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A NOTE ON GENERALIZATIONS OF TRANSITIVE SYSTEMS OF TRANSFORMATIONS

BY

J. ACZÉL (WATERLOO, ONT.) AND A. D. WALLACE (GAINESVILLE, FLA.)

The principal purpose of this paper is to extend a result of Hosszu's which concerns what might be called a "transitive system of transformations" and to state the interrelation of our generalization of transitive systems of transformations with that of R. D. Luce.

In order that accumulated genitives should not complicate our statements all too much we will abbreviate "systems of transformations" into "formations".

Defitition 1. A *formation* is a continuous function $T\times X \to X$ such that T and X are non-void Hausdorff spaces.

The value of the anonymous function at the place (t, x) will be written tx. If $Z\subset T$ and if $a\in X$ or $A\subset X$, then $Za = \{za \mid z\in Z\}$ and $ZA = \{za \mid z\in Z \text{ and } a\in A\}$, respectively. Universal quantifiers will generally be omitted.

Definition 2. A *semigroup* is a non-void Hausdorff space together with a continuous associative multiplication, generally denoted by juxtaposition. More formally, a semigroup is such a continuous function $S\times S\to S$ that S is a non-void Hausdorff space and denoting the value of the function at (x, y) by xy, we have $x(yz) = (xy)z$ for all $x, y, z\in S$.

Thus a semigroup is a special instance of a formation. Background material on semigroups may be found in Paalman de Miranda [7] and all results of a topological nature used here can be found in Kelley [5]. This latter also contains requisite definitions and results on relations. For discrete semigroups one may consult Clifford and Preston [2].

It may be helpful to make explicit the fact that a semigroup, as defined in either of these books, is a special instance of a semigroup as defined in this note. In fact, in the principal result, it is explicitly assumed that certain spaces are either *compact* or *discrete*.

The literature contains a variety of conflicting terms for the concepts used here, and nomenclature and terminology are so far from being uniform as to be confusing. Because of the brevity of this note and with a desire to spare the reader, technical words are kept to the barest minimum essential for adequate communication — thus hypotheses are generally written in terms of the symbolism of the above definitions rather than in words.

A *morphism* is a continuous function $k\colon T \to S$, T and S being semigroups, such that $k(xy) = k(x)k(y)$ for all $x, y \in T$. If k is also a homeomorphism onto S, then k is termed an *isomorphism*.

THEOREM 1. *In the formation $T \times X \to X$ assume that each of T and X is either compact or discrete, and that*

$$(1) \qquad t(t'x) = t'(tx) \quad \textit{for all } t, t' \in T \textit{ and all } x \in X,$$

$$(2) \qquad Ta = X \quad \textit{for some } a \in X.$$

Then X is a commutative semigroup under an operation $\circ$ with a as unit, such that

$$tx = (ta) \circ x \quad \textit{for all } t \in T \textit{ and all } x \in X.$$

Conversely, suppose that X is a commutative semigroup with unit a under an operation $\circ$ and that there is a continuous function from T onto X whose value at t is denoted by ta. If, for each $x \in X$, we define

$$tx = (ta) \circ x,$$

then $T \times X \to X$ is a formation such that (1) and (2) hold.

Proof. Let $\mathscr{E} \subset T \times T$ be defined by

$$(t, t') \in \mathscr{E} \quad \text{iff} \quad g(t) = g(t')$$

where $g(t) = ta$. Then (Kelley [5]) there is an analytic diagram

$$\begin{tikzcd} T/\mathscr{E} & \\ T \arrow[u, "\varphi"] \arrow[r, "g"'] & X \arrow[ul, "\psi"'] \end{tikzcd}$$

i.e., $\varphi = \psi g$. Moreover, $T/\mathscr{E}$ is Hausdorff because $\mathscr{E}$ is closed (g is continuous by hypothesis), and also ψ is a homeomorphism onto because it is one-to-one (as is readily verified using (2)) and g and φ are continuous, T and X being either both discrete or both compact.

In the next diagram

$$\begin{tikzcd}
T/\mathscr{E} \times T/\mathscr{E} \arrow[r, "n"] & T/\mathscr{E} \\
T \times T \arrow[u, "\varphi\times\varphi"] \arrow[ur, "m"'] &
\end{tikzcd}$$

the function m is defined by

$$m(t, t') = \psi\big(tg(t')\big) = \psi\big(t(t'a)\big)$$

and is certainly continuous. If $t_1 a = t_1' a$ and $t_2 a = t_2' a$, then we have $t_1(t_2 a) = t_1'(t_2' a)$, at once from (1). Since, by definition, $(\varphi\times\varphi)(t, t') = \big(\varphi(t), \varphi(t')\big)$ this implies, in virtue of the analyticity of the first diagram, that from $(\varphi\times\varphi)(t_1, t_2) = (\varphi\times\varphi)(t_1', t_2')$ we may infer that $m(t_1, t_2) = m(t_1', t_2')$. Thus there is a function n such that this diagram is analytic, $m = (\varphi\times\varphi)n$, and n is continuous in virtue of the continuity of the other two functions and the compactness of T and T/E, see Kelley [5]. (In the discrete case there is nothing to be proved about continuity.)

For esthetic reasons we write

$$z\circ w = n(z, w) \qquad \text{for all } z, w \in T/\mathscr{E}$$

and, with the aid of the homeomorphism ψ, define a continuous operation, also denoted by $\circ$, on X by

$$x\circ y = \psi^{-1}\big(\psi(x)\circ\psi(y)\big)$$

so that ψ is an iseomorphism from the binary algebra $(X, \circ)$ onto the binary algebra $(T/\mathscr{E}, \circ)$.

It should be observed that

$$\varphi(t)\circ\varphi(t') = \psi\big(tg(t')\big) = \psi\big(t(t'a)\big),$$

as follows at once from the analyticity of the diagram and the notation just introduced.

To show that each of the operations $\circ$ is associative it is enough to prove that one of them is, and to this end, noting that the function g is *onto*, we have, for $p, q \in X$

$$p = g(u) \text{ and } q = g(v) \qquad \text{for some } u, v \in T.$$

Now

$$\begin{aligned} p\circ q &= \psi^{-1}(\psi(p))\circ\psi^{-1}(\psi(q)) \\ &= \psi^{-1}(\varphi(u))\circ\psi^{-1}(\varphi(v)) \\ &= \psi^{-1}(\varphi(u)\circ\varphi(v)) \\ &= \psi^{-1}\big(\psi(ug(v))\big) \\ &= ug(v) = uq. \end{aligned}$$

Hence, for $x, y, z \epsilon X$, we get

$$x\circ(y\circ z) = x\circ(tz) = t'(tz),$$

$$y = g(t), \qquad x = g(t')$$

and also

$$(x\circ y)\circ z = t''z, \qquad x\circ y = g(t''),$$

so that

$$g(t'') = x\circ y = t'y = t'g(t).$$

For some $t_0 \epsilon T$, $z = t_0 a$ so that

$$\begin{aligned} t''z &= t''\cdot t_0 a = t_0\cdot t'' a \\ &= t_0\cdot t'(ta) = t'\cdot t_0(ta) \\ &= t'\cdot t(t_0 a) = t'\cdot tz, \end{aligned}$$

from which we conclude that

$$x\circ(y\circ z) = (x\circ y)\circ z.$$

To show that $\circ$ is commutative, the earlier notation is employed, so that

$$p\circ q = uq, \quad \text{and} \quad q\circ p = vp$$

and

$$uq = u(va) = v(ua) = vp.$$

It is clear that a is a right unit for x because

$$x\circ a = ta \text{ with } x = ta, \text{ i.e., } x\circ a = x.$$

The proof of the first part is complete and the converse is immediate.

Remark 1. In the discrete case and under the assumption that

(3) $$Tx = X \quad \text{for all } x \epsilon X,$$

the above was proved by Hosszu [4] and a small modification of his proof is employed here.

Remark 2. Formations satisfying (3) are called *transitive*. (2) is evidently a weaker condition than (3). R. D. Luce has considered another generalization of transitive formations (see (4), below) in [6] (p. 380-398; esp. 390-392), but only in the case where T and X are real intervals. In a conversation between him and one of us (J. Aczél) at the Symposium on Mathematical Learning Processes (Stanford, California 1964), various problems arose and one of them is solved below.

We will need the following definitions:

A *relation* on a space X is a subset R of $X \times X$. If R is *closed* (in the usual topology of $X \times X$), *reflexive* (in the sense that $(x, x) \in R$ for all $x \in X$) and *transitive* (in the sense that $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$, whatever be $x, y, z \in X$) and if X is compact Hausdorff, then there is at least one *R-maximal* element a. That is to say, there is an element $a \in X$ such that $(a, x) \in R$ implies $(x, a) \in R$, e.g. [8]. Of course X is assumed to be non-void.

THEOREM 2. *If, in the formation* $T \times X \to X$, *it is supposed that* T *and* X *are compact and that*

$$\text{(4)} \qquad \text{for any } x, y \in X \text{ either } x \in Ty \text{ or } y \in Tx$$

and that

$$T(Tx) \subset Tx \text{ for each } x \in X,$$

then $Ta = X$ *for some* $a \in X$.

Proof. A relation R on X is defined by $(x, y) \in R$ iff $Tx \subset Ty$. It is clear that R is reflexive and transitive and to see that R is closed one may apply an argument well-known in connection with semigroups. If (x, y) is not in R, then it may be assumed that tx is not in Ty for some $t \in T$ and hence that there are disjoint open sets U' and W containing tx and Ty respectively, because Ty is compact and hence closed. From the fact that T is compact and W is open there is an open set V about y with $TV \subset W$, using the assumed continuity of the formation. Similarly, there is an open set U about x with $tU \subset U'$. Then $U \times V$ is an open set in $X \times X$ which contains (x, y) and which does not intersect R.

Now let a be an R-maximal element and assume that some x is not contained in Ta. Then, by (4), $a \in Tx$ and thus $Ta \subset T(Tx) \subset Tx$ by assumption. Since a is R-maximal we have $Ta \supset Tx$ and thus $Ta = Tx$. But $x \in Tx$.

It follows from Theorem 2 that condition (2) of Theorem 1 can be weakened somewhat and the conditions of Theorem 2 used in its stead.

A. Hajnal has remarked that a maximality argument similar to that used in the proof of Theorem 2 was earlier used to prove that a finite complete graph contains an element which may be joined to any other element, e.g. [3] (p. 115-116). This observation has been extended to the infinite case under suitable topological conditions (without them it surely is *not* true, cf. [3]) in a paper by A. R. Bednarek and A. D. Wallace [1].

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THE UNIVERSITY OF WATERLOO IN WATERLOO, ONT., CANADA
THE UNIVERSITY OF FLORIDA IN GAINESVILLE, FLA., U.S.A.

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