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THE WEAK BASIS THEOREM

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1. Introduction. The purpose of this paper * is to present another proof of the theorem of Bessaga and Pełczyński ([4], Theorem 4) which states that a weak basis in a Fréchet space is a basis. Let $(E, \mathcal{T})$ be a linear topological space over the real or complex scalar field. A *topological* basis, or more briefly a *$\mathcal{T}$-basis*, is a sequence $\{x_i\}$ in the vector space E such that to each $x \in E$ there corresponds a unique sequence of scalars $\{f_i(x)\}$ for which the series $\sum_{i=1}^{\infty} f_i(x) x_i$ converges to x in the topology $\mathcal{T}$ of E. For each positive integer i, the correspondence from x to the i'th coefficient $f_i(x)$ in the basis expansion for x defines a linear functional f_i on E. If a basis has the property that each of its coefficient functionals f_i is continuous, i.e., $f_i \in E^*$, the space of continuous linear functionals on E, then the basis is called a *Schauder basis*. Banach [3], p. 238, first noted that a sequence is a norm basis for a Banach space if it is a basis for the space with its weak topology. Karlin [7], Theorem 1, sketched a proof of this fact. Edwards [6], p. 453-457, gives a detailed proof of the theorem of Bessaga and Pełczyński that a $w(E, E^*)$-basis for a Fréchet space $(E, \mathcal{T})$ is a $\mathcal{T}$-Schauder basis. In this paper we prove a more general version of the theorem, namely, a weak basis of closed subspaces in a Fréchet space is a Schauder basis of subspaces. Ruckle [9], Theorem 1.20, has given an elegant proof of this result for Banach spaces.

If $(E, \mathcal{T})$ is a linear topological space, then a sequence of non-trivial subspaces $\{M_i\}$ of $(E, \mathcal{T})$ is a *basis of subspaces* for $(E, \mathcal{T})$ if and only if to each $x \in E$ corresponds a unique sequence $\{E_i(x)\}$, $E_i(x) \in M_i$, such that the series $\sum_{i=1}^{\infty} E_i(x)$ converges to x in the topology $\mathcal{T}$. Note that for a basis of subspaces the correspondence from x to the i'th term in its series expansion defines a linear transformation E_i from E to itself which

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satisfies $E_i(E) = M_i$, $E_i^2 = E_i$, and $E_iE_j = 0$ if $i \neq j$. A basis of subspaces with the property that each E_i is continuous is called a *Schauder basis of subspaces.* It is evident that a topological basis may be thought of as the special case of a basis of subspaces $\{M_i\}$ in which each M_i has dimension one. In this paper the notation and terminology of [8] will be followed.

2. A weak Schauder basis is a basis. For completeness a proof of the following main lemma is included. Proofs of it for the topological basis case have been given by Dieudonné [5], Proposition 5, and Arsove and Edwards [2], Theorem 11.

LEMMA 1. *A $w(E, E^*)$-Schauder basis of subspaces for a barrelled (tonnelé) space $(E, \mathscr{T})$ is a $\mathscr{T}$-Schauder basis of subspaces.*

Proof. Let $\{M_i\}$ be a weak basis of subspaces for $(E, \mathscr{T})$ and $\{E_i\}$ its associated sequence of $w(E, E^*)$-continuous projections. In a barrel space $(E, \mathscr{T})$ a linear transformation is $w(E, E^*)$-continuous if and only if it is $\mathscr{T}$-continuous ([8], 18.9 (iii), 21.4 (i)). Let

$$S_n(x) = \sum_{i=1}^{n} E_i(x), \qquad x \in E.$$

Since each E_i is linear and $\mathscr{T}$-continuous, it follows that S_n is linear and $\mathscr{T}$-continuous, $n \in w$. It is clear also that $S_n(S_m) = S_{\min\{m,n\}}$. In a locally convex space $(E, \mathscr{T})$ the $w(E, E^*)$-closure and $\mathscr{T}$-closure of a subspace coincide. From this and the fact that $\{M_i\}$ is a $w(E, E^*)$-basis for E it follows that the linear span of $\bigcup_{i=1}^{\infty} M_i$ is dense in $(E, \mathscr{T})$. For each $x \in E$ the sequence $\{S_n(x)\}_{n \in w}$ is weakly convergent to x, so it is weakly bounded, hence bounded. By a generalization of the Banach-Steinhaus theorem valid for barrelled spaces ([8], 12.3) $\{S_n\}$ is equicontinuous. Suppose now that $x \in E$ and V is a neighborhood of 0. There exists a neighborhood U of 0 such that $U+U \subset V$ and by equicontinuity of $\{S_n\}$ there exists a neighborhood W of 0 such that $S_n(y) \in U$ for all n if $y \in W$. Since the linear span of $\bigcup_{i=1}^{\infty} M_i$ is dense in E, there exists $y = \sum_{i=1}^{m} x_i$, $x_i \in M_i$, such that $x - y \in U \cap W$. Note that $S_n(y) = y$ if $n \geqslant m$. Thus if $n \geqslant m$,

$$x - S_n(x) = (x-y) + S_n(y-x) \in U+U \subset V,$$

so x is the $\mathscr{T}$-limit of $S_n(x)$.

3. A weak basis is a basis.

LEMMA 2. *Let $\{M_i\}$ be a $w(E, E^*)$-basis of subspaces for a locally convex Hausdorff space $(E, \mathscr{T})$ and let $\{E_i\}$ be its associated sequence of* *projections. Then there exists a topology* $\mathscr{T}'$ *for* E *such that:*

(1) $(E, \mathscr{T}')$ *is a locally convex Hausdorff space with* $\mathscr{T} \subset \mathscr{T}'$.

(2) *Each* E_i *is* $\mathscr{T}'$*-continuous.*

(3) *If* $(E, \mathscr{T})$ *is metrizable, so is* $(E, \mathscr{T}')$.

(4) *If* $(E, \mathscr{T})$ *is complete and each* M_i *is closed, then* $(E, \mathscr{T}')$ *is complete.*

Proof. Let $\mathscr{V} = \{V\}$ denote a local base for $\mathscr{T}$ consisting of closed, convex, circled neighborhoods of zero. Let

$$\mathscr{V}' = \{V' : V \in \mathscr{V}\}, \qquad \text{where} \qquad V' = \{x \in E : \sum_{i=1}^{n} E_i(x) \in V, \text{ all } n\}.$$

Note the following properties of $\mathscr{V}'$:

(i) If U' and V' belong to $\mathscr{V}'$, then $U \cap V \in \mathscr{V}$ and $(U \cap V)' \subset U' \cap V'$.

(ii) If $U' \in \mathscr{V}'$, then there exists $V \in \mathscr{V}$ such that $V + V \subset U$ and then $V' + V' \subset U'$.

(iii) If $|a| \leqslant 1$ and $V' \in \mathscr{V}'$, then $aV' \subset V'$.

(iv) If $V' \in \mathscr{V}'$ and $x \in E$, then the sequence $\{\sum_{i=1}^{n} E_i(x)\}_{n \in \omega}$ is bounded, so there is a scalar a such that $\sum_{i=1}^{n} E_i(x) \in aV$ for all n. Thus, $x \in aV'$.

It follows ([8], 5.1) that $\mathscr{V}'$ is a local base for a vector topology $\mathscr{T}'$ for E. Each $V' \in \mathscr{V}'$ is locally convex since V is and each E_i is linear; hence $(E, \mathscr{T}')$ is a locally convex space. We assert that $V' \subset V$ for each $V \in \mathscr{V}$. This is so because if $x \in V'$, then $\sum_{i=1}^{n} E_i(x) \in V$ for all n. Now x is the weak limit of this sequence in V, so x belongs to the weak closure of V. Since V is convex and closed, V is equal to its weak closure, i.e., $x \in V$, so $V' \subset V$. This establishes the fact that $\mathscr{T} \subset \mathscr{T}'$. For $(E, \mathscr{T}')$ to be a Hausdorff space it is necessary and sufficient that $\bigcap \{V' : V' \in \mathscr{V}'\} = \{0\}$. This is satisfied since

$$\bigcap \{V' : V' \in \mathscr{V}'\} \subset \bigcap \{V : V \in \mathscr{V}\} = \{0\}.$$

To see that E_i is $\mathscr{T}'$-continuous let $V' \in \mathscr{V}'$ be given. Then there exists $U' \in \mathscr{V}'$ such that $U' + U' \subset V'$. Thus if $x \in U'$, then

$$E_i(x) = \sum_{j=1}^{i} E_j(x) - \sum_{j=1}^{i-1} E_j(x) \in U' + U' \subset V'.$$

If $(E, \mathscr{T})$ is metrizable, we may choose a countable local base $\mathscr{V}$ which generates $\mathscr{T}$ ([8], 6.7) and thus the dependent local base $\mathscr{V}'$ which generates $\mathscr{T}'$ is also countable, so $(E, \mathscr{T}')$ is metrizable ([8], 6.7).

It remains to show that if $(E, \mathscr{T})$ is complete and each M_i is closed, then $(E, \mathscr{T}')$ is complete. Suppose $\{x_\alpha\}$ is a Cauchy net in $(E, \mathscr{T}')$ and $V \in \mathscr{V}$. There exists $\alpha(V)$ such that $\alpha, \beta \geqslant \alpha(V)$ implies $x_\alpha - x_\beta \in V' \subset V$; so $\{x_\alpha\}$ is a Cauchy net in $(E, \mathscr{T})$. Let x be the $\mathscr{T}$-limit of $\{x_\alpha\}$. We wish to show that $\lim_\alpha x_\alpha = x$ relative to $\mathscr{T}'$. Now if $\alpha, \beta \geqslant \alpha(V)$, then

$$\sum_{i=1}^{n} E_i(x_\alpha - x_\beta) = \Big(\sum_{i=1}^{n} E_i(x_\alpha) - \sum_{i=1}^{n} E_i(x_\beta)\Big) \in V$$

for all n. Thus, for each fixed n, $\{\sum_{i=1}^{n} E_i(x_\alpha)\}_\alpha$ is a Cauchy net in $(E, \mathscr{T})$. Therefore, $\{E_n(x_\alpha)\}_\alpha$ is a Cauchy net in $(E, \mathscr{T})$ being the difference of the two Cauchy nets $\{\sum_{i=1}^{n} E_i(x_\alpha)\}$ and $\{\sum_{i=1}^{n-1} E_i(x_\alpha)\}$. Since M_n is a closed subspace of the complete Hausdorff space $(E, \mathscr{T})$, it too is complete, so there exists $x_n \in M_n$ such that $x_n = \lim_\alpha E_n(x_\alpha)$ relatively to $\mathscr{T}$. We next show that x is the weak limit of the series $\sum_{n=1}^{\infty} x_n$. To this end let $f \in E^*$ and $\varepsilon > 0$ be given. Then there exists $U \in \mathscr{V}$ such that $|f(y)| < \varepsilon/3$ if $y \in U$. Also there exists $\alpha(U)$ such that $\alpha, \beta \geqslant \alpha(U)$ implies $x_\alpha - x_\beta \in U'$, i.e.,

$$\Big(\sum_{i=1}^{n} E_i(x_\alpha) - \sum_{i=1}^{n} E_i(x_\beta)\Big) \in U \qquad \text{for all } n.$$

Thus, passing to the limit with β we obtain

$$\Big(\sum_{i=1}^{n} E_i(x_\alpha) - \sum_{i=1}^{n} x_i\Big) \in U$$

for all n if $\alpha \geqslant \alpha(U)$. Since $\lim_\alpha x_\alpha = x$ relatively to $\mathscr{T}$ and f is $\mathscr{T}$-continuous, there exists α_0 with $\alpha_0 \geqslant \alpha(U)$ such that $|f(x) - f(x_{\alpha_0})| < \varepsilon/3$. Since x_{α_0} is the weak limit of $\sum_{i=1}^{\infty} E_i(x_{\alpha_0})$, there exists $n(\alpha_0)$ such that $n \geqslant n(\alpha_0)$ implies

$$\Big|f(x_{\alpha_0}) - f\Big(\sum_{i=1}^{n} E_i(x_{\alpha_0})\Big)\Big| < \varepsilon/3.$$

Thus if $n \geqslant n(\alpha_0)$,

$$\begin{aligned}&\Big|f(x) - f\Big(\sum_{i=1}^{n} x_i\Big)\Big| \\ &\leqslant |f(x) - f(x_{\alpha_0})| + \Big|f(x_{\alpha_0}) - f\Big(\sum_{i=1}^{n} E_i(x_{\alpha_0})\Big)\Big| + \Big|f\Big(\sum_{i=1}^{n} E_i(x_{\alpha_0}) - \sum_{i=1}^{n} x_i\Big)\Big| < \varepsilon.\end{aligned}$$

We have shown that x is the weak limit of the series $\sum_{i=1}^{\infty} x_i$, $x_i \in M_i$, so $x_i = E_i(x)$ because the weak basis of subspaces expansion for x is unique. Furthermore, for the V and $\alpha(V)$ at the beginning of this completeness proof we have

$$\Big(\sum_{i=1}^{n} E_i(x_\alpha) - \sum_{i=1}^{n} E_i(x_\beta)\Big) \in V$$

for all n if $\alpha, \beta \geqslant \alpha(V)$. Passing to the limit with β and using the fact that $x_i = E_i(x)$ yields

$$\Big(\sum_{i=1}^{n} E_i(x_\alpha) - \sum_{i=1}^{n} E_i(x)\Big) \in V$$

for all n if $\alpha \geqslant \alpha(V)$, i.e., $x_\alpha - x \in V'$ if $\alpha \geqslant \alpha(V)$, so $\lim_\alpha x_\alpha = x$ relatively to $\mathcal{T}'$.

THEOREM 1. *A $w(E, E^*)$-basis of closed subspaces for a Fréchet space $(E, \mathcal{T})$ is a $\mathcal{T}$-Schauder basis of subspaces.*

Proof. Let $\{M_i\}$ be a weak basis of closed subspaces and $\{E_i\}$ its associated projections. It is immediate from (1), (3), and (4) of Lemma 2 that since $(E, \mathcal{T})$ is a Fréchet space, so is $(E, \mathcal{T}')$ and $\mathcal{T} \subset \mathcal{T}'$. It is clear from $\mathcal{T} \subset \mathcal{T}'$ that the identity mapping I from $(E, \mathcal{T}')$ into $(E, \mathcal{T})$ is continuous. By the open mapping theorem ([8], 11.4) I is bicontinuous, so $\mathcal{T} = \mathcal{T}'$. From this and (2) of Lemma 2 each E_i is $\mathcal{T}$-continuous. Now a Fréchet space is barrelled ([8], p. 104); so $\{M_i\}$ is a weak Schauder basis of subspaces and therefore, by Lemma 1, a $\mathcal{T}$-Schauder basis of subspaces.

4. The continuity theorem. By omitting the terms "$w(E, E^*)$" and "locally convex", from Lemma 2 a valid Lemma 3 is obtained. The topology $\mathcal{T}'$ in Lemma 3 is defined in the same manner as in Lemma 2. The proof of Lemma 3 is much the same as Lemma 2 except simpler. From Lemma 3 and the open mapping theorem the following continuity theorem is immediate. Banach [3], p. 110, proved the theorem for bases of vectors in a Banach space. It was generalized to the following form by S. Mazur in a seminar on functional analysis in 1955 in Warsaw (Arsove [1], Theorem 2, has given a proof of the theorem for bases of vectors):

THEOREM 2. *A basis of closed subspaces for a complete metric linear space is a Schauder basis of subspaces.*

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