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ω_0-*CATEGORICITY OF GENERALIZED PRODUCTS*

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The main results of this paper are the following theorems:

Every finite generalized product (in the sense of [1]) *of ω_0-categorical relational structures is ω_0-categorical. The limit reduced power of ω-categorical structure is ω_0-categorical if the corresponding limit reduced power of two-element Boolean algebra has finitely many atoms.*

These results are a little bit stronger than theorems proved by Grzegorczyk in [3] and by Węglorz and the author in [5]. Moreover, the proofs given in this paper are more elementary and comprehensive. We use only the characterisation of ω_0-categoricity given by Ryll-Nardzewski [4] and the technique of the main theorem of Feferman and Vaught [1].

1. Preliminaries. By $\boldsymbol{T}$ we denote a theory in a countable first order language $\boldsymbol{L}$. A relational structure of the similarity type $\boldsymbol{L}$ and with the universe A is denoted by $\mathfrak{A}$. The theory of $\mathfrak{A}$ we denote by $\mathrm{Th}(\mathfrak{A})$, and the set of all models of a given $\boldsymbol{T}$ – by $\mathrm{Mod}(\boldsymbol{T})$. For any formula φ of $\boldsymbol{L}$ $\boldsymbol{T} \vdash \varphi$ means that the universal closure of φ is a theorem of $\boldsymbol{T}$. s

By $\mathscr{F}_n(\boldsymbol{L})$ we denote the set of all formulas of $\boldsymbol{L}$ with free variablee among $v_0, v_1, \ldots, v_{n-1}$. $\mathscr{F}_0(\boldsymbol{L})$ is the set of all sentences of $\boldsymbol{L}$. In the whol paper we shall identify two formulas φ_1 and φ_2 provided $\vdash \varphi_1 \leftrightarrow \varphi_2$. With this identification $\mathscr{F}_n(\boldsymbol{L})$ can be considered to be a Boolean algebra with logical connectives as operations. If $\boldsymbol{T}$ is a theory in $\boldsymbol{L}$, then the equivalence relation on a set of formulas defined as $\varphi_1 \sim_{\boldsymbol{T}} \varphi_2$ if and only if $\boldsymbol{T} \vdash \varphi_1 \leftrightarrow \varphi_2$ is a congruence on $\mathscr{F}_n(\boldsymbol{L})$. The quotient Boolean algebra $\mathscr{F}_n(\boldsymbol{L})/\sim_{\boldsymbol{T}}$ will be denoted by $\mathscr{F}_n(\boldsymbol{T})$, but if $\boldsymbol{T} = \mathrm{Th}(\mathfrak{A})$, we write $\mathscr{F}_n(\mathfrak{A})$. Elements of $\mathscr{F}_n(\boldsymbol{T})$ will be denoted by $\varphi/\boldsymbol{T}$. We identify formula with its equivalence class if it does not lead to missunderstanding.

Let us recall that a theory is called *ω_0-categorical* if all of its at most countable models are isomorphic. A relational structure will be called *ω_0-categorical* if its theory is ω_0-categorical. The ω_0-categoricity can be characterized in terms of $\mathscr{F}_n$-algebras as follows:

THEOREM (cf. Ryll-Nardzewski [4]). *A theory $\boldsymbol{T}$ is ω_0-categorical if and only if it is complete, and every algebra $\mathscr{F}_n(\boldsymbol{T})$ is finite.*

In this paper we use the notions and results of the paper [1] the notation being slightly modified. Let 2 denotes the two-element Boolean algebra. Algebra 2^I will be identified with the algebra of all subsets of I with the usual set theoretical operations. Any expansion of this algebra will be called a *subset algebra over I*. For any such algebra $\mathfrak{S}$, a sequence $\zeta = \langle \Phi, \vartheta_0, \ldots, \vartheta_{k-1}\rangle$ is called *acceptable* if Φ is a formula of the language of $\mathfrak{S}$ with k free variables, and ϑ_i are formulas of L. Let $\langle \mathfrak{A}_i\colon i\in I\rangle$ be a family of relational structures of the similarity type L, and let $\mathfrak{A}$ be the direct product of this family. If ϑ is a formula of L and $\bar{f} = \langle f_n\colon n<\omega\rangle$ is a sequence of elements of $\mathfrak{A}$, then we define

$$K[\vartheta, \mathfrak{A}, \bar{f}] = \{i\in I\colon \mathfrak{A}_i \models \vartheta[\bar{f}(i)]\}$$

(where $\bar{f}(i) = \langle f_n(i)\colon n<\omega\rangle$).

If $\mathscr{X}$ is a set of acceptable sequences and $\mathfrak{S}$ is a subset algebra over I, then the $(\mathfrak{S}, \mathscr{X})$-product of the family $\langle \mathfrak{A}_i\colon i\in I\rangle$ is the structure $\langle A, Q_\zeta\rangle_{\zeta\in\mathscr{X}}$, where

(i) A is the direct product of the family $\langle A_i\colon i\in I\rangle$,

(ii) if $\zeta = \langle \Phi, \vartheta_0, \ldots, \vartheta_{k-1}\rangle$, then Q_ζ is fulfilled by $\bar{f}$ if and only if

$$\mathfrak{S} \models \Phi\big[K[\vartheta_0, \mathfrak{A}, \bar{f}], \ldots, K[\vartheta_{k-1}, \mathfrak{A}, \bar{f}]\big].$$

We assume that Q_ζ has finite arity equal to $\min\{n\colon \vartheta_i\in\mathscr{F}_n(L)$ for every $i<k\}$.

Any relational structure defined in this way will be called a *generalized product.* If $\mathscr{X}$ is the set of all acceptable sequences, then a generalized product is called *full.* If all factors are equal, then it is called a *generalized power.* A substructure of a generalized product with the universe definable in this product is called a *relativised generalized product.*

We use the following version of the theorem of Feferman and Vaught (cf. [1], Theorem 3.1), that is, a bit weaker than the original one:

THEOREM (S. Feferman and R. L. Vaught). *Let the type of subset algebra be given. Then for every formula ϑ of the language of full generalized product there exists an acceptable sequence ζ such that ϑ is equivalent to Q_ζ in any full generalized $\mathfrak{S}$-product* (*for every $\mathfrak{S}$ of the given similarity type*).

Such an acceptable sequence will be called an *F. V.-reduction* of ϑ. It is obvious that if ϑ has n free variables, then every member of the F. V.-reduction of ϑ, which is a formula of L, belongs to $\mathscr{F}_n(L)$.

The power of a set A will be denoted by $|A|$.

2. ω_0-categoricity of generalized powers and finite generalized products.

THEOREM 1. *If $\mathfrak{S}$ and $\mathfrak{A}$ are ω_0-categorical, then any generalized $\mathfrak{S}$-power of $\mathfrak{A}$* (*relativised or not*) *is ω_0-categorical.*

Proof. We give a proof for the full $\mathfrak{S}$-power of $\mathfrak{A}$, denoted by $\mathfrak{A}^{\mathfrak{S}}$. Proofs for other cases can be easily obtained from it.

Let the sequence $\langle\vartheta_0, \ldots, \vartheta_{m-1}\rangle$ consists of exactly one representative from every element of $\mathscr{F}_n(\mathfrak{A})$. It is easy to see that from every element of $\mathscr{F}_n(\mathfrak{A}^{\mathfrak{S}})$ we can choose a representative of the form Q_ζ such that $\zeta = \langle\Phi_\zeta, \vartheta_0, \ldots, \vartheta_{m-1}\rangle$. Moreover, if $\mathfrak{S} \models \Phi_\zeta \leftrightarrow \Phi_\xi$, then $\mathfrak{A}^{\mathfrak{S}} \models Q_\zeta \leftrightarrow Q_\xi$. So Q_ζ depends only on the equivalence class of Φ_ζ. From this it follows that $|\mathscr{F}_n(\mathfrak{A}^{\mathfrak{S}})| \leqslant |\mathscr{F}_m(\mathfrak{S})|$ for $m = |\mathscr{F}_n(\mathfrak{A})|$.

THEOREM 2. *Every (relativised or not) finite generalized product of ω_0-categorical structures is ω_0-categorical.*

Proof. For the sake of the simplicity of notation, we prove this theorem for a full product $\mathfrak{A}$ of two structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Let $\sim$ be the intersection of equivalence relations $\sim_{\mathrm{Th}(\mathfrak{A}_1)}$ and $\sim_{\mathrm{Th}(\mathfrak{A}_2)}$, defined on $\mathscr{F}_n(\boldsymbol{L})$. From this definition we have

$$|\mathscr{F}_n(\boldsymbol{L})/\sim| = |\{\varphi/\mathrm{Th}(\mathfrak{A}_1) \cap \psi/\mathrm{Th}(\mathfrak{A}_2) \colon \varphi, \psi \in \mathscr{F}_n(\boldsymbol{L})\}| \leqslant |\mathscr{F}_n(\mathfrak{A}_1)| \cdot |\mathscr{F}_n(\mathfrak{A}_2)|.$$

So we proved that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are ω_0-categorical, then $\mathscr{F}_n(\boldsymbol{L})/\sim$ is finite. Let the sequence $\langle\vartheta_0, \ldots, \vartheta_{l-1}\rangle$ contain exactly one representative from every equivalence class of $\sim$. As previously, one can choose from every element of $\mathscr{F}_n(\mathfrak{A})$ a representative of the form Q_ζ such that $\zeta = \langle\Phi_\zeta, \vartheta_0, \ldots, \vartheta_{l-1}\rangle$. The proof can be completed in the same way as the proof of the previous theorem.

From Theorems 1 and 2 one can immediately obtain results on ω_0-categoricity for all particular cases of generalized products (numerous examples of such operations are given in [1]). As one can see, Theorem 1 follows from Theorem 2.

PROPOSITION 1. *The subset algebra is ω_0-categorical if and only if it is finite.*

Proof. The necessity is obvious. To prove sufficiency, we take the set of formulas of the language of Boolean algebras $\{\varphi_n(X) \colon n < \omega\}$, where $\varphi_n(X)$ means: *X is a union of n atoms.* Of course, neither of two such formulas is equivalent to the other in any subset algebra over an infinite set.

We finish this section with the example of an application of Theorem 2.

COROLLARY 1 (Theorem 2 in [3]). *The class of ω_0-categorical similar structures is closed under the operations of finite direct union and finite direct product.*

The second part of this corollary was also proved in [5].

3. Infinite reduced powers. In [1] the operation of reduced product was considered as a special case of generalized products, but, by Proposition 1, the immediate application of Theorem 1 gives no information about preserving of the ω_0-categoricity under this operation. Nevertheless, the proof of Theorem 1 can be applied to every operation, for which an analogue of the Feferman and Vaught theorem can be proved. As it was observed by J. Weinstein (for details, see Theorem 4.10 in [2]), such a theorem holds for a reduced power $\mathfrak{A}^I_{\mathscr{D}}$ if one takes the Boolean algebra $2^I_{\mathscr{D}}$ instead of a subset algebra (appropriate modifications of notation and definitions are necessary). Analogous observation for limit powers has been done in [7]. Putting these two results together, one can easily prove the following theorem:

THEOREM 3. *Let $\mathscr{D}$ be a filter over I, and $\mathscr{F}$ a filter over I^2. If $\mathfrak{A}$ and $2^I_{\mathscr{D}} \mid \mathscr{F}$ are ω_0-categorical, then $\mathfrak{A}^I_{\mathscr{D}} \mid \mathscr{F}$ is ω_0-categorical.*

This theorem for the case of reduced powers was proved in [5] but, using results of [6], one can extend the proof to the general case.

Remark. The proof of Theorem 1 works also for infinite products provided all factors are models of the same theory $\boldsymbol{T}$ such that

(i) *$\mathscr{F}_n(\boldsymbol{T})$ is finite for every n and if the corresponding Boolean algebra is ω_0-categorical.*

The proof of this fact easily follows from the previous theorems, because condition (i) is equivalent to the conjunction of the following two statements:

(ii) *Every complete extension of $\boldsymbol{T}$ is ω_0-categorical.*

(iii) *There is only a finite number of complete extensions of $\boldsymbol{T}$.*

As an example of the theory of equality shows, (ii) does not imply (iii). So condition (ii) seems to be an interesting generalization of the notion of ω_0-categoricity.

4. Final remarks. Let us denote by $\dim(\mathfrak{B})$ the cardinality of the smallest set of generators of a finite Boolean algebra $\mathfrak{B}$ (the *dimension* of $\mathfrak{B}$).

PROPOSITION 2. a. $\dim(\mathfrak{B}) = -E\big(-\log_2\log_2(\mathfrak{B})\big)$.

b. $\max\big(\dim(\mathfrak{B}_1), \dim(\mathfrak{B}_2)\big) \leqslant \dim(\mathfrak{B}_1 \times \mathfrak{B}_2) \leqslant \max\big(\dim(\mathfrak{B}_1), \dim(\mathfrak{B}_2)\big) + 1$.

Proof. a. If $\mathfrak{B} \simeq 2^m$, then $\dim(\mathfrak{B})$ is a non-decreasing function of m. The largest algebra of dimension k has 2^k atoms. So, $2^{\dim(\mathfrak{B})-1} < m \leqslant 2^{\dim(\mathfrak{B})}$ which completes the proof of a. b follows immediately from a.

From the proofs of Theorems 1 and 2 the following proposition, a bit stronger than these theorems, follows:

PROPOSITION 3. a. *$|\mathscr{F}_n(\mathfrak{A}^{\mathfrak{S}})| \leqslant |\mathscr{F}_k(\mathfrak{S})|$ for $k = \dim\big(\mathscr{F}_n(\mathfrak{A})\big)$.*

b. *$|\mathscr{F}_n(\mathfrak{A}_1 \times \mathfrak{A}_2)| \leqslant |\mathscr{F}_k(2^2)|$ for $k = \dim\big(\mathscr{F}_n(\mathfrak{A}_1) \times \mathscr{F}_n(\mathfrak{A}_2)\big)$.*

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