COLLOQUIUM MATHEMATICUM

VOL. XXXV 1976 FASC. 1

PRIME IDEALS IN A SEMIGROUP OF MEASURES

BY

H. L. CHOW (HONG KONG)

Throughout S denotes a compact topological semigroup, i.e. a compact Hausdorff space with a jointly continuous associative multiplication, and $P(S)$ the set of probability measures on S. It is well known that $P(S)$ is a compact semigroup with convolution and the weak* topology (see [1]). If the support of a measure $\mu \in P(S)$ is denoted by $\operatorname{supp}\mu$, we have [1]

$$\operatorname{supp}\mu\nu = \operatorname{supp}\mu \operatorname{supp}\nu \quad (\mu, \nu \in P(S)).$$

For a subset $\Delta \subset P(S)$, let

$$\mathscr{S}(\Delta) = \bigcup_{\mu\in\Delta} \operatorname{supp}\mu.$$

It follows that $\mathscr{S}(\Delta)$ is an ideal of S, provided Δ is an ideal of $P(S)$.

An ideal Q of S is said to be *prime* if $AB \subset Q$ implies that $A \subset Q$ or $B \subset Q$, A, B being ideals of S. It is the purpose of this paper to investigate prime ideals in $P(S)$ and their intersection. In particular, if a prime ideal Q of $P(S)$ is open, we show that Q is convex and dense, and $\mathscr{S}(Q) = S$; moreover, the intersection of all open prime ideals in $P(S)$ also has similar properties.

LEMMA. *Given a non-empty subset Δ of $P(S)$, let*

$$V(\Delta) = \{t\mu + (1-t)\nu\colon \mu \in \Delta,\ \nu \in P(S),\ 0 < t < 1\}.$$

Then $V(\Delta)$ is convex and dense, and $\mathscr{S}\big(V(\Delta)\big) = S$. Further, $V(\Delta)$ is an ideal if Δ is an ideal.

Proof. It is easy to check that $V(\Delta)$ is a convex set. To show that $V(\Delta)$ is dense, take $\nu \in P(S)$ and $\mu \in \Delta$. Let

$$\mu_n = \frac{1}{n}\mu + \frac{n-1}{n}\nu.$$

Obviously, $\mu_n \in V(\Delta)$ and $\mu_n \to \nu$, so that $V(\Delta)$ is dense in $P(S)$. Now take any $a \in S$ and consider the Dirac measure $\delta(a)$ at a. Since

$$\tfrac{1}{2}\big(\mu + \delta(a)\big) \in V(\Delta) \quad \text{and} \quad a \in \operatorname{supp} \tfrac{1}{2}\big(\mu + \delta(a)\big),$$

we see that $\mathcal{S}(V(\Delta)) = S$ follows. Finally, if Δ is an ideal, it is clear that $V(\Delta)$ is an ideal, completing the proof.

THEOREM 1. *If Δ is an open prime ideal of $P(S)$, then $\Delta = V(\Delta)$. Hence, the open prime ideal Δ is convex and dense, and $\mathcal{S}(\Delta) = S$.*

Proof. Evidently, the theorem holds if $\Delta = P(S)$. Now suppose Δ is a proper open prime ideal of $P(S)$. By virtue of Theorem 2 of [2], there exists an idempotent $\tau \in P(S)$ such that Δ is the largest ideal contained in the set $P(S)\setminus\{\tau\}$. We assert that $\tau \notin V(\Delta)$. Suppose not, i.e. $\tau \in V(\Delta)$, and we have $\tau = t\mu + (1-t)\nu$ for some $\mu \in \Delta$, $\nu \in P(S)$, $0 < t < 1$. It follows that $\operatorname{supp}\tau \supset \operatorname{supp}\mu$. This together with Lemma 3 of [3] gives $\tau\mu\tau = \tau$, whence $\tau \in \Delta$, a contradiction. Thus $\tau \notin V(\Delta)$, i.e. the ideal $V(\Delta)$ is contained in $P(S)\setminus\{\tau\}$, implying that $V(\Delta) \subset \Delta$. The result follows from the fact that $V(\Delta) \supset \Delta$.

COROLLARY 1. *Suppose Γ is the intersection of all open prime ideals of $P(S)$; then $\Gamma = V(\Gamma)$. Hence Γ is convex and dense, and $\mathcal{S}(\Gamma) = S$.*

Proof. It is clear that Γ is non-empty. Let $\tau \in V(\Gamma)$, i.e. $\tau = t\mu + (1-t)\nu$ for some $\mu \in \Gamma$, $\nu \in P(S)$, $0 < t < 1$. For any open prime ideal Δ, we have $\mu \in \Delta$, so that $\tau \in V(\Delta) = \Delta$. Consequently, $\tau \in \Gamma$ and so $V(\Gamma) = \Gamma$.

Note that Theorem 1 and Corollary 1 do not hold for general prime ideals in $P(S)$. For instance, consider the semigroup $S = [0,1]$ with usual topology and multiplication. As is easily seen, $\{\delta(0)\}$ is a prime ideal of $P(S)$ and also the intersection of prime ideals in $P(S)$. It is trivial that $\{\delta(0)\}$ is not dense in $P(S)$ and $\mathcal{S}(\{\delta(0)\}) = \{0\} \neq S$.

For a non-empty subset $A \subset S$, let $P(A) = \{\mu \in P(S)\colon \operatorname{supp}\mu \subset A\}$.

THEOREM 2. *Let $I \subset S$ be non-empty. Then I is a prime ideal of S if and only if $P(I)$ is a prime ideal of $P(S)$.*

Proof. First we suppose that I is a prime ideal of S. It is obvious that $P(I)$ is an ideal of $P(S)$. Let Δ_1, Δ_2 be ideals of $P(S)$ such that $\Delta_1\Delta_2 \subset P(I)$. Then

$$\mathcal{S}(\Delta_1)\mathcal{S}(\Delta_2) = \mathcal{S}(\Delta_1\Delta_2) \subset I.$$

Since $\mathcal{S}(\Delta_1)$ and $\mathcal{S}(\Delta_2)$ are ideals of S, and I is a prime ideal, we have $\mathcal{S}(\Delta_1) \subset I$ or $\mathcal{S}(\Delta_2) \subset I$. It follows that $\Delta_1 \subset P(I)$ or $\Delta_2 \subset P(I)$, i.e. $P(I)$ is a prime ideal of $P(S)$. Conversely, suppose I is a non-empty subset of S such that $P(I)$ is a prime ideal of $P(S)$. For $a \in I$ and $b \in S$, we have $\delta(a) \in P(I)$ and $\delta(b) \in P(S)$. Since $\delta(ab) = \delta(a)\delta(b) \in P(I)$ and $\delta(ba) \in P(I)$, we see that $ab \in I$ and $ba \in I$, i.e. I is an ideal of S. To show that I is prime, let A, B be ideals of S such that $AB \subset I$. Hence

$$P(A)P(B) \subset P(AB) \subset P(I).$$

Since $P(A)$ and $P(B)$ are ideals of $P(S)$, $P(A) \subset P(I)$ or $P(B) \subset P(I)$. Therefore, $A \subset I$ or $B \subset I$, and the theorem is proved.

We notice that the prime ideal $P(I)$ is not open when I is a proper open prime ideal of S. For, if $a \in S \setminus I$ and $\mu \in P(I)$, then

$$\mu_n = \frac{n-1}{n}\mu + \frac{1}{n}\delta(a) \in P(S) \setminus P(I),$$

but $\mu_n \to \mu$, i.e. $P(S)\setminus P(I)$ is not closed. So $P(I)$ is not open.

COROLLARY 2. *Let $\{I_a\}$ and $\{J_\beta\}$ be the families of all prime ideals in S and $P(S)$, respectively. Then*

$$\bigcap_a I_a \supset \mathscr{S}\Big(\bigcap_\beta J_\beta\Big).$$

Proof. Take any prime ideal I_a of S. It follows from Theorem 2 that $P(I_a)$ is a prime ideal of $P(S)$, so that $P(I_a) \supset \bigcap_\beta J_\beta$, whence

$$I_a \supset \mathscr{S}\Big(\bigcap_\beta J_\beta\Big),$$

and the result is immediate.

REFERENCES

[1] I. Glicksberg, *Convolution semigroups of measures*, Pacific Journal of Mathematics 9 (1959), p. 51-67.
[2] K. Numakura, *Prime ideals and idempotents in compact semigroups*, Duke Mathematical Journal 24 (1957), p. 671-680.
[3] J. S. Pym, *Idempotent probability measures on compact semitopological semigroups*, Proceedings of the American Mathematical Society 21 (1969), p. 499-501.

DEPARTMENT OF MATHEMATICS
CHUNG CHI COLLEGE
THE CHINESE UNIVERSITY OF HONG KONG

Reçu par la Rédaction le 20. 9. 1974