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CONVERGENCE OF TRUNCATED SINGULAR INTEGRALS WITH TWO WEIGHTS

BY

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1. Introduction. By $\mathbf{R}^n$ we denote the n-dimensional euclidean space and for x and y in $\mathbf{R}^n$ we set $(x \cdot y) = \sum_{i=1}^n x_i y_i$ and $|x| = (x \cdot x)^{1/2}$. If E is a Lebesgue measurable set contained in $\mathbf{R}^n$, $|E|$ stands for its Lebesgue measure. We shall say that an ordered pair of non-negative measurable functions $(v(x), u(x))$ defined on $\mathbf{R}^n$ belongs to the class $A(p,q)$, $1 \le p \le \infty$ and $1 \le q \le \infty$, if there exists a finite constant C such that for every ball (or every cube) $B \subset \mathbf{R}^n$

$$\left(|B|^{-1} \int_B u(x)^{-p'}\,dx\right)^{1/p'} \left(|B|^{-1} \int_B v(x)^q\,dx\right)^{1/q} \le C. \tag{1.1}$$

Here and in the sequel, p' denotes the conjugate exponent of p, i.e. $p' = p/(p-1)$.

Let $w(x)$ be a non-negative measurable function. We denote by $L^p(w)$, $1 \le p \le \infty$, the class of measurable functions f such that

$$\|f\|_{L^p(w)} = \Big(\int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx\Big)^{1/p} < \infty.$$

If $w = 1$, we simply write L^p. The element of surface area of the unit sphere $\Sigma = \{x : |x| = 1,\ x \in \mathbf{R}^n\}$ will be denoted by $d\sigma(x)$. The space $L^p(\Sigma)$, $1 \le p \le \infty$, is the class of all measurable functions f defined on Σ such that

$$\|f\|_{L^p(\Sigma)} = \Big(\int_\Sigma |f(x)|^p\,d\sigma(x)\Big)^{1/p} < \infty.$$

Let $k(x) = \Omega(x)/|x|^n$ be a function defined on $\mathbf{R}^n \setminus \{0\}$. We shall say that k is an *L^r-Dini singular integral kernel* (see [2] and [4]) if $\Omega(x)$ satisfies the following assumptions:

(i) $\Omega(x)$ is a positively homogeneous function of degree zero,

(ii) $\Omega \in L^r(\Sigma)$, $1 \le r \le \infty$, and if

$$w_r(\delta) = \sup_{|\rho|<\delta} \Big(\int_\Sigma |\Omega(\rho x) - \Omega(x)|^r \, d\sigma(x) \Big)^{1/r},$$

where ρ denotes a rotation and $|\rho| = \sup_{x\in\Sigma} |\rho x - x|$, then

$$\int_0^1 w_r(\delta)\delta^{-1}\, d\delta < \infty,$$

(iii) $\int_\Sigma \Omega(x)\, d\sigma(x) = 0$.

Let k be an L^r-Dini singular integral kernel and $\eta > 0$. The function $k_\eta(x)$ defined as

$$\begin{aligned} k_\eta(x) &= k(x) && \text{if } |x| \ge \eta, \\ k_\eta(x) &= 0 && \text{otherwise,} \end{aligned}$$

will be called the *truncated singular integral kernel* of k. The *truncated singular integral* of a function f is defined as

$$K_\eta(f)(x) = \int_{\mathbf{R}^n} k_\eta(x-y) f(y)\, dy = \int_{|x-y|>\eta} k(x-y) f(y)\, dy,$$

and the *singular integral* of f is defined as the limit .

$$K(f)(x) = \lim_{\eta\to 0} K_\eta(f)(x) = \lim_{\eta\to 0} \int_{|x-y|>\eta} k(x-y) f(y)\, dy$$

whenever this limit exists.

The main result of this paper is the following theorem:

(1.2) THEOREM A. *Let $k(x)$ be an L^r-Dini singular integral kernel and $(v,u) \in A(p, p(r/p)')$, $1 \le p \le r < \infty$. If f and $K(f)$ belong to $L^p(u^p)$, then*

(i) $$\|K_\varepsilon(f)\|_{L^p(v^p)} \le C\{\|f\|_{L^p(u^p)} + \|K(f)\|_{L^p(u^p)}\}$$

for every $\varepsilon > 0$ with a constant C not depending on ε and f. Moreover,

(ii) $$\lim_{\varepsilon\to 0} \|K_\varepsilon(f) - K(f)\|_{L^p(v^p)} = 0.$$

For the case $v = u$, $1 = p < r$, Theorem A is known (see [3]), for $1 < p < r$, the continuity of the maximal singular integral K^* holds and implies Theorem A (see [1] and [3]). In the case of different weights, $v \ne u$, we do not even have in general the continuity of $K(f)$. Thus, the proof of Theorem A does not distinguish whether p is equal to one or not. We shall use in the proof a generalization of a remarkable result of B. Muckenhoupt (see [5]) which we believe has an independent interest; we state it in the following theorem:

(1.3) THEOREM B. *Let $F(x)$ be a function defined on $\mathbf{R}^n$. For $k \in \mathbf{Z}$, set*

$$c_k = \Big(\int\limits_{2^k < |x| \le 2^{k+1}} |F(x)|^r \, dx \Big)^{1/r} .$$

Assume that

$$A = \sum_{k \in \mathbf{Z}} c_k 2^{kn/r'} < \infty .$$

Then, if $F_\varepsilon(x) = \varepsilon^{-n} F(x/\varepsilon)$, $1 \le p < r \le \infty$ or $1 \le p = r < \infty$, $(v,u) \in A(p, p(r/p)')$ and $f \in L^p(u^p)$, we have

(i) $$\|F_\varepsilon * f\|_{L^p(v^p)} \le c\, A \|f\|_{L^p(u^p)}$$

with a constant c depending on n, r, p and the constant C of (1.1) only. Moreover, for $a = \int_{\mathbf{R}^n} F(x)\, dx$,

(ii) $$\lim_{\varepsilon \to 0} \|F_\varepsilon * f - af\|_{L^p(v^p)} = 0 .$$

2. Proof of the Muckenhoupt-type theorem on the approximation of the identity. The main property of the L^r-Dini singular integral kernels that will be used in the sequel is the following lemma due to D. S. Kurtz and R. L. Wheeden, see [4]:

(2.1) LEMMA. *Let $k(x)$ be an L^r-Dini singular integral kernel and $|y| < R/2$. Then*

$$\Big(\int\limits_{R<|x|<2R} |k(x-y) - k(x)|^r \, dx \Big)^{1/r} \le c\, R^{-n/r'} \Big(|y|/R + \int\limits_{|y|/2R < \delta < |y|/R} w_r(\delta) \delta^{-1} \, d\delta \Big),$$

where c does not depend on $R > 0$.

Proof of Theorem B, (1.3). (i) Let χ_k be the characteristic function of the annulus $2^k < |x| \le 2^{k+1}$, $k \in \mathbf{Z}$. Let $\{Q_j\}$ be a partition of $\mathbf{R}^n$ into cubes with sides of length $2^{k+1}\varepsilon$. Then we have

(2.2) $$\|(F\chi_k)_\varepsilon * f\|^p_{L^p(v^p)} \le \varepsilon^{-np} \sum_j \int\limits_{Q_j} \Big(\int |(F\chi_k)((z-x)/\varepsilon)|\, |f(x)| \, dx \Big)^p v(z)^p \, dz .$$

By Minkowski's integral inequality, (2.2) is bounded by

$$\varepsilon^{-np} \sum_j \Big[\int\limits_{3Q_j} |f(x)| \Big(\int\limits_{Q_j} |(F\chi_k)((z-x)/\varepsilon)|^p v(z)^p \, dz \Big)^{1/p} dx \Big]^p .$$

For each j, by Hölder's inequality, we have

$$\Big(\int_{Q_j} |(F\chi_k)((z-x)/\varepsilon)|^p v(z)^p\, dz\Big)^{1/p}$$
$$\le \varepsilon^{n/r}\Big(\int_{2^k<|z|\le 2^{k+1}} |F(z)|^r\, dz\Big)^{1/r}\Big(\int_{Q_j} v(z)^{p(r/p)'}\, dz\Big)^{1/p(r/p)'}$$
$$= \varepsilon^{n/r} c_k \Big(\int_{Q_j} v(z)^{p(r/p)'}\, dz\Big)^{1/p(r/p)'}.$$

Thus, (2.2) is bounded by

$$\varepsilon^{-np/r'} c_k^p \sum_j \Big(\int_{Q_j} v(z)^{p(r/p)'}\, dz\Big)^{1/(r/p)'}\Big(\int_{3Q_j} |f(x)|\, dx\Big)^p.$$

Since, by Hölder's inequality,

$$\Big(\int_{3Q_j} |f(x)|\, dx\Big)^p \le \Big(\int_{3Q_j} |f(x)|^p u(x)^p\, dx\Big)\Big(\int_{3Q_j} u(x)^{-p'}\, dx\Big)^{p/p'},$$

and taking into account that $(v,u)\in A(p,p(r/p)')$, it follows that (2.2) is bounded by a constant times

$$c_k^p 2^{k\,np/r'}\|f\|^p_{L^p(u^p)}.$$

Then, by Minkowski's inequality, we obtain

$$\|F_\varepsilon * f\|_{L^p(v^p)} \le \sum_{k\in\mathbf{Z}} \|(F\chi_k)_\varepsilon * f\|_{L^p(u^p)} \le c\Big(\sum_{k\in\mathbf{Z}} c_k 2^{kn/r'}\Big)\|f\|_{L^p(u^p)}.$$

(ii) Assume first that f is a bounded function with bounded support and N is a number large enough so that $|x|\le N$ contains the support of f, and

$$\int_{|x|\le N} v(x)^p\, dx > 0, \qquad \int_{|x|\le N} u(x)^{-p'}\, dx > 0.$$

We observe that it is always possible to find N unless $v=0$ a.e. or $u=\infty$ a.e., the cases that are trivial. We have

$$\|F_\varepsilon * f - af\|^p_{L^p(v^p)}$$
$$\le \Big(\int_{|x|\le 3N} + \int_{|x|>3N}\Big)|F_\varepsilon * f - af|^p v^p\, dx = I_1^p(\varepsilon) + I_2^p(\varepsilon).$$

Given $\eta>0$, let M satisfy

$$\int_{|z|>M} |F(z)|dz < \eta.$$

Such an M exists since our assumption on $\{c_k\}$ implies that F is integrable on $\mathbf{R}^n$. From the fact that v^p is locally integrable and that f is a bounded function with bounded support it follows easily that

$$\left(\int |f(x-\varepsilon z)-f(x)|^p v(x)^p dx\right)^{1/p} < \eta$$

if ε is small enough. Then

$$I_1(\varepsilon) \le \left(\int_{|x|\le 3N}\left(\int_{|z|>M} |F(z)|\,|f(x-\varepsilon z)-f(x)|\,dz\right)^p v(x)^p\,dx\right)^{1/p}$$
$$+\left(\int_{|x|\le 3N}\left(\int_{|z|\le M} |F(z)|\,|f(x-\varepsilon z)-f(x)|\,dz\right)^p v(x)^p\,dx\right)^{1/p}.$$

Since f is bounded, applying Minkowski's integral inequality we get

$$I_1(\varepsilon) \le 2\|f\|_\infty \int_{|z|>M} |F(z)|\,dz\left(\int_{|x|\le 3N} v(x)^p\,dx\right)^{1/p}$$
$$+\int_{|z|\le M} |F(z)|\left(\int |f(x-\varepsilon z)-f(x)|^p v(x)^p\,dx\right)^{1/p} dz$$
$$\le \left\{2\|f\|_\infty\left(\int_{|x|\le 3N} v(x)^p\,dx\right)^{1/p} + \|F\|_{L^1}\right\}\eta,$$

showing that $I_1(\varepsilon)$ tends to zero with ε. Let us estimate $I_2(\varepsilon)$. By Minkowski's integral inequality, we have

$$\text{(2.3)}\qquad \left(\int_{|x|>3N} |(F\chi_k)_\varepsilon * f|^p v^p dx\right)^{1/p}$$
$$\le \varepsilon^{-n}\int |f(z)|\left(\int_{|x|>3N} |(F\chi_k)((x-z)/\varepsilon)|^p v(x)^p\,dx\right)^{1/p} dz.$$

If z belongs to the support of f and $|x| > 3N$ it follows that $|z| \le N$ and $|x - z| > 2N$. Then, the right hand side of (2.3) is equal to zero for $\varepsilon 2^{k+1} \le 2N$, that is to say, for $k \le \log_2(N/\varepsilon)$. For $k > \log_2(N/\varepsilon)$, by Hölder's inequality, we get

$$\text{(2.4)}\qquad \left(\int_{|x|>3N} |(F\chi_k)((x-z)/\varepsilon)|^p v(x)^p\,dx\right)^{1/p}$$
$$\le \varepsilon^{n/r}\left(\int_{2^k<|z|\le 2^{k+1}} |F(z)|^r\,dz\right)^{1/r}\left(\int_{|x-z|\le 2^{k+1}\varepsilon} v(x)^{p(r/p)'}\,dx\right)^{1/p(r/p)'}.$$

Multiplying and dividing by $(\int_{|x|<N} u(x)^{-p'}\,dx)^{1/p'}$ and taking into account

that $|x| \le N$ implies $|x - z| < 2^{k+1}\varepsilon$ whenever $|z| < N$, we find that (2.4) is bounded by

$$c\varepsilon^n c_k 2^{kn/r'} \Big(\int_{|x|<N} u(x)^{-p'} dx \Big)^{-1/p'} .$$

Thus, for (2.3) we have the bound

$$c\|f\|_{L^1} \Big(\int_{|x|<N} u(x)^{-p'} dx \Big)^{-1/p'} c_k 2^{kn/r'} .$$

This shows that $I_2(\varepsilon)$ is smaller than a constant times $\sum_{k>\log_2(N/\varepsilon)} c_k 2^{kn/r'}$, which goes to zero with ε. We have shown that (ii) holds for functions f bounded and with bounded support.

It is easy to see that the functions g in $L^p(u^p)$ bounded and with bounded support are dense in $L^p(u^p)$. On the other hand, we observe that by the Lebesgue differentiation theorem, the condition $(v, u) \in A(p, p(r/p)')$ implies that $v(x) \le c\, u(x)$ a.e. Thus,

$$\text{(2.5)} \quad \|F_\varepsilon * f - af\|_{L^p(v^p)} \\ \le \|F_\varepsilon * (f - g)\|_{L^p(v^p)} + \|F_\varepsilon * g - ag\|_{L^p(v^p)} + |a|\, \|f - g\|_{L^p(v^p)} .$$

Then, by parts (i) and (ii) already proved for bounded functions with bounded support, we see that (2.5) is bounded by

$$c\|f - g\|_{L^p(u^p)} + \eta$$

if ε is close to zero. The density of the functions g in $L^p(u^p)$ completes the proof of (ii).

(2.6) Corollary. *Let $H(x)$ be a positively homogeneous function of degree zero belonging to $L^r(\Sigma)$, $1 \le r \le \infty$, and φ a function with least decreasing radial majorant function ψ ($|\varphi(x)| \le \psi(|x|)$) integrable on $\mathbf{R}^n$. If (v, u) belongs to $A(p, p(r/p)')$ and $f \in L^p(u^p)$, $1 \le p < r \le \infty$ or $1 \le p = r < \infty$, then*

$$\text{(i)} \qquad \|(H\varphi)_\varepsilon * f\|_{L^p(v^p)} \le c\|H\|_{L^r(\Sigma)} \int_0^\infty \psi(t) t^{n-1}\, dt \cdot \|f\|_{L^p(u^p)} ,$$

and moreover if $a = \int_{\mathbf{R}^n} H(x)\varphi(x)\, dx$, then

$$\text{(ii)} \qquad \lim_{\varepsilon \to 0} \|(H\varphi)_\varepsilon * f - af\|_{L^p(v^p)} = 0 .$$

Proof. Let $F = H\varphi$. For this F we estimate the sequence c_k of Theorem B, (1.3). We have

$$c_k = \Big(\int_{2^k < |x| \le 2^{k+1}} |(H\varphi)(x)|^r dx \Big)^{1/r} \le c_{n,r} \psi(2^k) \|H\|_{L^r(\Sigma)} \cdot 2^{kn/r} .$$

Thus,

$$\begin{aligned}\sum_{k\in\mathbf{Z}} c_k 2^{kn/r'} &\le c_{n,r}\|H\|_{L^r(\Sigma)} \sum_{k\in\mathbf{Z}} \psi(2^k)2^{kn} \\ &\le c'_{n,r}\|H\|_{L^r(\Sigma)} \int_0^\infty \psi(t)t^{n-1}\,dt\,.\end{aligned}$$

(2.7) COROLLARY. *Let $k(x)$ be an L^r-Dini singular integral kernel and $\varphi(x) \in C^1$ a function supported in the unit ball with $\int \varphi(x)\,dx = 1$. Define*

$$\delta(x) = K(\varphi)(x) - k_1(x)\,,$$

where $k_1(x) = k(x)$ if $|x| \ge 1$ and $k_1(x) = 0$ otherwise. Then the kernel $\delta_\varepsilon(x) = \varepsilon^{-n}\delta(x/\varepsilon)$ satisfies

$$\text{(i)} \qquad \|\delta_\varepsilon * f\|_{L^p(v^p)} \le c\|f\|_{L^p(u^p)}\,,$$

$$\text{(ii)} \qquad \lim_{\varepsilon\to 0}\|\delta_\varepsilon * f\|_{L^p(v^p)} = 0\,,$$

whenever (v,u) belongs to $A(p,p(r/p)')$, $1 \le p < r \le \infty$ or $1 \le p = r < \infty$.

Proof. Let $\delta^{(1)}(x) = \delta(x)$ for $|x| \le 4$ and $\delta^{(1)}(x) = 0$ if $|x| \ge 4$. Since $\varphi \in C^1$ it follows that $|\delta^{(1)}(x)| \le c(1 + |\Omega(x)|)$ for $|x| \le 4$. Then, by Corollary (2.6), we find that $\delta^{(1)}$ satisfies (i). Let $\delta^{(2)}(x) = \delta(x) - \delta^{(1)}(x)$. Then we have

$$|\delta^{(2)}(x)| \le \|\varphi\|_\infty \int_{|y|\le 1} |k(x-y) - k(x)|\,dy\,.$$

Let us estimate the sequence c_k of Theorem B, (1.3), for $F(x) = \delta^{(2)}(x)$. We observe that $c_k = 0$ for $k < 2$. In the case $k \ge 2$, we have

$$\begin{aligned}c_k &\le c\Big(\int_{2^k<|x|\le 2^{k+1}} \Big(\int_{|y|\le 1} |k(x-y) - k(x)|\,dy\Big)^r dx\Big)^{1/r} \\ &\le c\int_{|y|\le 1}\Big(\int_{2^k<|x|\le 2^{k+1}} |k(x-y) - k(x)|^r\,dx\Big)^{1/r} dy\,.\end{aligned}$$

Thus, by Lemma (2.1), we get

$$c_k \le c\cdot 2^{-kn/r'}(2^{-k} + w_r(2^{-k}))\,.$$

Therefore,

$$\sum_{k\in\mathbf{Z}} c_k 2^{kn/r'} \le c\sum_{k\ge 2}(2^{-k} + w_r(2^{-k})) = c\Big(1 + \int_0^1 w_r(t)\,dt/t\Big) < \infty\,.$$

Thus, we can apply Theorem B, (1.3), to $F = \delta^{(2)}$ and obtain part (i) of Corollary (2.7) for $\delta^{(2)}$. This, together with the result already obtained for $\delta^{(1)}$, proves (i) for $\delta = \delta^{(1)} + \delta^{(2)}$. As for part (ii), it is enough to prove that

$\int \delta(x)dx = 0$. Let $N > 0$. Then, since $\int_\Sigma k(x)d\sigma(x) = 0$ and recalling that $\varphi \in C^1$, we have

$$\int_{|x|<N} \delta(x)\,dx = \int_{|x|<N} K(\varphi)(x)\,dx$$
$$= \int_{|x|<N} dx \int_{|y|<N+1} k(y)[\varphi(x-y) - \varphi(x)]\,dy.$$

Changing the order of integration, we get

$$\int_{|x|<N} \delta(x)\,dx = \int_{|y|<N+1} k(y)\Big(\int_{|x|<N} [\varphi(x-y) - \varphi(x)]\,dx\Big)\,dy$$
$$= \int_{N-1<|y|<N+1} + \int_{|y|\le N-1} = I_1 + I_2\,.$$

It is easy to check that $I_2 = 0$. For I_1 we have

$$|I_1| \le 2\|\varphi\|_{L^1} \int_{N-1<|y|<N+1} |k(y)|\,dy \le c\|\varphi\|_{L^1}\|\Omega\|_{L^1(\Sigma)}N^{-1}\,.$$

Therefore,

$$\int_{\mathbf{R}^n} \delta(x)\,dx = \lim_{N\to\infty} \int_{|x|<N} \delta(x)\,dx = 0\,,$$

as we wanted to show.

3. Proof of the main result. First of all, we shall show in Proposition (3.4) that $K_\eta(f)(x)$ and $K(f)(x)$ are defined almost everywhere on $\mathbf{R}^n$. For this purpose we shall need the next two lemmas.

(3.1) LEMMA. *Let $1 < p \le r < \infty$. Then for (v,u) belonging to $A(p, p(r/p)')$,*

$$\int u(x)^{-p'}(1+|x|)^{-np'/r'-p'/r}\,dx < \infty\,.$$

The proof is simple and will not be given.

(3.2) LEMMA. *If $(v,u) \in A(p, p(r/p)')$, $1 \le p \le r < \infty$, then*

$$\|f\|_{L^1((1+|x|)^{-n/r'-1/r})} \le c\|f\|_{L^p(u^p)}\,, \tag{3.3}$$

where the constant c does not depend on f.

Proof. If $1 < p \le r$, by Hölder's inequality we get

$$\int |f(x)|(1+|x|)^{-n/r'-1/r}\,dx$$
$$\le \Big(\int |f(x)|^p u(x)^p\,dx\Big)^{1/p}\Big(\int u(x)^{-p'}(1+|x|)^{-np'/r'-p'/r}\,dx\Big)^{1/p'}\,.$$

Then, by Lemma (3.1), it follows that (3.3) holds. In the case $p = 1$, we observe that $(v, u) \in A(1, r')$ implies that $(1 + |x|)^{-n/r'-1/r} \le c\, u(x)$ a.e. Thus (3.3) also holds in this case.

We observe that Lemma (3.2) implies, in particular, that a function belonging to $L^p(u^p)$ with (v, u) in $A(p, (p(r/p)')$ belongs locally to L^1.

(3.4) PROPOSITION. *Let k be an L^r-Dini singular integral kernel. If f belongs to $L^1((1+|x|)^{-n/r'-1/r})$, then the truncated singular integral $K_\eta(f)(x)$ and the singular integral $K(f)(x)$ exist for almost every x. In particular, by Lemma (3.2), this holds if $f \in L^p(u^p)$, $(v, u) \in A(p, p(r/p)')$ and $1 \le p \le r < \infty$.*

Proof. Let $T > 1 > \eta$. For $|x| < T$, we set

$$I(x) = \int_{|y|>3T} |k(x-y)|\,|f(y)|\,dy.$$

Then, integrating $I(x)$, we get

$$\int_{|x|<T} I(x)\,dx \le c \int_{|x|<T} \Big(\int_{|y|>3T} |y|^{-n} |\Omega(x-y)|\,|f(y)|\,dy \Big)\,dx$$

$$= c \int_{|y|>3T} |f(y)|\,|y|^{-n} \Big(\int_{|x|<T} |\Omega(x-y)|\,dx \Big)\,dy.$$

By Hölder's inequality and enlarging the domain of integration, we obtain

$$\int_{|x|<T} |\Omega(x-y)|\,dx \le c \Big(\int_{|y|-T<|z|<|y|+T} |\Omega(z)|^r\,dz \Big)^{1/r}.$$

Recalling that $\Omega(z)$ is a homogeneous function of degree zero we have

$$\int_{|x|<T} |\Omega(x-y)|\,dx \le c' \|\Omega\|_{L^r(\Sigma)} |y|^{(n-1)/r}. \tag{3.5}$$

Thus, by Lemma (3.2) it follows that

$$\int_{|x|<T} I(x)\,dx \le c \int_{|y|>3T} |f(y)|\,|y|^{-n+(n-1)/r}\,dy$$

$$\le c'' \int |f(y)| (1 + |y|)^{-n/r'-1/r}\,dy < \infty,$$

which shows that $I(x)$ is finite almost everywhere on $|x| < T$. On the other hand, since f belongs locally to L^1 as we observed after the proof of Lemma (3.2), we find that

$$\int_{\substack{|x-y|>\eta \\ |y|<3T}} k(x-y) f(y)\,dy$$