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A CATEGORY THEORETICAL BACKGROUND FOR HOMOMORPHISM THEOREMS

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IN MEMORIAM
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1. This paper is motivated by the fact that in connection with partial algebras we know at least two so-called "homomorphism theorems" in the category $\mathfrak{P}^{\Delta}$ consisting of all partial algebras of any given similarity type Δ and all homomorphisms φ: $\boldsymbol{A} \to \boldsymbol{B}$ ($\boldsymbol{A}, \boldsymbol{B} \in \mathfrak{P}^{\Delta}$), where φ is a mapping from the carrier set A of $\boldsymbol{A}$ into the carrier set B of $\boldsymbol{B}$ carrying the structure of $\boldsymbol{A}$ into that of $\boldsymbol{B}$ (for the fundamental definitions concerning universal and partial algebras cf. [1]–[3], [6], [7] or [10]; for category theoretical notions see [9] and [10]). Namely, the first "homomorphism theorem" characterizes, up to unique isomorphism, all full and surjective homomorphisms starting from a given partial algebra $\boldsymbol{A}$ by their congruence relations induced on $\boldsymbol{A}$ (cf. [12], Section 3). The second one, presented by Schmidt in [13] characterizes, up to unique isomorphism, all almost surjective homomorphisms φ starting from a given partial algebra $\boldsymbol{A}$ by the closed congruence relations which are induced by the closed homomorphic extensions $\tilde{\varphi}$ of φ on the initial segments of a fixed free completion $\hat{\boldsymbol{A}}$ of $\boldsymbol{A}$.

Now a question arises whether there is a common category theoretical background for these two results. This problem is partly stimulated by (and vice versa – it stimulates) a project to get a survey of the concrete meanings of category theoretical notions in categories of partial algebras (cf. [6] and [11]).

We think that the presented translation of the J. Schmidt kernel into category theoretical concepts yields some interesting new insights into partial algebras and leads to some purely category theoretical problems.

2. We recall [13] the following three "basic statements" in connection with the homomorphism theorem in the category $\mathfrak{T}^{\Delta}$ of all total algebras of type Δ:

I. *Each homomorphism* φ: $\boldsymbol{A} \to \boldsymbol{B}$ *induces a congruence relation* R_{φ} *in* $\boldsymbol{A}$.

II. *Each congruence relation R in A is induced by a surjective homomorphism φ: $A \to \boldsymbol{B}$ for some $\boldsymbol{B}$.*

III (diagram completion). *For a surjective homomorphism φ: $A \to \boldsymbol{B}$ and an arbitrary homomorphism ψ: $A \to \boldsymbol{C}$ there is a — necessarily unique — homomorphism ω: $\boldsymbol{B} \to \boldsymbol{C}$ such that $\psi = \omega \circ \varphi$ iff $R_\varphi \subseteq R_\psi$.*

If we replace here "total algebras" by "partial algebras" and "surjective homomorphism" everywhere by "full and surjective homomorphism" (*full* means that the structure on $\varphi[B]$ is totally induced through φ by the structure of A), we get a corresponding triplet of statements for partial algebras (cf. [12]). In both cases, Statements II and III just say that the surjective homomorphisms in $\mathfrak{T}^{\Delta}$ (respectively, the full and surjective homomorphisms in $\mathfrak{P}^{\Delta}$)are exactly the regular epimorphisms of the corresponding category.

For the results of [13] the situation is somewhat different: namely, a given (almost surjective) homomorphism φ: $A \to \boldsymbol{B}$ is first factorized into an embedding of A into an A-generated relative subalgebra A' of the free completion $\hat{A} := \boldsymbol{F}(A, \boldsymbol{T}^{\Delta})$ of A followed by a closed homomorphism $\tilde{\varphi}$: A' $\to \boldsymbol{B}$, and then the induced congruence relation of $\tilde{\varphi}$ on A' is defined as the kernel of φ (we shall call it the *J. Schmidt-kernel* or *S-kernel* of φ, briefly: S-ker φ):

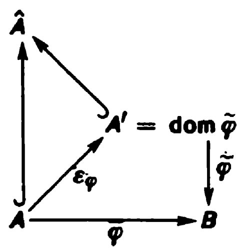

Diagram 1

(Note that the graph of $\tilde{\varphi}$ is given by $\tilde{\varphi} = \mathrm{C}_{\hat{A}\times B}\varphi$, whence it is uniquely determined by A, $\boldsymbol{B}$ and φ.) Then the statements corresponding to I, II and III are the following (cf. [13]) for $A, \boldsymbol{B}, \boldsymbol{C} \in \mathfrak{P}^{\Delta}$:

2.1. THEOREM. *The S-kernel of an arbitrary homomorphism φ: $A \to \boldsymbol{B}$ is a closed congruence on* **dom** $\tilde{\varphi}$ (**dom** $\tilde{\varphi}$ *is an intermediate initial segment of* $\hat{A}$).

2.2. THEOREM. *Each closed congruence relation in an intermediate initial segment of $\hat{A}$ is the S-kernel of some almost surjective homomorphism φ: $A \to \boldsymbol{B}$ for some $\boldsymbol{B}$.*

2.3. THEOREM. *Let φ: $A \to \boldsymbol{B}$ be an almost surjective and ψ: $A \to \boldsymbol{C}$ an arbitrary homomorphism. Then there is a — necessarily unique — homomorphism ω: $\boldsymbol{B} \to \boldsymbol{C}$ such that $\psi = \omega \circ \varphi$ iff*

$$R_{\tilde{\varphi}} \subseteq R_{\tilde{\psi}}, \quad \textit{i.e.,} \quad \text{S-ker}\,\varphi \subseteq \text{S-ker}\,\psi.$$

3. In order to translate these results to arbitrary categories we have to find suitable category theoretical concepts behind the notions and constructions used above (possibly in such a way that also I–III for full and surjective homomorphisms in $\mathfrak{P}^{\Delta}$ fit into the pattern). Our main clue is the observation from [4] that in connection with Theorems 2.1–2.3 we deal with a factorization system (see Definition 3.2 below).

3.1. THEOREM. *In the factorization system*

$(\mathscr{E}(\mathfrak{T}^{\Delta}), \mathscr{P}(\mathfrak{T}^{\Delta})) =$ ($\mathfrak{T}^{\Delta}$*-extendable epimorphisms*, $\mathfrak{T}^{\Delta}$*-perfect homomorphisms*)

in $\mathfrak{P}^{\Delta}$ *(which exists according to results in* [8] *or* [9]), *the class of all* $\mathfrak{T}^{\Delta}$*-perfect homomorphisms is exactly the class of all closed homomorphisms.*

3.2. DEFINITION (cf. [10] or [11], slightly narrower in [8], [9]). A *factorization system* $(\mathscr{H}, \mathscr{S})$ in a category $\mathfrak{C}$ consists of two classes $\mathscr{H}$ and $\mathscr{S}$ of $\mathfrak{C}$-morphisms such that

(i) $\mathscr{S} \circ \mathscr{H} = \mathrm{Mor}\, \mathfrak{C}$, where $\mathscr{S} \circ \mathscr{H} = \{s \circ h \mid h \in \mathscr{H}, s \in \mathscr{S}\}$;

(ii) $\mathrm{Iso}\, \mathfrak{C} \subseteq \mathscr{H} \cap \mathscr{S}$;

(iii) $\mathscr{H} \circ \mathscr{H} \subseteq \mathscr{H}$ and $\mathscr{S} \circ \mathscr{S} \subseteq \mathscr{S}$;

(iv) every $(\mathscr{H}, \mathscr{S})$-factorization, according to (i), is unique up to unique isomorphism, i.e., if $f = s \circ h = s' \circ h'$ are two $(\mathscr{H}, \mathscr{S})$-factorizations of $f \in \mathrm{Mor}\, \mathfrak{C}$ $(h, h' \in \mathscr{H}, s, s' \in \mathscr{S})$ then there exists a unique isomorphism i such that $s = s' \circ i$ and $h = i^{-1} \circ h'$ (this means that Diagram 2 is commutative).

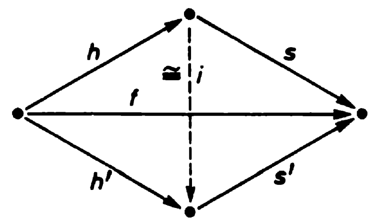

Diagram 2

3.3. OBSERVATIONS. (i) A congruence relation corresponds to the kernel pair of a morphism f, i.e., to the pullback $(K; k_1, k_2) = \ker f$ of Diagram 3.

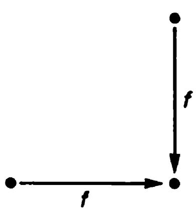

Diagram 3

(ii) Every closed epimorphism in $\mathfrak{P}^{\Delta}$ is a full and surjective, and hence a regular epimorphism (i.e., the coequalizer (or pushout) of a pair of morphisms).

(iii) Every regular epimorphism in $\mathfrak{P}^{\Delta}$ is the coequalizer of its kernel pair (if this exists) (cf. [9], Proposition 21.16).

(iv) S-ker φ is just the congruence relation of $\tilde{\varphi}$ in the $(\mathscr{E}(\mathfrak{T}^{\Delta}), \mathscr{P}(\mathfrak{T}^{\Delta}))$-factorization $(\varepsilon_{\varphi}, \tilde{\varphi})$ of φ.

(v) If $(r, r'; L)$ is the pushout of the kernel pair $(K; k_1, k_2)$ of f, then $r = r'$ (Diagram 4).

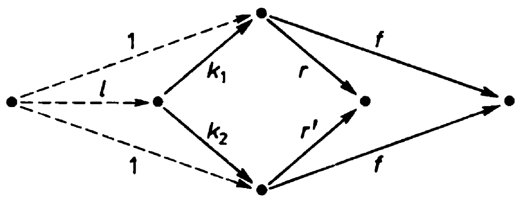

Diagram 4

Since $f \circ 1 = f \circ 1$, there is an l such that $k_1 \circ l = k_2 \circ l = 1$. Hence $r \circ k_1 = r' \circ k_2$ implies

$$r = r \circ k_1 \circ l = r' \circ k_2 \circ l = r'.$$

Thus we are led to the following conditions on a category to become amenable to our generalizations.

3.4. Claim. In what follows we want the category $\mathfrak{C}$ to have a factorization system $(\mathscr{H}, \mathscr{S})$ satisfying the following conditions:

(1) for every $s \in \mathscr{S}$ there exists a congruence relation ker s in $\mathfrak{C}$, i.e., a kernel pair $(K; k_1, k_2)$;

(2) the kernel pair of an $\mathscr{S}$-morphism is also the kernel pair of a regular epimorphism.

It is obvious that (1) corresponds to Statement I and Theorem 2.1, while (2) corresponds to Statement II and Theorem 2.2. An analogue to Statement III and Theorem 2.3 is given by Theorem 4.3 below.

4. Before we formulate the General Diagram Completion Theorem we would like to extend the notion of J. Schmidt-kernel to categories.

4.1. Definition. Let $\mathfrak{C}$ be a category which satisfies Claim 3.4 with respect to a factorization system $(\mathscr{H}, \mathscr{S})$ and let $f \in \mathrm{Mor}_{\mathfrak{C}}(A, B)$ for any two objects $A, B \in \mathrm{Ob}\,\mathfrak{C}$; moreover, let (h, s) be an $(\mathscr{H}, \mathscr{S})$-factorization of f, i.e., $f = s \circ h$, $s \in \mathscr{S}$, $h \in \mathscr{H}$. A *category theoretical J. Schmidt-kernel* of f (with respect to the factorization system $(\mathscr{H}, \mathscr{S})$) is any kernel pair $(K; k_1, k_2)$ of the morphism s, i.e., we have the commutative Diagram 5.

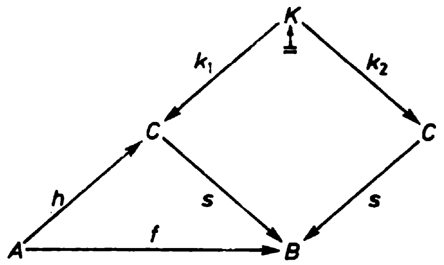

Diagram 5

We write $(K; k_1, k_2) \approx (\mathscr{H}, \mathscr{S})\text{-ker}\, f$, where $\approx$ indicates equality up to unique isomorphism (see [9]).

4.2. Remarks. This definition is a common generalization of the notion of the J. Schmidt-kernel for partial algebras and of the notion of a usual kernel for partial algebras or total algebras. In fact:

(a) In the case of partial algebras we have

$$\text{S-ker}\,\varphi = (\mathscr{E}(\mathfrak{T}^\Delta), \mathscr{P}(\mathfrak{T}^\Delta))\text{-ker}\,\varphi.$$

(b) In the case of the usual kernel we take in $\mathfrak{P}^\Delta$:

$$(\mathscr{H}, \mathscr{S}) := (\text{Iso}\,\mathfrak{P}^\Delta, \ \text{Hom}\,\mathfrak{P}^\Delta),$$

and in $\mathfrak{T}^\Delta$:

$$(\mathscr{H}, \mathscr{S}) := (\text{Iso}\,\mathfrak{T}^\Delta, \text{Hom}\,\mathfrak{T}^\Delta),$$

the trivial factorization systems. Thus in both cases we have

$$\ker\varphi = (\mathscr{H}, \mathscr{S})\text{-ker}\,\varphi$$

for any $\varphi \in \text{Hom}\,\mathfrak{P}^\Delta$ $(\varphi \in \text{Hom}\,\mathfrak{T}^\Delta)$.

We should observe that in all these cases $\mathscr{H}$ consists of bimorphisms, which is not required in general.

We are now able to formulate the General Diagram Completion Theorem.

4.3. General Diagram Completion Theorem. Let $\mathfrak{C}$ be a category which satisfies Claim 3.4 with respect to a factorization system $(\mathscr{H}, \mathscr{S})$. Let

$$A, B, B' \in \text{Ob}\,\mathfrak{C}, \quad e \in \text{Mor}_{\mathfrak{C}}(A, B) \cap \text{Epi}\,\mathfrak{C}, \quad f \in \text{Mor}_{\mathfrak{C}}(A, B'),$$

and let (h, s) and (h', s') be $(\mathscr{H}, \mathscr{S})$-factorizations of e and f, respectively:

$$e = s \circ h, \quad f = s' \circ h'.$$

Moreover, let

$$\text{Dom}\,s := C, \quad \text{Dom}\,s' := C'$$

and

$$(K; k_1, k_2) :\approx \ker s \ (\approx (\mathscr{H}, \mathscr{S})\text{-}\ker e),$$

$$(K'; k_1', k_2') :\approx \ker s' \ (\approx (\mathscr{H}, \mathscr{S})\text{-}\ker f);$$

further, let l: $C \to K$ and l': $C' \to K'$ be sections which belong to $(K; k_1, k_2)$ and $(K'; k_1', k_2')$, respectively (cf. Observation 3.3 (v)):

$$k_j \circ l = 1_C, \quad k_j' \circ l' = 1_{C'} \quad \text{for } j \in \{1, 2\}.$$

Let us consider the following statements:

(i) *There exists a unique* g: $B \to B'$ *in* Mor $\mathfrak{C}$ *such that* $g \circ e = f$.

(ii) *There exists a unique* m: $K \to K'$ *in* Mor $\mathfrak{C}$ *such that the following equations hold for* $j \in \{1, 2\}$:

$$(*) \qquad k_j' \circ m \circ l \circ h = h',$$

$$(**) \qquad k_j' \circ m \circ l \circ k_j = k_j' \circ m.$$

THEOREM. *With the above assumptions we have* (i) $\Rightarrow$ (ii). *If, in addition,* s *is a regular epimorphism, then also* (ii) $\Rightarrow$ (i).

The commutative Diagram 6 illustrates this situation.

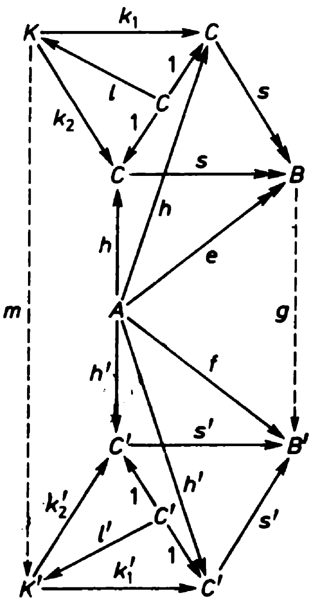

Diagram 6

The proof of Theorem 4.3 will follow from the results in the next sections. Moreover, we shall see that our notion of $(\mathscr{H}, \mathscr{S})$-kernel is natural in so far as different $(\mathscr{H}, \mathscr{S})$-factorizations of the same morphism yield isomorphic $(\mathscr{H}, \mathscr{S})$-kernels. In connection with additional assumptions on our category or on the factorization system we shall get some additional statements which one usually considers in connection with such diagram completion theorems (for instance, when g is an epimorphism, monomorphism or isomorphism, respectively). Throughout the rest of this paper we keep to the assumptions and to the notation introduced in connection with Theorem 4.3.

5. Existence and properties of m.

5.1. "4.3 (i) $\Rightarrow$ 4.3 (ii)". Consider Diagram 7 which extends Diagram 6 in

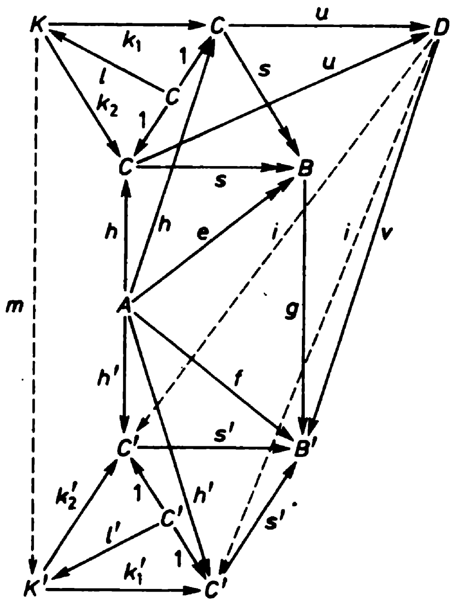

Diagram 7

the way described below. Note that $s \in \mathrm{Epi}\,\mathfrak{C}$, since $e \in \mathrm{Epi}\,\mathfrak{C}$. We assume $(u, v) \in \mathscr{H} \times \mathscr{S}$ to be an $(\mathscr{H}, \mathscr{S})$-factorization pair for the morphism $g \circ s$, i.e.,

$$(1) \qquad g \circ s = v \circ u, \quad u\colon C \to D, \quad v\colon D \to B'.$$

Then $f = g \circ e = g \circ s \circ h = v \circ (u \circ h)$, where $v \in \mathscr{S}$ and $u \circ h \in \mathscr{H}$. Since also $f = s' \circ h'$, $s' \in \mathscr{S}$ and $h' \in \mathscr{H}$, there exists – according to 3.2 (iv) – a unique isomorphism $i\colon D \to C'$ such that

$$(2) \qquad h' = i \circ (u \circ h), \quad v = s' \circ i.$$

We now define two morphisms $p_1, p_2\colon K \to C'$ as follows:

$$(3) \qquad p_1 := i \circ u \circ k_1, \quad p_2 := i \circ u \circ k_2.$$

Since, by (2) and (1),

$$s' \circ p_1 = s' \circ i \circ u \circ k_1 = v \circ u \circ k_1 = g \circ s \circ k_1$$
$$= g \circ s \circ k_2 = \ldots = s' \circ p_2,$$

and since $(K'; k'_1, k'_2)$ is a pullback of $(s', s'; B')$, there exists a unique morphism $m\colon K \to K'$ such that

$$(4) \qquad k'_j \circ m = p_j, \quad j \in \{1, 2\}.$$

Now, by $k_j \circ l = 1$, formulas (4), (3) and (2), we have

$$k'_j \circ m \circ l \circ h = i \circ u \circ k_j \circ l \circ h = i \circ u \circ h = h' \quad \text{for } j \in \{1, 2\},$$

i.e. (*) of condition (ii) holds. Further, because of (4) and (3), we obtain

$$k'_j \circ m \circ l \circ k_j = i \circ u \circ k_j \circ l \circ k_j = i \circ u \circ k_j = k'_j \circ m,$$

whence (**) of condition (ii) is satisfied. Thus one implication of Theorem 4.3 has been proved.

5.2. Lemma. *Given g, for every $m\colon K \to K'$ satisfying (*) and (**) we get $m \in \operatorname{Mono} \mathfrak{C}$ if $\mathscr{H} \subseteq \operatorname{Mono} \mathfrak{C}$.*

Proof. Let $a, b\colon L \to K$ be morphisms such that $m \circ a = m \circ b$. Then for $j \in \{1, 2\}$ we have $k'_j \circ m \circ a = k'_j \circ m \circ b$ and, by (3) and (4),

$$k'_j \circ m \circ a = i \circ u \circ k_j \circ a = i \circ u \circ k_j \circ b.$$

Since i is an isomorphism and $u \in \mathscr{H} \subseteq \operatorname{Mono} \mathfrak{C}$ by assumption, for $j \in \{1, 2\}$ we get $k_j \circ a = k_j \circ b$. Since $s \circ k_1 = s \circ k_2$, we infer that

$$s \circ k_1 \circ a = s \circ k_2 \circ a = s \circ k_2 \circ b = s \circ k_1 \circ b;$$

and since $(K; k_1, k_2)$ is the kernel pair of s, there exists a unique morphism p such that $k_j \circ p = k_j \circ a = k_j \circ b$, whence $p = a = b$, i.e., $m \in \operatorname{Mono} \mathfrak{C}$.

Note that so far neither s has been required to be regular nor e to be an epimorphism.

5.3. Remark and example. In order to motivate in 5.2 the assumption "$\mathscr{H} \subseteq \operatorname{Mono} \mathfrak{C}$" consider the class $\mathfrak{K}$ of all mono-unary universal algebras $(A; \varphi^A)$ (with one unary operation) satisfying, say, the identity $\varphi(x) = \varphi(y)$. Let $\boldsymbol{T} = (T; \varphi^T)$ be the absolutely free mono-unary algebra with basis $\{x, y\}$. Moreover, let $\boldsymbol{F} = (F; \varphi^F)$ be the $\mathfrak{K}$-free $\mathfrak{K}$-algebra with $\mathfrak{K}$-basis $\{\bar{x}, \bar{y}\}$, i.e.,

$$F = \{\bar{x}, \bar{y}, \bar{z}\}, \quad \varphi^F(\bar{x}) = \varphi^F(\bar{y}) = \varphi^F(\bar{z}) = \bar{z}.$$

In Diagram 6 we choose

$$A = B := \boldsymbol{T} = C, \quad e := \mathrm{id}_T = h = s, \quad B' = C' := \boldsymbol{F},$$

let f stand for the homomorphic extension of $(x \to \bar{x}, y \to \bar{y}) = g = h'$, and $s' = \mathrm{id}_F$. Then $\boldsymbol{K}$ and $\boldsymbol{K}'$ have as carrier sets in each case the diagonal of $T \times T$ and of $F \times F$, respectively, and k_1, k_2, k'_1, k'_2 are isomorphisms. (h, s) and (h', s') are ($\mathfrak{K}$-extendable, $\mathfrak{K}$-perfect)-factorizations of e and f, respectively (cf. [8]). Note that m is isomorphic to f which is not a monomorphism.

A consequence of the results obtained so far is a proof of the categoricity of our notion of $(\mathscr{H}, \mathscr{S})$-kernel:

5.4. LEMMA. *Let* (h, s) *and* (h', s') *be two* $(\mathscr{H}, \mathscr{S})$-*factorizations of the same* $\mathfrak{C}$-*morphism* $e: A \to B$ *and let* $\mathfrak{C}$ *satisfy Claim* 3.4. *Then the kernels* $(K; k_1, k_2)$ *of* s *and* $(K'; k'_1, k'_2)$ *of* s' *are isomorphic by* $m: K \to K'$ *such that both* m *and* m^{-1} *satisfy* $(*)$ *and* $(**)$ *of statement* (ii) *in* 4.3.

Proof. As we noted at the end of 5.2 we have not needed in 5.1 the morphism e to be an epimorphism. Since 1_B plays the rôle of g in 5.1, we get the existence of unique $m: K \to K'$ and $m': K' \to K$ both satisfying $(*)$ and $(**)$. Since $m' \circ m$ and 1_K (respectively, $m \circ m'$ and 1_K) satisfy $(*)$ and $(**)$ for the identical factorizations (h, s), (h, s) and (h', s'), (h', s') of e, respectively, we conclude that $m' \circ m = 1_K$ and $m \circ m' = 1_{K'}$, whence $m' = m^{-1}$, and both m and m' are isomorphisms satisfying $(*)$ and $(**)$.

6. Existence and properties of g.

6.1. "4.3 (ii) $\Rightarrow$ 4.3 (i)". Using the notation of Theorem 4.3 and Diagram 6, assume that for given e, f, etc., there exists a unique $m: K \to K'$ satisfying

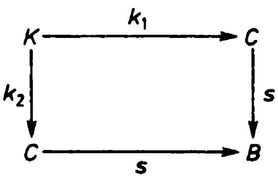

Diagram 8

$(*)$ and $(**)$. Let s be a regular epimorphism. Hence Diagram 8 is a pushout diagram ($s \approx \mathrm{Coeq}(k_1, k_2)$; cf. [9], Proposition 21.16).

We consider $p' := s' \circ k'_1 \circ m \circ l (= s' \circ k'_2 \circ m \circ l$, since $s' \circ k'_1 = s' \circ k'_2)$. Hence

$$p' \circ k_1 = s' \circ k'_1 \circ m \circ l \circ k_1 = s' \circ k'_1 \circ m = s' \circ k'_2 \circ m$$
$$= s' \circ k'_2 \circ m \circ l \circ k_2 = p' \circ k_2$$

by $(**)$. Thus there exists a unique $g: B \to B'$ such that

$$g \circ s = p' = s' \circ k'_j \circ m \circ l,$$

whence

$$g \circ e = g \circ s \circ h = s' \circ k'_j \circ m \circ l \circ h = s' \circ h' = f$$

by $(*)$. (The uniqueness of g also follows, since e is an epimorphism.)

That the assumption on s to be regular is not too strong is shown by the following

6.2. LEMMA. *With the notation of Theorem* 4.3 *for a given* e *assume that* $\mathcal{M} \subseteq \mathrm{Epi}\,\mathfrak{C}$ *and that* (ii) $\Rightarrow$ (i) *holds for every* f: $A \to C'$. *Then* s *is a regular epimorphism.*

Proof. Let the assumptions of Lemma 6.2 be satisfied. Then from Claim 3.4 (2) we infer that there exists a regular epimorphism, say s'': $C \to B''$, such that $(K; k_1, k_2) \approx \ker s''$. Thus there exists a unique (epi)-morphism w: $B'' \to B$ such that $w \circ s'' = s$ (note that $e \in \mathrm{Epi}\,\mathfrak{C}$ implies $s \in \mathrm{Epi}\,\mathfrak{C}$, whence $w \in \mathrm{Epi}\,\mathfrak{C}$). From [11], Satz 1.7 (9), and $\mathcal{M} \subseteq \mathrm{Epi}\,\mathfrak{C}$ we may infer that $s'' \in \mathcal{S}$ (cf. Diagram 9). Thus f: $s'' \circ h$ (i.e., $B' := B''$) yields the

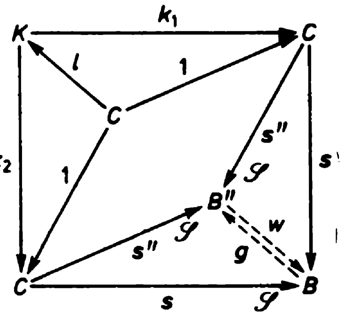

Diagram 9

situation of Theorem 4.3, where $(K'; k_1', k_2') = (K; k_1, k_2)$, whence $m = 1_K$ satisfies (*) and (**). Consequently, there exists a unique g: $B \to B''$ such that $g \circ e = f$, i.e., $g \circ s \circ h = s'' \circ h$. Since f is also an epimorphism, by considering the pairs (e, e) and (f, f) (instead of (e, f) in Theorem 4.3) we get $g \circ w = 1_{B''}$ and $w \circ g = 1_B$, whence s and s'' are isomorphic, i.e., s is regular.

6.3. PROPOSITION. *Under all the assumptions of Theorem* 4.3 *the following statements hold*:

(i) $g \in \mathrm{Epi}\,\mathfrak{C}$ *iff* $f \in \mathrm{Epi}\,\mathfrak{C}$.

(ii) $g \in \mathrm{Iso}\,\mathfrak{C}$ *iff*

(a) $m \in \mathrm{Iso}\,\mathfrak{C}$ (m *satisfies* (*) *and* (**)),

(b) s' *is a regular epimorphism,*

(c) m^{-1} *satisfies the identities corresponding to* (*) *and* (**).

(iii) *If* $\mathcal{M} \subseteq \mathrm{Mono}\,\mathfrak{C}$, *then* (a) *and* (b) *of* (ii) *imply* (c).

Proof. (i) is obvious. In order to prove (ii) assume first that $g \in \mathrm{Iso}\,\mathfrak{C}$. Then $g \in \mathcal{S}$, whence there exists an isomorphism i: $C \to C'$ such that $i \circ h = h'$, and $s' \circ i = g \circ s$, i.e., $s' = g \circ s \circ i^{-1}$ is isomorphic to s, and therefore is a regular epimorphism like s. Thus (b) holds. Since now also s' is a regular epimorphism, we infer easily (by applying Theorem 4.3 in the opposite direction and observing the uniqueness statements) that m^{-1} exists and satisfies (*) and (**).

Now let (a), (b) and (c) be satisfied. Then, by Theorem 4.3, there exists a $g'\colon B' \to B$ such that $g' \circ f = e$, and by the uniqueness statement in Theorem 4.3 and further applications of this theorem we see that g' is an inverse of g, i.e., that g is an isomorphism. Finally, assume in (ii) that (a) and (b) are satisfied and that $\mathscr{H} \subseteq \text{Mono}\,\mathfrak{C}$. Then, for $j \in \{1, 2\}$ we get

$$(k_j' \circ m \circ l) \circ (k_j \circ m^{-1} \circ l') = k_j' \circ m \circ m^{-1} \circ l' = k_j' \circ l' = 1_{C'},$$

whence $k_j \circ m^{-1} \circ l'$ is a section and $k_j' \circ m \circ l$ is the corresponding retraction. We use g, u, v, i as in 5.1, whence by (3) and (4) in 5.1 we have

$$i \circ u \circ k_j = k_j' \circ m = k_j' \circ m \circ l \circ k_j.$$

Now, $k_j \in \text{Epi}\,\mathfrak{C}$ (since $k_j \circ l = 1_C$), whence $i \circ u = k_j' \circ m \circ l$. But $i \in \text{Iso}\,\mathfrak{C}$ and $u \in \mathscr{H} \subseteq \text{Mono}\,\mathfrak{C}$, and so

$$i \circ u = k_j' \circ m \circ l \in \text{Mono}\,\mathfrak{C}.$$

Since it is also a retraction, it is an isomorphism (cf. [9], Proposition 67). This shows that

$$k_j \circ m^{-1} \circ l' = (k_j' \circ m \circ l)^{-1}, \quad \text{i.e.} \quad 1_C = k_j \circ m^{-1} \circ l' \circ k_j' \circ m \circ l,$$

whence

$$h = k_j \circ m^{-1} \circ l' \circ k_j' \circ m \circ l \circ h = k_j \circ m^{-1} \circ l' \circ h'$$

by (*), and so (*) also holds for m^{-1}. From the above equations we also get

$$i \circ u \circ k_j \circ m^{-1} = k_j',$$

whence

$$\begin{aligned} k_j \circ m^{-1} \circ l' \circ k_j' &= (k_j \circ m^{-1} \circ l') \circ (i \circ u) \circ k_j \circ m^{-1} \\ &= (k_j' \circ m \circ l)^{-1} \circ (k_j' \circ m \circ l) \circ k_j \circ m^{-1} = k_j \circ m^{-1}. \end{aligned}$$

Thus we also have (**) for m^{-1}.

7. Characterization of monomorphisms by ($\mathscr{H}$, $\mathscr{S}$)-kernels.

7.1. While epimorphisms and isomorphisms could be described above in general categories, we can describe monomorphisms only in special concrete categories. Note that in the category $\mathfrak{P}^{\Delta}$ of all partial algebras of some given type Δ, for the epimorphism e and the homomorphism f determined as in Theorem 4.3, g is a monomorphism iff

$$\text{S-ker}\, e = C \times C \cap \text{S-ker}\, f.$$

In order to get similar statements in a greater generality we extend Diagram 7 to Diagram 10 below by adding the direct products $C \times C$ and

$C' \times C'$ and the morphism m^* between them induced by $i \circ u \circ q_j$ $(j \in \{1, 2\})$. The morphisms k and k' are induced by k_1, k_2 and k_1', k_2', respectively. q_1, q_2, q_1', q_2' are the corresponding projections. For other commutativities compare the previous sections, e.g., we assume that (*) and (**) of 4.3 (ii) hold. We abbreviate

(5) $$x := s' \circ i \circ u \ (= v \circ u = g \circ s).$$

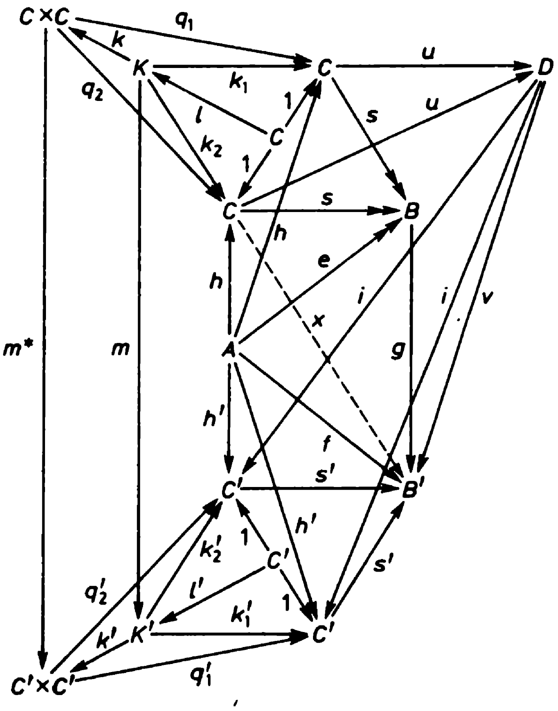

Diagram 10

PROPOSITION. *Under the assumptions of Theorem* 4.3 *and with the notation of Diagram* 10 *and the statements*

(i) *g is a monomorphism in $\mathfrak{C}$: $g \in \mathrm{Mono}\,\mathfrak{C}$;*

(ii) *$(K; k_1, k_2)$ is a kernel pair of x: $(K; k_1, k_2) \approx \ker x$;*

(iii) *$(K; k, m)$ is a pullback for $K' \xrightarrow{k'} C' \times C' \xleftarrow{m^*} C \times C$;*

for finitely complete categories $\mathfrak{C}$ we have

(a) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(b) *If, in addition, $(\mathfrak{C}, \mathscr{U})$ is a concrete category, and if its forgetful functor $\mathscr{U}$ preserves finite products, pullbacks and coequalizers, then* (ii) *implies* (i); *if, moreover, $\mathscr{M} \subseteq \mathrm{Mono}\,\mathfrak{C}$, then* (iii) *implies* (i), *so* (i), (ii) *and* (iii) *are then equivalent.*

Proof. (a) Let us first prove that (i) $\Rightarrow$ (ii). Assume that g is a monomorphism and let

$$F \overset{r_1}{\underset{r_2}{\rightrightarrows}} C$$

so that $x \circ r_1 = x \circ r_2$. Since, by (5), $x = g \circ s$, and since $g \in \mathrm{Mono}\,\mathfrak{C}$, we have $s \circ r_1 = s \circ r_2$. Now the rest of the proof of $(K; k_1, k_2) \approx \ker x$ is a consequence of $(K; k_1, k_2) \approx \ker s$ and the fact that

$$x \circ k_1 = g \circ s \circ k_1 = g \circ s \circ k_2 = x \circ k_2.$$

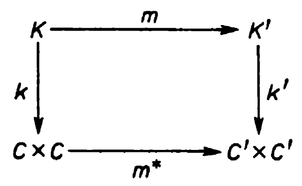

Diagram 11

Let us now assume that $(K; k_1, k_2) \approx \ker x$. In order to prove that Diagram 11 is a pullback square, consider $G \xrightarrow{a} K'$ and $G \xrightarrow{b} C \times C$ in $\mathrm{Mor}\,\mathfrak{C}$ such that

(6) $$k' \circ a = m^* \circ b.$$

Hence for $j \in \{1, 2\}$ we have

$$s' \circ q'_j \circ k' \circ a = s' \circ k'_j \circ a = s' \circ q'_j \circ m^* \circ b = s' \circ i \circ u \circ q_j \circ b.$$

Since $s' \circ k'_1 = s' \circ k'_2$, we even get

$$s' \circ q'_1 \circ k' \circ a = s' \circ q'_2 \circ k' \circ a = s' \circ i \circ u \circ q_1 \circ b$$
$$= s' \circ i \circ u \circ q_2 \circ b = x \circ q_1 \circ b = x \circ q_2 \circ b.$$

Since $(K; k_1, k_2) \approx \ker x$, there exists a unique t: $G \to K$ such that

(7) $$k_j \circ t = q_j \circ b \quad \text{for } j \in \{1, 2\}.$$

But $k_j = q_j \circ k$ $(j \in \{1, 2\})$, whence

$$q_1 \circ k \circ t = q_1 \circ b \quad \text{and} \quad q_2 \circ k \circ t = q_2 \circ b.$$

Consequently, by the properties of the projections of a product,

(8) $$k \circ t = b.$$

We now consider for $j \in \{1, 2\}$ (cf. (3) and (4) in 5.1 and (6) and (7) above)

(9) $$k'_j \circ m \circ t = i \circ u \circ k_j \circ t = i \circ u \circ q_j \circ b = q'_j \circ m^* \circ b$$
$$= q'_j \circ k' \circ a = k'_j \circ a.$$

Since $(K; k'_1, k'_2)$ is a kernel pair, we conclude from (9) that

(10) $$m \circ t = a.$$

From (8) and (10) we infer that Diagram 11 is really a pullback-diagram (the uniqueness of t was noted before (7)).

(b) Let the assumptions of (b) be true and let Diagram 11 be a pullback-diagram. Moreover, let

$$H \overset{t_1}{\underset{t_2}{\rightrightarrows}} B$$

be $\mathfrak{C}$-morphisms such that

(11) $$g \circ t_1 = g \circ t_2.$$

We have to show that $t_1 = t_2$ or (since the forgetful functor $\mathscr{U}$ is faithful) that $\mathscr{U}(t_1) = \mathscr{U}(t_2)$. Because of the properties required for $\mathscr{U}$, Diagram 10 is transferred by $\mathscr{U}$ into a diagram with the same properties in the category **Set** of all sets where the usual mappings are the morphisms; only epimorphisms may not be transferred into surjective mappings, but finite products, pullbacks (e.g., kernel pairs and monomorphisms), regular epimorphisms and all commutativities are preserved. Thus, assuming (ii), we have

(12) $$\ker \mathscr{U}(s) \approx (\mathscr{U}(K);\ \mathscr{U}(k_1),\ \mathscr{U}(k_2)) \approx \ker \mathscr{U}(g \circ s) \approx \ker(\mathscr{U}(g) \circ \mathscr{U}(s)).$$

But among sets this means that $\mathscr{U}(g)$ is injective, i.e., a monomorphism in **Set**. Since $\mathscr{U}$ reflects monomorphisms (cf. [9], Proposition 12.8), g is a monomorphism.

Let us finally assume that (iii) holds and $\mathscr{M} \subseteq \text{Mono}\,\mathfrak{C}$. Then $m \in \text{Mono}\,\mathfrak{C}$ by Lemma 5.2, and $m^* \in \text{Mono}\,\mathfrak{C}$ (since m^* is induced by $i \circ u \circ q_j$ ($j \in \{1, 2\}$), where $i \in \text{Iso}\,\mathfrak{C}$, $u \in \mathscr{M} \subseteq \text{Mono}\,\mathfrak{C}$, and (q_1, q_2) is a "mono-pair", m^* itself is a monomorphism). We want to show that also $k' \in \text{Mono}\,\mathfrak{C}$, whence (cf. [9], Proposition 21.13) $k \in \text{Mono}\,\mathfrak{C}$ and our assumption that Diagram 11 is a pullback square means that $\mathscr{U}(K)$ is (isomorphic to) the intersection of $\mathscr{U}(C \times C) \cong \mathscr{U}(C) \times \mathscr{U}(C)$ and $\mathscr{U}(K')$, whence $\mathscr{U}(g)$ is injective (applying the usual set theoretical arguments and observing that $\mathscr{U}(s)$ is a regular, therefore surjective epimorphism); this would prove that $g \in \text{Mono}\,\mathfrak{C}$. Thus assume that

$$k' \circ a' = k' \circ b' \quad \text{for } Q \overset{a'}{\underset{b'}{\rightrightarrows}} K'.$$

Then

$$q_j' \circ k' \circ a' = k_j' \circ a' = q_j' \circ k' \circ b' = k_j' \circ b'.$$

Since $(K';\ k_1',\ k_2')$ is a pullback (of $(s', s';\ B')$), this yields $a' = b'$, whence k' is a monomorphism.

As a consequence of Proposition 7.1 and the last part of the proof above we get the following

7.2. THEOREM. *Under the assumptions of Theorem* 4.3 *and with the notation of Diagram* 10 *let* $\mathfrak{C}$ *be in addition a concrete category with finite products and with forgetful functor* $\mathscr{U}$ *preserving finite products, pullbacks and coequalizers. Let* $\mathscr{M} \subseteq \text{Mono}\,\mathfrak{C}$. *Then the following statements are equivalent:*

(i) $g \in \mathrm{Mono}\,\mathfrak{C}$;

(ii) $(K; m, k)$ *is the intersection* (*up to unique isomorphism*) *of* $(k', m^*; C' \times C')$.

It is quite likely that one can still manipulate with the assumptions of the above theorem, but for most applications it will be sufficient. For instance, most of the interesting examples of factorization systems in a category $\mathfrak{C}$ of all partial algebras of some given similarity type satisfy $\mathscr{H} \subseteq \mathrm{Bi}\,\mathfrak{C}$, and $(\mathfrak{C}, \mathscr{U})$ satisfies the assumptions above. Moreover (cf. [9], Theorem 27.7), if $\mathscr{U}$ has a left adjoint, then $\mathscr{U}$ preserves limits. Hence the only requirement on $\mathscr{U}$ which then remains is that $\mathscr{U}$ has to preserve coequalizers (regularity). That this is not always satisfied, even when $\mathscr{U}$ has a left adjoint, is shown e.g. in [11], Satz 6.6, in the case of the category **Cat** of small categories:

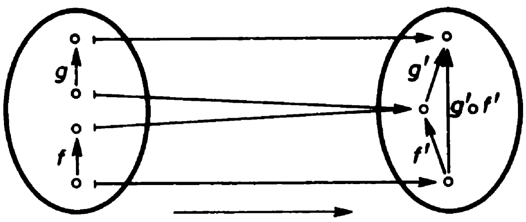

Diagram 12

Diagram 12 shows a regular morphism in **Cat**, but not surjective, i.e., $\mathscr{U}$ does not preserve regularity but it has a left adjoint.

8. Concluding remarks. Quite a few questions could still be posed and answered in general in connection with $(\mathscr{H}, \mathscr{S})$-kernels.

8.1. For instance, one can observe that exactly those epimorphisms starting from some object A in $\mathfrak{C}$ which have an $(\mathscr{H}, \mathscr{S})$-factorization (h, s) in which s is a regular epimorphism are characterized by their $(\mathscr{H}, \mathscr{S})$-kernel up to unique isomorphism. In the case where the class of these epimorphisms forms itself a partner in a factorization system – and this quite often happens, e.g., in the category of all partial algebras of some given type, as will be shown in some later paper – the $(\mathscr{H}, \mathscr{S})$-kernel is a convenient tool for describing this class $(\mathrm{Reg}\,\mathfrak{C} \cap \mathscr{S}) \circ \mathscr{H} \cap \mathrm{Mor}(A, \mathfrak{C})$ for objects A in $\mathfrak{C}$.

8.2. Assume $\mathscr{H} \subseteq \mathrm{Bi}\,\mathfrak{C}$, i.e., $\mathscr{H}$ consists only of bimorphisms of $\mathfrak{C}$, and let $\mathfrak{C}$ have finite products and a terminal object T. Then for every $A \in \mathrm{ob}\,\mathfrak{C}$ there exists a unique morphism $t_A\colon A \to T$ and in its $(\mathscr{H}, \mathscr{S})$-factorization

$$(h_T^A, s_T^A)\colon A \xrightarrow{h_T^A} A_T \xrightarrow{s_T^A} T$$

the object A_T plays an important rôle, since then the $(\mathscr{H}, \mathscr{S})$-kernel-object for each $((\mathrm{Reg}\,\mathfrak{C} \cap \mathscr{S}) \circ \mathscr{H})$-epimorphism starting from A is a subobject of $A_T \times A_T$ (this is an easy consequence of what has been shown in Section 7).

Note that the total one-element algebra of type Δ is a terminal object in the category $\mathfrak{P}^\Delta$ of all partial algebras of some given type Δ and that for every $A \in \mathrm{ob}\,\mathfrak{P}^\Delta$ its free completion (cf. [5]) $\hat{A}$ plays the rôle of A_T in the case of ($\mathfrak{T}^\Delta$-extendable epimorphisms, $\mathfrak{T}^\Delta$-perfect homomorphisms)-factorizations ($\mathfrak{T}^\Delta$ being the full subcategory of all total algebras of type Δ). This fact explains the rôle of $\hat{A}$ in [13] in connection with the results of this note.

8.3. Finally, let us observe that we can characterize all epimorphisms starting from A by their $(\mathscr{H}, \mathscr{S})$-kernel if every $\mathscr{S}$-epimorphism is regular. In $\mathfrak{P}^\Delta$ and for $\mathscr{H} \subseteq \mathrm{Bi}\,\mathfrak{P}^\Delta$ this is the case if $\mathscr{S}$ consists of all closed homomorphisms (or at least contains this class), and this is just the situation considered in [13].

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