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LINEARIZATION OF A CONTRACTIVE HOMEOMORPHISM

BY

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Let X be a topological space and $\varphi\colon X \to X$ a continuous selfmapping of X. We say that φ is *linearized in L by Φ* if there exists a topological embedding $\mu\colon X \to L$ of the space X into the linear topological vector space L such that $\mu(\varphi(x)) = \Phi(\mu(x))$ for all $x \in X$, where Φ is a continuous linear operator on L.

Let X now be metrizable and let $a \in [0, 1)$. We say that $\varphi\colon X \to X$ is a *topological a-contraction on X* if there exists a metric $\varrho(x, y)$ on X inducing the given topology such that

$$\forall x,\ y \in X\colon\ \varrho(\varphi(x), \varphi(y)) \leqslant a\varrho(x, y).$$

If φ is a homeomorphism and at the same time a topological a-contraction, we will say φ is a *topologically a-contractive homeomorphism.*

Let now $a > 0$. We will say that φ is a *topological a-homothety on X* if there is a metric $\varrho(x, y)$ on X inducing the given topology such that

$$\forall x,\ y \in X\colon\ \varrho(\varphi(x), \varphi(y)) = a\varrho(x, y).$$

The main objective of this paper will be to prove the following

THEOREM. *If X is a compact metrizable space $a \in (0, 1)$, and $\varphi\colon X \to X$ a topologically a-contractive homeomorphism, then φ can be linearized in a separable Hilbert space as a homothety. More precisely, for every $\beta \in (0, 1)$ there exists a topological embedding $\mu\colon X \to H$ of X into a separable Hilbert space H such that $\forall x \in X\colon \mu(\varphi(x)) = \beta\mu(x)$.*

According to the theorem proved in [1], the mapping φ is a topological a-homothety for every $a \in (0, 1)$, i.e., there exists a metric ϱ on X such that

$$\forall x,\ y \in X\colon\ \varrho(\varphi(x), \varphi(y)) = a\varrho(x, y).$$

We will show first of all that (X, ϱ) can be embedded isometrically in the larger metric space (X^*, ϱ^*) over which φ can be extended as an a-homothety *onto.*

Let us write $A_0 = X - \varphi(X)$, $A_{n+1} = \varphi(A_n)$ for $n = 0, 1, 2, \ldots$

We observe that the sets A_n are all mutually homeomorphic and disjoint, and that X can be represented in the form

$$X = \Big[\bigcup_{n=0}^{\infty} A_n\Big] \cup \{a\},$$

where $a \in X$ is the fixed point of φ. The mapping φ has an inverse on $\varphi(X)$ and we have $\varphi^{-1}(A_n) = A_{n-1}$ for $n = 1, 2, \ldots$

Let us now introduce the family of sets $A_{-1}, A_{-2}, \ldots$ as mutually disjoint abstract copies of A_0 and disjoint with X. Let us introduce mappings $\varphi_n\colon A_n \to A_{n+1}$ for $n = -1, -2, \ldots$ to be one-to-one and onto. Now we can introduce the set X^* as

$$\Big[\bigcup_{-\infty}^{+\infty} A_n\Big] \cup \{a\}$$

and the mapping $\varphi^*\colon X^* \to X^*$ in the following way: if $x \in A_n$ for $n \geqslant 0$, we put $\varphi^*(x) = \varphi(x)$; if $x \in A_n$ for $n < 0$, we put $\varphi^*(x) = \varphi_n(x)$; and finally we put $\varphi^*(a) = \varphi(a)$.

It is obvious that φ^* is one to one and maps X^* onto itself.

Define $n(x) = n$ for $x \in A_n$ and

$$n(a) = \infty, \qquad n(x, y) = \min\{n(x), n(y)\}.$$

With this denotation we define a metric ϱ^* on X^* by the formula:

$$\varrho^*(x, y) = a^n \varrho\big(\varphi^{*-n}(x), \varphi^{*-n}(y)\big),$$

where $n = n(x, y)$.

Due to the fact that $n\big(\varphi^*(x)), \varphi^*(y)\big) = 1 + n(x, y)$ we see that $\varrho^*\big(\varphi^*(x), \varphi^*(y)\big) = a\varrho^*(x, y)$ for all $x, y \in X^*$. In the sequel we will denote the function φ^* again by φ, and ϱ^* we will denote by ϱ on the whole X^*.

Our next objective will be to show that for every $\beta \in (0, 1)$ there exists a countable family of functions $f_i(x) \in C(X^*)$ such that

1. $f_i\big(\varphi(x)\big) = \beta f_i(x)$ for all $i = 1, 2, \ldots$;

2. the family is uniformly bounded on X, i.e. $\exists M \geqslant 0$ such that $|f_i(x)| \leqslant M$ for all $i = 1, 2, \ldots$ and all $x \in X$;

3. the family is point separating on X, i.e. for any $t_1, t_2 \in X$ there exists an index i such that $f_i(t_1) \neq f_i(t_2)$.

Let $x \in A_0$ and denote by $d(x)$ the distance between x and $\varphi(X)$: $d(x) = \varrho\big(x, \varphi(X)\big)$. The function $d(x)$ is positive, because $\varphi(X)$ is compact. Denote by $N(x, r)$ a spherical neighborhood of the radius $r > 0$ about $x \in A_0$ in X^* (we are working in X^*):

$$t \in N(x, r) \Leftrightarrow \varrho(x, t) < r.$$

If $r < d(x)$, then $N(x, r) \cap \varphi(X) = 0$ and it is easily seen that if $r < d(x)/2$, then all images $\varphi^n(N(x, r))$ are mutually disjoint.

Let us denote by $\mathfrak{R}_x$ the set of all $r > 0$ such that

(i) $N(x, r) \cap \varphi(X) = 0$,

(ii) the family $\varphi^n(N(x, r))$ is disjoint.

It is easy to see that $\mathfrak{R}_x$ is an interval $(0, R_x]$, where $R_x > 0$.

Let us now associate to every $x \in A_0$ and every $r \in (0, R_x]$ a continuous function $g(x, r; t)$ of t on X^* in such a way that:

(i) $g(x, r; t)\colon X^* \to [0, 1]$,

(ii) $g(x, r; t) = 1$ for $t \in N(x, r/2)$

(iii) $g(x, r; t) = 0$ for $t \in N^C(x, r)$ (complement of $N(x, r)$ in X^*).

The fact that all $\varphi^n[N(x, r)]$ are disjoint enables us to define the function $f(x, r; t)\colon X^* \to [0, \infty)$ putting

$$f(x, r; t) = \begin{cases} \beta^n g(x, r; \varphi^{-n}(t)) & \text{if} \quad t \in \varphi^n[N(x, r)], \\ 0 & \text{if} \quad t \notin \bigcup_{-\infty}^{+\infty} \varphi^n[N(x, r)]. \end{cases}$$

The number β is chosen arbitrarily from $(0, 1)$. Continuity of $f(x, r; t)$ can be easily seen because it can be represented in the form,

$$\sum_{-\infty}^{+\infty} \beta^n g(x, r; \varphi^{-n}(t)),$$

the sum being uniformly converging on each set of the form

$$\Big[\bigcup_{i=n}^{\infty} A_k\Big] \cup \{a\}.$$

The function $f(x, r; t)$ satisfies obviously the equation:

$$f(x, r; \varphi(t)) = \beta f(x, r; t)$$

and is bounded on X because $\varphi^{-n}[N(x, r)] \cap X = 0$ for all $n = 1, 2, \ldots$ and therefore

$$\sup_{t \in X} f(x, r; t) = \sup_{t \in X} g(x, r; t) = 1.$$

Let Q be a dense and countable subset of A_0, and let $t_1, t_2 \in X$ be arbitrarily given different points of X: $t_1 \neq t_2$. Then at least one of them, say $t_2 \neq a$ and therefore there exists $x \in A_0$ such that $\varphi^n(x) = t_2$ for some $n \geqslant 0$. Consider the neighborhood $N(x, r)$ for some rational $r \in (0, R_x]$ and choose $q \in Q$, $q \in N(x, r/4)$. Then evidently $r/2 \in (0, R_q]$ and the function $f(q, r/2; t)$ separates points t_2 and a, because $f(q, r/2; t_2) = a^n f(q, r/2; x) > 0$ because $x \in N(q, r/4)$, and $f(q, r/2; a) = 0$, so if

$t_1 = a$, we are done. If $t_1 \neq a$, then $y = \varphi^{-m}(t_1)$ for some $y \in A_0$ and $m \geqslant 0$. If we choose the rational number r such that $r \in (0, R_x]$, $r < \varrho(x, y)$ and choose $q \in N(x, r/4)$, then we have again $f(q, r/2; t_2) \neq 0$, $f(q, r/2; y) = 0$ and therefore $f(q, r/2; t_1) = 0$, and we have shown that the family $f(q, r; t)$, where $q \in Q$ and r is a rational number from $(0, R_q]$, satisfies our conditions. If we index this function of our family by natural numbers $f_n(t)$, we may construct the desired embedding μ: $X \to H$ by the formula

$$\mu(t) = \{f(t), \tfrac{1}{2}f_2(t), \tfrac{1}{3}f_3(t), \ldots\}.$$

REFERENCES

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[2] — *One-to-one contractive mappings on compact spaces*, Notices of the American Mathematical Society, January 1967, p. 133, 67 T — 21.

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