($n \geqslant 1$), $f$ a non-constant map of $S$ onto $Y$ and $\dim Y = k \leqslant n-1$. Then the set $F = \{y \in Y \mid n-k \leqslant \dim f^{-1}(y)\}$ has dimension $> 0$.*

Proof. Suppose $\dim F \leqslant 0$. Since $f$ is non-constant, let $p$ and $q$ be distinct points of $Y$. By Lemma 9, $F$ is $F_\sigma$ in $Y$ and by a theorem of T. Nishiura (see [6], p. 222), we obtain a closed separator $C$ of $Y$ between $p$ and $q$, $\dim C \leqslant k-1$ and $C \cap F = \emptyset$. Let $g$ be $f$ restricted to $f^{-1}[C]$. Then $g$ is a continuous function of the compact set $f^{-1}[C]$ onto $C$ and as $C \cap F = \emptyset$, $\dim g^{-1}(y) \leqslant n-k-1$, for any $y \in C$. Hence Hurewicz' Theorem gives $\dim f^{-1}[C] \leqslant k-1+n-k-1 = n-2$ which is a contradiction since $f^{-1}[C]$ separates the $n$-dimensional Cantor-manifold $S$. Hence $\dim F > 0$.

Remark. Now an affirmative answer to question (P) follows from Theorem 5 in case $n \geqslant 1$, since $S^n$ is an $n$-dimensional Cantor-manifold. In the case when $n = 0$, (P) is trivially true.

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