COLLOQUIUM MATHEMATICUM

VOL. LIII 1987 FASC. 2

THE GEOMETRY OF A SEMI-DIRECT EXTENSION OF A HEISENBERG TYPE NILPOTENT GROUP

BY

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The notion of nilpotent group of type H introduced by Kaplan [5] has attracted considerable attention ([6], [7], [9], [10], [14], [15]). These groups include the nilpotent subgroups N of semisimple rank one Lie groups G which appear in the Iwasawa decomposition $G = NAK$.

Following ideas of Cygan and Kaplan and Putz [8] we go a step further and we look at the groups $S = NA$, where N is a type H group and A a group of dilations of N, equipped with a suitable left-invariant metric. We thus obtain a generalization of rank one symmetric spaces

The aim of this paper is to describe the group $I(S)$ of isometries of S. As expected it turns out that only in the classical case, i.e., where $S = G/K$, $I(S)$ is large. In all the remaining cases $I(S)$ appears to be as small as possible, i.e., the semi-direct product of the group $A(S)$ of automorphisms of S which preserve the inner product (cf. definitions below) and the group S itself (Theorem 4.4).

The main idea of the proof is to describe the set $\{d\eta_e\colon \eta \in I(S),\ \eta(e) = e\}$. Our reasoning is based on the fact that $d\eta_e$ is orthogonal and it preserves the connection ∇, the curvature tensor R, and its covariant derivative ∇R. For the non-classical group S these conditions imply that $d\eta_e$ must be an automorphism of $\mathfrak{s}$ in the following way ($\mathfrak{s}$, $\mathfrak{n}$, and $\mathfrak{a}$ denote Lie algebras of S, N and A, respectively).

The main point is Theorem 4.2 which states that for non-classical cases $\nabla R(x) = 0$ iff $x \in \mathfrak{a}$. To prove this we consider two cases:

(i) $\mathfrak{n} = \boldsymbol{O}^n \times \boldsymbol{O}$, $\boldsymbol{O}$ being the octonions, $n > 1$;

(ii) $\dim Z \neq 0, 1, 3, 7$, Z being the center of $\mathfrak{n}$

(cf. Propositions 4.2 and 4.3). Now, from Theorem 4.2 we conclude that $d\eta_e(\mathfrak{a}) \subset \mathfrak{a}$, $d\eta_e(V) \subset V$, $d\eta_e(Z) \subset Z$ and, finally, that $d\eta_e$ is an automorphism (Theorem 4.3).

Our choice of invariants ∇, R, and ∇R is somewhat arbitrary and it is quite likely that by selecting other invariants one could obtain a simpler proof.

The author is grateful to Jacek Cygan, Witold Roter, and Andrzej Hulanicki for helpful suggestions and remarks concerning this paper.

1. Introduction. Let $\mathfrak{n}$ be a nilpotent 2-step algebra with an inner product. Denote by V the orthogonal complement to its center Z. Then, for every $v \in V$, ad_v maps V into Z and we have the orthogonal decomposition of V given by

$$V = \mathrm{Ker}\,\mathrm{ad}_v \oplus D(v).$$

$\mathfrak{n}$ is said to be of *Heisenberg type* (shortly, of *type H*) if for every unit vector $v \in V$ the mapping ad_v: $D(v) \to Z$ is a surjective isometry.

Every Lie algebra of type H arises as follows [5]. Let U and V be vector spaces with positive definite quadratic forms $|\cdot|^2$. By definition, a composition of these quadratic forms is a bilinear map μ: $U \times V \to V$ which satisfies

$$|\mu(u, v)| = |u|\,|v|, \qquad u \in U,\ v \in V,$$

and for a u_0

$$\mu(u_0, v) = v, \qquad v \in V.$$

Let Z be the orthogonal complement to $\boldsymbol{R}u_0$ and let π: $U \to Z$ be the orthogonal projection. Define a bilinear map Φ: $V \times V \to U$ by

$$\langle u, \Phi(v, v')\rangle = \langle \mu(u, v), v'\rangle. \tag{1.1}$$

Then $\pi\Phi$ is skew-symmetric [5] and $\mathfrak{n} = V \times Z$ with the bracket

$$[(v, z), (v', z')] = \big(0, \pi\Phi(v, v')\big),$$

and the inner product

$$\langle (v, z), (v', z')\rangle = \langle v, v'\rangle + \langle z, z'\rangle$$

is an algebra of type H.

Let ϱ be a function defined on nonnegative integers by the condition: if $n = (\mathrm{odd})\,2^{4p+q}$, $0 \leqslant q \leqslant 3$, then

$$\varrho(n) = 8p + 2^q.$$

An algebra of type H with $\dim V = n$ and $\dim Z = m-1$ exists if and only if $m \leqslant \varrho(n)$ (see [2]). In particular, the equality $m = n$ yields $n = 1, 2, 4, 8$.

Let N be a connected and simple connected Lie group whose Lie algebra is $\mathfrak{n}$. If we identify N with $\mathfrak{n}$ by the exponential map, the multiplication in N is given by

$$(v, z)(v', z') = \big(v + v', z + z' + \tfrac{1}{2}\pi\Phi(v, v')\big).$$

We denote by A the multiplicative group of $\boldsymbol{R}^+$. Let

$$S = NA \tag{1.2}$$

be a semi-direct product of N and A, A acting on N as dilations $\delta_a(v, z) = (av, a^2 z)$. Thus we identify S with $V \times Z \times A$ and

$$(v, z, a)(v', z', a') = \left(v + av',\ z + a^2 z' + \tfrac{1}{2} a\pi\Phi(v, v'),\ aa'\right).$$

S has a Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ with the bracket

$$[v + z + rh_0,\ v' + z' + r' h_0] = rv' - r'v + 2rz' - 2r'z + \pi\Phi(v, v'), \tag{1.3}$$

where $\mathfrak{a}$ is the Lie algebra of A and $h_0 \in \mathfrak{a}$ is such that $[h_0, v] = v$.

In the Lie algebra $\mathfrak{s}$ we select an inner product

$$\langle v + z + rh_0,\ v' + z' + r' h_0 \rangle_S = \langle v, v' \rangle + \langle z, z' \rangle + 4rr',$$

and the left-invariant metric it defines on S we denote also by $\langle \cdot, \cdot \rangle_S$.

In the next part we shall prove that the above construction includes the noncompact rank one symmetric spaces (that is, hyperbolic spaces) as particular cases.

2. Spaces $(S, \langle \cdot, \cdot \rangle_S)$ as a generalization of hyperbolic spaces. Let G denote a connected semisimple Lie group, $\mathfrak{g}$ its Lie algebra, B the Killing form of $\mathfrak{g}$, θ the Cartan involution, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. Let K be the connected subgroup of G with the Lie algebra $\mathfrak{k}$ and let φ: $G \to G/K$ be given by $\varphi(g) = gK$. The space G/K with G-invariant Riemannian structure is a symmetric space and it does not depend on the choice of the Cartan decomposition and G-invariant metric Q. If G is one of the groups $SO_0(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$, $F_{4(-20)}$ (see [1], [3], [16]), and $Q_K = B \circ (d\varphi_e|_{\mathfrak{p}})^{-1}$, we get all, up to isometry, noncompact rank one symmetric spaces [3].

Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. For the hyperbolic spaces the subalgebra $\mathfrak{a}$ is one-dimensional; hence for a root α the positive part of the set of restricted roots is either $\{\alpha\}$ or $\{\alpha, 2\alpha\}$. The root spaces $\mathfrak{g}^{-\alpha}$ and $\mathfrak{g}^{-2\alpha}$ corresponding to $-\alpha$ and -2α are orthogonal relatively to the inner product $\langle \cdot, \cdot \rangle_\theta = -B(\cdot, \theta \cdot)$. Then

$$\mathfrak{n} = \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{-2\alpha}$$

is a nilpotent algebra with the center $\mathfrak{g}^{-2\alpha}$ and we have the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

$\mathfrak{n}$ and $\mathfrak{a}$ are $\langle \cdot, \cdot \rangle_\theta$-orthogonal ([17], p. 163–168).

Let now N and A be the connected subgroups of G with Lie algebras $\mathfrak{n}$ and $\mathfrak{a}$, respectively. The map $(n, a, k) \to nak$ is a diffeomorphism of $N \times A \times K$ on G, and $S = NA$ is a closed subgroup of G [17]. Obviously, S with the metric f^*Q induced from G/K by the diffeomorphism $f = \varphi|_S$ is a symmetric space.

PROPOSITION 2.1. *f^*Q is a left-invariant metric given by*

$$(f^*Q)_e(x_1+y_1,\, x_2+y_2) = \tfrac{1}{2}\langle x_1, x_2\rangle_\theta + \langle y_1, y_2\rangle_\theta, \tag{2.1}$$

where $x_i \in \mathfrak{n}$ *and* $y_i \in \mathfrak{a}$ (see [8]).

Proof. Let $L(s)\colon S \to S$ and $\tau(s)\colon G/K \to G/K$ be defined by

$$L(s)(s_1) = ss_1 \quad \text{and} \quad \tau(s)(gK) = sgK.$$

Then $\tau(s)\circ f = f\circ L(s)$, whence

$$d\tau(s)\circ df = df\circ dL(s).$$

Let X and X' be left-invariant vector fields on S. Then

$$d\tau(s)\big(df(X)\big) = df\big(dL(s)(X)\big) = df(X)^{\cdot}$$

and

$$\begin{aligned}(f^*Q)_s(X, X') &= Q_{sk}\big(df_s(X_s), df_s(X'_s)\big)\\ &= Q_{sk}\big(d\tau(s)_k\circ df_e(X_e), d\tau(s)_k\circ df_e(X'_e)\big)\\ &= Q_K\big(df_e(X_e), df_e(X'_e)\big)\\ &= B\big(\tfrac{1}{2}(X_e-\theta X_e), \tfrac{1}{2}(X'_e-\theta X'_e)\big)\\ &= -\tfrac{1}{2}\langle X_e, \theta X'_e\rangle_\theta + \tfrac{1}{2}\langle X_e, X'_e\rangle_\theta.\end{aligned}$$

Now, putting in the above formula left-invariant vector fields belonging to $\mathfrak{n}$ or $\mathfrak{a}$, we obtain immediately (2.1).

If $\mathfrak{n}$ and N and, consequently, $\mathfrak{s}$ and S are of the forms just described, we call them *classical* and we shall often write $\mathfrak{n}\in\mathscr{C}$, $N\in\mathscr{C}$, $S\in\mathscr{C}$, $\mathfrak{s}\in\mathscr{C}$.

PROPOSITION 2.2. *If $S\in\mathscr{C}$, then (S, f^*Q) is a particular case of construction* (1.2).

Proof. $\mathfrak{n}$ with the inner product

$$\langle\cdot,\cdot\rangle_{\mathfrak{n}} = \frac{1}{m_\alpha + 4m_{2\alpha}}\langle\cdot,\cdot\rangle_\theta$$

is an algebra of type H (see [10]), where $m = \dim \mathfrak{g}^{-\alpha}$ and $m_{2\alpha} = \dim \mathfrak{g}^{-2\alpha}$. Let $h\in\mathfrak{a}$ be such that $\alpha(h) = 1$ and let $v\in\mathfrak{g}^{-\alpha}$, $z\in\mathfrak{g}^{-2\alpha}$. Then

$$[v+z+lh,\, v'+z'+l'h] = -lv'+l'v-2lz'+2l'z+[v, v']$$

and (cf. [8])

$$\langle h, h\rangle_\theta = B(h, h) = 2(m_\alpha+4m_{2\alpha}).$$

Hence

$$\langle v+z+lh,\, v'+z'+lh'\rangle_\theta = (m_\alpha+4m_{2\alpha})\big(\langle v+z, v'+z'\rangle_{\mathfrak{n}} + 2ll'\big).$$

According to Proposition 2.1, S with the left-invariant metric given by

$$\langle v+z+lh,\ v'+z'+lh'\rangle_S = \tfrac{1}{2}(m_\alpha+4m_{2\alpha})(\langle v+z,\ v'+z'\rangle_{\mathfrak{n}}+4ll')$$

is a symmetric space. Since a connected and simply connected Lie group is uniquely determined by its algebra, construction (1.2) includes (S, f^*Q), $S\in\mathscr{C}$.

3. Some properties of the composition of quadratic forms. From now on we assume that $u, u'\in U$, $z, z'\in Z$, $v, v', v''\in V$. In the sequel we need the following properties of the composition of quadratic forms:

(3.1) $$\langle \mu(u, v), \mu(u, v')\rangle = |u|^2\langle v, v'\rangle,$$

(3.2) $$\langle \mu(u, v), \mu(u', v)\rangle = |v|^2\langle u, u'\rangle.$$

Let $A_u\colon V\to V$ and $B_v\colon U\to U$ be the mappings defined by

$$A_u(v) = \mu(u, v) \quad \text{and} \quad B_v(u) = \mu(u, v).$$

(3.3) A_u is an isomorphism. If $|u| = 1$, then A_u is an orthogonal mapping.

(3.4) B_v is a monomorphism. If $|v| = 1$, then B_v is an orthogonal mapping.

(3.5) $$\mu(z, \mu(z, v)) = -|z|^2 v.$$

For the proof of (3.1)–(3.5) see [5] and [12].

(3.6) $$\mu(z, \mu(z', v)) = -\mu(z', \mu(z, v))-2\langle z, z'\rangle v.$$

We easily obtain (3.6) applying (3.5) to $z+z'$ and v.

(3.7) $$D(v) = \{\mu(z, v)\colon z\in Z\}.$$

Indeed, it is sufficient to notice that $\mu(z, v)\in D(v)$ and $\dim\{\mu(z, v)\colon z\in Z\} = \dim Z$.

(3.8) $$\langle \mu(u, v), \mu(u', v')\rangle+\langle \mu(u, v'), \mu(u', v)\rangle = 2\langle v, v'\rangle\langle u, u'\rangle.$$

Applying (3.2) to u and $v+v'$ we get (3.8).

(3.9) $$\begin{aligned}\pi\Phi(v, \mu(z, v')) &= -\pi\Phi(v', \mu(z, v))+2\langle v, v'\rangle z,\\ \pi\Phi(\mu(z, v'), v) &= -\pi\Phi(\mu(z, v), v')-2\langle v, v'\rangle z.\end{aligned}$$

To see (3.9) we use (1.1) and (3.8).

(3.10) $$\pi\Phi(v, \mu(z, v)) = |v|^2 z.$$

(3.11) $$\pi\Phi(\mu(z, v), \mu(z, v')) = -|z|^2\pi\Phi(v, v')+2\langle z, \pi\Phi(v, v')\rangle z.$$

We change the places of $\mu(z, v)$ and v' according to (3.9) and use (3.5).

Then we obtain (3.10) and (3.11).

$$\pi\Phi(\mu(z,v),\mu(z,v'))=0 \quad \text{iff} \quad \pi\Phi(v,v')=0. \tag{3.12}$$

This follows immediately from (3.11) and (3.5).

4. The group of isometries of $(S, \langle\cdot,\cdot\rangle_S)$. The left-invariant Riemannian connection on a Lie group with a left-invariant metric satisfies (see [13])

$$\langle \nabla_x y, u\rangle = \tfrac{1}{2}(\langle [x,y],u\rangle - \langle [y,u],x\rangle + \langle [u,x],y\rangle), \tag{4.1}$$

where x, y, and u belong to the Lie algebra. Let $e_0=\frac{1}{2}h_0$ (see (1.3)). Using (4.1) we obtain

$$\begin{aligned} &\nabla_{re_0}x=0, \qquad \nabla_v re_0=-\tfrac{1}{2}rv,\\ &\nabla_z re_0=-rz, \qquad \nabla_z v=\nabla_v z=-\tfrac{1}{2}\mu(z,v),\\ &\nabla_z z'=\langle z,z'\rangle e_0, \qquad \nabla_v v'=\tfrac{1}{2}\langle v,v'\rangle e_0+\tfrac{1}{2}\pi\Phi(v,v'). \end{aligned} \tag{4.2}$$

Now we calculate R and ∇R in the required cases:

$$R(e_0,v,v')=-\tfrac{1}{4}(\langle v,v'\rangle e_0+\pi\Phi(v,v')), \tag{4.3}$$

$$R(e_0,v,e_0)=\tfrac{1}{4}v, \tag{4.4}$$

$$R(e_0,z,e_0)=z, \tag{4.5}$$

$$R(v,e_0,z)=-\tfrac{1}{4}\mu(z,v), \tag{4.6}$$

$$R(z,v,v')=-\tfrac{1}{2}\langle v,v'\rangle z+\tfrac{1}{4}\langle \mu(z,v),v'\rangle e_0+\tfrac{1}{4}\pi\Phi(v,\mu(z,v')), \tag{4.7}$$

$$\begin{aligned} R(v,v',v'')={}&-\tfrac{1}{4}\langle v',v''\rangle v+\tfrac{1}{4}\langle v,v''\rangle v'-\tfrac{1}{4}\mu(\pi\Phi(v',v''),v)+\\ &+\tfrac{1}{4}\mu(\pi\Phi(v,v''),v')+\tfrac{1}{2}\mu(\pi\Phi(v,v'),v''), \end{aligned} \tag{4.8}$$

$$R(z,v,z')=-\tfrac{1}{4}\mu(z',\mu(z,v)), \tag{4.9}$$

$$R(z,e_0,v)=-\tfrac{1}{2}\mu(z,v), \tag{4.10}$$

$$R(z,z',v)=-\tfrac{1}{2}\mu(z',\mu(z,v))-\tfrac{1}{2}\langle z,z'\rangle v, \tag{4.11}$$

$$R(z,v,e_0)=-\tfrac{1}{4}\mu(z,v), \tag{4.12}$$

$$\nabla R(e_0,x,y,t)=0, \tag{4.13}$$

$$\nabla R(z,z',v,v')=0. \tag{4.14}$$

By (4.3) and (4.7) we get

$$\begin{aligned} \nabla R(z,z',v,v')={}&\big(-\tfrac{1}{4}\langle v,v'\rangle\langle z,z'\rangle+\tfrac{1}{8}\langle\mu(z,v),\mu(z',v')\rangle+\\ &+\tfrac{1}{8}\langle\mu(z',v),\mu(z,v')\rangle\big)e_0-\tfrac{1}{4}\langle\mu(z',v),v'\rangle z+\\ &+\tfrac{1}{4}\langle z,z'\rangle\pi\Phi(v,v')+\tfrac{1}{8}\pi\Phi(\mu(z,v),\mu(z',v'))+\\ &+\tfrac{1}{8}\pi\Phi(v,\mu(z',\mu(z,v'))\big). \end{aligned}$$

Applying (3.6) and (3.9) to $\pi\Phi(v, \mu(z', \mu(z, v')))$, and (3.8) to the expression by e_0 we get (4.14).

The map $u \to A_u$ (cf. (3.3)) defines a structure of a Clifford module over the Clifford algebra $C(Z, -|\cdot|^2)$. From the classification of these objects (see [4], [7], and [12]) one can see that for $\dim Z = 0, 1, 3, 7$ the type H algebra can be regarded as

$$\mathfrak{n} = F^n \times F_0 \tag{4.15}$$

with the bracket

$$[(q, p), (q', p')] = (0, 2\mathrm{Im}\, \bar{q}q')$$

and the inner product

$$\langle (q, p), (q', p') \rangle_{\mathfrak{n}} = 4\mathrm{Re}(\bar{q}q' + \bar{p}p'),$$

where F denotes the algebra of real ($\boldsymbol{R}$), complex ($\boldsymbol{C}$), quaternionic ($\boldsymbol{H}$) or Cayley numbers ($\boldsymbol{O}$), and

$$F_0 = \{p \in F\colon\ \bar{p} = -p\},$$

$$q = (q_1, \ldots, q_n), \qquad \bar{q} = (\bar{q}_1, \ldots, \bar{q}_n),$$

$$qq' = \sum_{i=1}^{n} q_i q_i'.$$

The suitable composition of quadratic forms $\mu\colon F \times F^n \to F^n$ is $\mu(p, q) = 2qp$. For $\dim Z = 0, 1, 3$ and $\dim Z = 7$, $n = 1$, in such a manner we obtain the classical algebras. They have the following property:

PROPOSITION 4.1. *Let*

$$O(v) = \{\mu(u, v)\colon\ u \in U\}. \tag{4.16}$$

If $\mathfrak{n} \in \mathscr{C}$, *then for every* v *we have*

(4.17) *If* v' *is orthogonal to* $O(v)$, *then* $O(v')$ *is orthogonal to* $O(v)$.

Proof. Let $F = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$. We put $v' = (q', 0)$, $v = (q, 0)$, $p = \overline{\bar{q}' q}$ and we assume $v' \perp O(v)$. Then

$$0 = \langle q', \mu(p, q) \rangle = 8\mathrm{Re}(\bar{q}'(qp)) = 8\mathrm{Re}((\bar{q}' q)(\overline{\bar{q}' q})).$$

Hence $\bar{q}' q = 0$ and

$$\langle \mu(p', q'), \mu(p, q) \rangle = 16\mathrm{Re}(\overline{q' p'} \cdot qp) = 16\mathrm{Re}(\bar{p}'(\bar{q}' q) p) = 0.$$

If $F = \boldsymbol{O}$ and $n = 1$, then $O(q) = F$, and (4.17) holds trivially.

Remark. If $F = \boldsymbol{O}$ and $n > 1$, then we can only say that (4.17) is satisfied by some v; for example, $v = (q_1, 0, \ldots, 0)$. At the same time we have

THEOREM 4.1. *If* (4.17) *holds for some* v, *then* $\dim Z = 0, 1, 3, 7$.

Proof. Let $v \in V$ be such that, for every v' orthogonal to $O(v)$, $O(v')$ is orthogonal to $O(v)$. First we show that $\mu(U \times O(v)) \subset O(v)$, that is $A_u(O(v)) \subset O(v)$ for $|u| = 1$.

Consider the orthogonal decomposition $V = O(v) \oplus V_1$. Let $w \in V_1$. Then $A_u(w) \in O(w)$ and $O(w) \perp O(v)$. Hence $A_u(w) \in V_1$. This implies that V_1 and $O(v)$ are invariant subspaces of A_u.

Let $\tilde{\mu} = \mu|_{U \times O(v)}$. Then $\tilde{\mu}$: $U \times O(v) \to O(v)$ has all the properties of the composition of quadratic forms and $\dim U = \dim V$. Hence $\dim Z = 0, 1, 3, 7$.

Proposition 4.1 and Theorem 4.1 imply that if $\mathfrak{n} \notin \mathscr{C}$, then either $\mathfrak{n}$ is of the form (4.15) for $F = \boldsymbol{O}$ and $n > 1$ or $\dim Z \neq 0, 1, 3, 7$, and then $\mathfrak{n}$ satisfies the condition

(4.18) For every $v \neq 0$ there is v' orthogonal to $O(v)$ such that $O(v')$ and $O(v)$ are not orthogonal to each other.

We will show that in both cases $\nabla R(x) = 0$ if and only if $x \in \mathfrak{a}$, where $\nabla R(x)$ denotes the tensor field arising from ∇R by fixing x at the first place. The proof is based on a common property contained in Proposition 4.2 for the first and in Proposition 4.3 for the second case.

The case $\dim Z = 7$. In this case we have the orthogonal decomposition of $V = V_1 \oplus \ldots \oplus V_n$, $n > 1$, such that $\mu(U \times V_i) \subset V_i$ and $\pi\Phi(w_i, w_j) = 0$ for $w_i \in V_i$, $w_j \in V_j$, $i \neq j$. We denote by v_i the i-th component of v according to the above decomposition.

First we show the following lemmas:

LEMMA 4.1. (a) *If* $w_i \in V_i$, $w_j \in V_j$, $i \neq j$, *then*

$$\begin{aligned}\nabla R(v, z, w_i, w_j) = {} & \tfrac{1}{4}\mu\big(\pi\Phi(\mu(z, v_i), w_i), w_j\big) + \tfrac{1}{4}\langle v_i, w_i\rangle \mu(z, w_j) + \\ & + \tfrac{1}{4}\mu\big(\pi\Phi(v_i, w_i), \mu(z, w_j)\big) + \tfrac{1}{4}\langle z, \pi\Phi(v_i, w_i)\rangle w_j + \\ & + \tfrac{1}{8}\langle \mu(z, v_j), w_j\rangle w_i + \tfrac{1}{8}\mu\big(\pi\Phi(\mu(z, v_j), w_j), w_i\big) + \\ & + \tfrac{1}{8}\mu\big(\pi\Phi(v_j, w_j), \mu(z, w_i)\big) + \tfrac{1}{8}\langle v_j, w_j\rangle \mu(z, w_i).\end{aligned}$$

(b) *If* $w_i, w_i' \in V_i$, $w_j \in V_j$, $i \neq j$, *then*

$$\begin{aligned}\nabla R(z, w_i, w_i', w_j) = {} & \tfrac{1}{2}\mu\big(\pi\Phi(w_i, w_i'), \mu(z, w_j)\big) + \\ & + \tfrac{1}{2}\mu\big(\pi\Phi(\mu(z, w_i), w_i'), w_j\big) + \\ & + \tfrac{1}{2}\langle w_i, w_i'\rangle \mu(z, w_j) + \tfrac{1}{2}\langle z, \pi\Phi(w_i, w_i')\rangle w_j.\end{aligned}$$

Proof. (a) By (4.7), $R(z, w_i, w_j) = 0$. In view of (4.2) and (4.8) we obtain

$$-R(\nabla_v z, w_i, w_j) = \tfrac{1}{2}R\big(\mu(z, v_i), w_i, w_j\big) + \tfrac{1}{2}R\big(\mu(z, v_j), w_i, w_j\big)$$

$$= \tfrac{1}{4}\mu\big(\pi\Phi\big(\mu(z, v_i), w_i\big), w_j\big) + \tfrac{1}{8}\langle \mu(z, v_j), w_j\rangle w_i + \\ + \tfrac{1}{8}\mu\big(\pi\Phi\big(\mu(z, v_j), w_j\big), w_i\big).$$

Moreover, by (4.10) and (4.11) we have

$$-R(z, \nabla_v w_i, w_j) = -R(z, \nabla_{v_i} w_i, w_j) = \tfrac{1}{4}\langle v_i, w_i\rangle \mu(z, w_j) + \\ + \tfrac{1}{4}\mu\big(\pi\Phi(v_i, w_i), \mu(z, w_j)\big) + \tfrac{1}{4}\langle z, \pi\Phi(v_i, w_i)\rangle w_j$$

and by (4.9) and (4.12) we get

$$-R(z, w_i, \nabla_v w_j) = -R(z, w_i, \nabla_{v_j} w_j) \\ = \tfrac{1}{8}\mu\big(\pi\Phi(v_j, w_j), \mu(z, w_i)\big) + \tfrac{1}{8}\langle v_j, w_j\rangle \mu(z, w_i).$$

Putting this together we obtain the assertion.

(b) We use (a) and the Bianchi identity

$$\nabla R(z, w_i, w_i', w_j) = \nabla R(w_i, z, w_i', w_j) - \nabla R(w_i', z, w_i, w_j).$$

LEMMA 4.2. (a) *For every $q \neq 0$ in $\boldsymbol{O}$ there are $p \in \boldsymbol{O}_0$ and $q_1, q_2 \in \boldsymbol{O}$ such that*

$$(q_1 p)(q q_2) - q_1\big((pq) q_2\big) \neq 0. \tag{4.19}$$

(b) *For every $p \neq 0$ in $\boldsymbol{O}_0$ there are $q_1, q_2, q_3 \in \boldsymbol{O}$ such that*

$$(q_1 p)(q_2 q_3) - q_1\big((p q_2) q_3\big) \neq 0. \tag{4.20}$$

Proof. We identify $\boldsymbol{O}$ with $\boldsymbol{H} + \boldsymbol{H}_e$ with the multiplication defined by

$$(be)c = (b\bar{c})e, \quad b(ce) = (cb)e, \quad (be)(ce) = -\bar{c}b \quad \text{for } b, c \in \boldsymbol{H}.$$

For $q = b + ce$, $q \neq 0$ in $\boldsymbol{O}$, the selection of q_1, p, and q_2 is shown in the following table:

Case	q_1	p	q_2
$b \neq 0$	e	i	$q_2 \in \boldsymbol{H}$ is such that $bq_2 = j$
$b = 0,\ c \notin \boldsymbol{R}$	e	$p \in \boldsymbol{H}_0$ is such that $cp - pc \neq 0$	1
$b = 0,\ c \in \boldsymbol{R}$	1	i	j

(b) For $p = b + ce$, $b \in \boldsymbol{H}_0$, the selection of q_1, q_2, and q_3 is shown in the following table:

Case	q_1	q_2	q_3
$b \neq 0$	$q_1 \in \boldsymbol{H}$ is such that $q_1 b - b q_1 \neq 0$	1	e
$b = 0,\ c \notin \boldsymbol{R}$	1	$q_2 \in \boldsymbol{H}$ is such that $c\bar{q}_2 - \bar{q}_2 c \neq 0$	e
$b = 0,\ c \in \boldsymbol{R}$	i	j	1

A simple calculation shows that with these values of p, q_1, q_2, and q_3, (4.19) and (4.20) do not vanish.

PROPOSITION 4.2. *If* $\mathfrak{n} = \boldsymbol{O}^n \times \boldsymbol{O}$, $n > 1$, *then*

(a) *for every* $v \neq 0$ *there are* z, v', v'' *such that*

$$\nabla R(v, z, v', v'') \neq 0.$$

(b) *for every* $z \neq 0$ *there are* v', v'', v''' *such that*

$$\nabla R(z, v', v'', v''') \neq 0.$$

Proof. (a) If $v = (q_1, \ldots, q_n)$, $q_i \neq 0$, we put $z = p \in \boldsymbol{O}_0$ and

$$v' = (0, \ldots, q_i', \ldots, 0), \quad v'' = (0, \ldots, q_j', \ldots, 0), \quad i \neq j.$$

Then Lemma 4.1 implies

$$\big(\nabla R(v, z, v', v'')\big)_j = 2\big((q_j p)(\bar{q}_i q_i') - q_j((p\bar{q}_i) q_i')\big).$$

By Lemma 4.2 the right-hand side does not vanish for suitable p, q_i', and q_j.

(b) If we put $z = p \in \boldsymbol{O}_0$ and

$$v' = (0, \ldots, q_i, \ldots, 0), \quad v'' = (0, \ldots, q_i', \ldots, 0), \quad v''' = (0, \ldots, q_j, \ldots, 0),$$

then by Lemma 4.1 we have

$$\nabla R(z, v', v'', v''') = 4\big((q_j p)(\bar{q}_i q_i') - q_j((p\bar{q}_i) q_i')\big).$$

The assertion follows now from Lemma 4.2.

The case $\dim Z \neq 0, 1, 3, 7$. Condition (4.18) is equivalent to

(4.21) For every $v \neq 0$ there is v' orthogonal to $O(v)$ such that $D(v) \subset \operatorname{Ker} \operatorname{ad}_{v'}$.

For the proof of this equivalence it is sufficient to notice that $O(v)$ is an orthogonal sum of $\operatorname{lin}\{v\}$ and $D(v)$.

Before showing in this case the analogy of Proposition 4.2 we prove a few lemmas.

LEMMA 4.3. *The following conditions are equivalent*:

(i) $\pi\Phi(v, v') = 0$ *and* v *is orthogonal to* v'.

(ii) v' *is orthogonal to* $O(v)$.

(iii) v *is orthogonal to* $O(v')$.

The simple proof is omitted.

LEMMA 4.4. *If* v' *is orthogonal to* $O(v)$ *and* $\pi\Phi(v', \mu(z, v)) \neq 0$, *then*

$$\nabla R(v, z, v', \mu(z, v)) \neq 0, \quad \nabla R(z, v, v', \mu(z, v)) \neq 0.$$

Proof. First we prove

$$\nabla R(v, z, v', \mu(z, v)) = \tfrac{3}{8}\mu(\pi\Phi(\mu(z, v), v')), \mu(z, v)),$$
$$(4.22) \qquad \nabla R(z, v, v', \mu(z, v)) = \tfrac{3}{4}\mu(\pi\Phi(\mu(z, v), v'), \mu(z, v)).$$

We can assume that $|v| = |z| = |v'| = 1$. By (4.7) and Lemma 4.3 we get

$$\nabla R(v, z, v', \mu(z, v)) = \tfrac{1}{2}\big(R(\mu(z, v), v', \mu(z, v)) - R(z, v', \pi\Phi(v, \mu(z, v)))\big).$$

Now applying (3.10), (4.8), and (4.9) we obtain the first equality of (4.22). Formulas (4.7)–(4.9) and (3.10) applied to $\nabla R(v', z, v, \mu(z, v))$ imply

$$\nabla R(v', z, v, \mu(z, v)) = \tfrac{1}{4}\mu(\pi\Phi(\mu(z, v'), v), \mu(z, v)) - \\ -\tfrac{1}{8}\mu(z, \mu(\pi\Phi(v', \mu(z, v)), v)).$$

Now we transform the first summand according to (3.9), the second according to (3.6) and we obtain

$$\nabla R(v', z, v, \mu(z, v)) = -\tfrac{3}{8}\mu(\pi\Phi(\mu(z, v), v'), \mu(z, v)).$$

The last result combined with the Bianchi identity

$$\nabla R(z, v, v', \mu(z, v)) = \nabla R(v, z, v', \mu(z, v)) - \nabla R(v', z, v, \mu(z, v))$$

gives the second equality of (4.22).

Now it is sufficient to notice that

$$\langle \mu(\pi\Phi(\mu(z, v), v'), \mu(z, v)), v' \rangle = |\pi\Phi(\mu(z, v), v')|^2.$$

LEMMA 4.5. *If* n *satisfies* (4.21), *then for every* $z \in Z$ *we have*

(4.23) *There are* $w, w' \in V$ *such that* w' *is orthogonal to* $O(w)$ *and*

$$\pi\Phi(v', \mu(z, v)) \neq 0.$$

Proof. By assumption, (4.23) holds for a z_0. We can assume $|z_0| = 1$. Let v and v' be such that v' is orthogonal to $O(v)$ and $\pi\Phi(v', \mu(z, v)) \neq 0$. It is sufficient to prove (4.23) when $|z| = 1$, $z \notin \operatorname{lin}\{z_0\}$ and $\pi\Phi(v', \mu(z, v)) = 0$. We put

$$w = \mu(z - z_0, v) \quad \text{and} \quad w' = \mu(z - z_0, v').$$

By (3.1), w is orthogonal to w' and, by (3.12), $\pi\Phi(w, w') = 0$. Hence w' is orthogonal to $O(w)$. Moreover, $\langle z, z - z_0 \rangle \neq 0$ and

$$\pi\Phi(\mu(z - z_0, v'), \mu(z, \mu(z - z_0, v))) \\ = -\pi\Phi(\mu(z - z_0, v'), \mu(z - z_0, \mu(z, v))) - \\ -2\langle z, z - z_0 \rangle \pi\Phi(\mu(z - z_0, v'), v)$$

$$= 2\langle z, z-z_0\rangle \pi\Phi(\mu(z-z_0, v), v')$$
$$= 2\langle z, z-z_0\rangle \pi\Phi(v', \mu(z_0, v)) \neq 0.$$

From Lemmas 4.4 and 4.5 we conclude

PROPOSITION 4.3. *If* $\dim Z \neq 0, 1, 3, 7$, *then*

(a) *for every* v *there are* z, v', *and* v'' *such that*
$$\nabla R(v, z, v', v'') \neq 0;$$

(b) *for every* z *there are* v', v'', *and* v''' *such that*
$$\nabla R(z, v', v'', v''') \neq 0.$$

This settles the second case.

THEOREM 4.2. *If* $\mathfrak{s} \notin \mathscr{C}$, *then, for every* $x \notin \mathfrak{a}$, $\nabla R(x) \neq 0$.

Proof. Let $x = v+z+re_0$. If $v \neq 0$, then, by Propositions 4.2 (a) and 4.3 (a), $\nabla R(v, z, v', v'') \neq 0$ for suitable z, v', v''. Hence, also $\nabla R(x, z, v', v'') \neq 0$ in virtue of (4.13) and (4.14).

If $v = 0$ and $z \neq 0$, then by Propositions 4.2 (b) and 4.3 (b) there are v', v'', v''' such that $\nabla R(z, v', v'', v''') \neq 0$. Hence $\nabla R(x, v', v'', v''') \neq 0$.

Now we shall describe $I(S)$ for non-classical S.

THEOREM 4.3. *If* $\mathfrak{s} \notin \mathscr{C}$ *and* L: $\mathfrak{s} \to \mathfrak{s}$ *is an orthogonal mapping such that*

$$L(\nabla(x, y)) = \nabla(L(x), L(y)), \quad x, y \in \mathfrak{s}, \tag{4.24}$$

then L *is an automorphism of* $\mathfrak{s}$.

Proof. Obviously,

$$L(R(x, y, t)) = R(L(x), L(y), L(t)), \tag{4.25}$$
$$L(\nabla R(x, y, t, w)) = \nabla R(L(x), L(y), L(t), L(w)), \tag{4.26}$$

and

$$\nabla R(L(e_0), x, y, t) = L(\nabla R(e_0, L^{-1}(x), L^{-1}(y), L^{-1}(t))) = 0.$$

Hence $L(e_0) = e_0$, $\varepsilon = \pm 1$, and $L(\mathfrak{n}) \subset \mathfrak{n}$. Now we prove the following:

(a) $L(V) \subset V$ and $L(Z) \subset Z$,
(b) $L([x, y]) = \varepsilon [x, y]$, $x, y \in \mathfrak{s}$,
(c) $L(\nabla(x, y)) = \varepsilon \nabla(L(x), L(y))$, $x, y \in \mathfrak{s}$.

(a) Let $L(v) = v'+z'$. Then by (4.4) and (4.5) we have
$$L(R(e_0, v, e_0)) = \tfrac{1}{4}(v'+z'), \qquad R(L(e_0), L(v), L(e_0)) = \tfrac{1}{4}v'+z'.$$

Hence $z' = 0$; $L(V) \subset V$, and $L(Z) \subset Z$.

(b), (c). By (4.6) and the last result we have

$$L(R(v, e_0, z)) = -\tfrac{1}{4}L(\mu(z, v)),$$
$$R(L(v), L(e_0), L(z)) = -\tfrac{1}{4}\varepsilon\mu(L(z), L(v)).$$

Hence

$$L(\mu(z, v)) = \varepsilon\mu(L(z), L(v)). \tag{4.27}$$

Combining (4.27) and (1.1) we get

$$L(\pi\Phi(v, v')) = \varepsilon\pi\Phi(L(v), L(v')), \tag{4.28}$$

which implies immediately (b), while (b) with (4.1) gives (c). But in view of (4.24) we have $\varepsilon = 1$, and L is an automorphism.

Remark. It is worth to notice here that the orthogonal automorphisms of $\mathfrak{s}$ (without any assumptions on S) preserve V and Z and are identities on $\mathfrak{a}$. This means that they are completely determined by the orthogonal automorphisms of $\mathfrak{n}$, and these have been investigated in [14].

Denote by $A(S)$ the group of automorphisms of S preserving the inner product $\langle \cdot, \cdot \rangle_S$. We summarize the results above in the following

THEOREM 4.4. *If $\mathfrak{s} \notin \mathcal{C}$, then $I(S)$ is a semi-direct product $A(S) \times S$ (S acting by left translations).*

COROLLARY 4.1. *If $\mathfrak{s} \notin \mathcal{C}$, then S is not a generalized symmetric space* (for definition see [11]).

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Reçu par la Rédaction le 5. 8. 1982;
en version modifiée le 29. 6. 1983