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INVARIANT METRICS ON THE TANGENT BUNDLE OF A HOMOGENEOUS SPACE

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Brockett and Sussmann [1] have proved that tangent bundles of homogeneous spaces are homogeneous spaces. More precisely, if H is a closed subgroup of a Lie group G, then the tangent bundle TM of the homogeneous space $M = G/H$ is the homogeneous space $\tilde{G}/\tilde{H}$, where $\tilde{G} = G \times g$, $\tilde{H} = H \times h$, g and h are Lie algebras of G and H, respectively, and the group structure on $\tilde{G}$ is defined by

$$(a, X)\cdot(a', X') = \big(aa', X+\mathrm{ad}(a)X'\big).$$

The group $\tilde{G}$, called the *tangent group of a Lie group* G, was defined by Kobayashi [3]. In the present paper, invariant metrics on the space $\tilde{G}/\tilde{H}$ are studied.

THEOREM 1. *If* $[g, h] \subset h$, *then any two* G-*invariant indefinite Riemannian metrics on* M *induce a* $\tilde{G}$-*invariant metric on* TM. *If* $[h, g] \not\subset h$, *then there exists no* $\tilde{G}$-*invariant positive-definite metric on* TM.

Proof. In view of the natural correspondence between G-invariant indefinite Riemannian metrics on M and $\mathrm{ad}(H)$-invariant non-degenerate symmetric bilinear forms on g/h (see [4], p. 200) we have to prove the following:

(a) If $[h, g] \subset h$ and B, B' are such forms on g/h, then the form $\tilde{B}$ on $\tilde{g}/\tilde{h} = g/h \times g/h$ defined by

$$\tilde{B}\big((\bar{X}, \bar{X}'), (\bar{Y}, \bar{Y}')\big) = B(\bar{X}, \bar{Y})+B'(\bar{X}', \bar{Y}'),$$

where $\bar{X}, \bar{X}'$, etc. are elements of g/h represented by $X, X' \in g$, is $\mathrm{ad}(\tilde{H})$-invariant.

(b) If $[h, g] \not\subset h$, then there exists no $\mathrm{ad}(\tilde{H})$-invariant positive-definite symmetric bilinear form on $\tilde{g}/\tilde{h}$.

Let us take an arbitrary element $\tilde{a} = (a, X)$ of $\tilde{H}$ and two left-invariant vector fields Y and Z on G. Denote by (a_t) and (b_t) the 1-parameter subgroups of G generated by Y and Z, respectively. Then

$$\begin{aligned}\mathrm{ad}(\tilde{a})(Y, Z) &= \frac{d}{dt}\tilde{a}(a_t, b_t)\tilde{a}^{-1}|_{t=0}\\ &= \frac{d}{dt}(a, X)(a_t, b_t)\big(a^{-1}, -\mathrm{ad}(a^{-1})X\big)\big|_{t=0}\\ &= \frac{d}{dt}\big(\mathrm{ad}(a)a_t, X+\mathrm{ad}(a)b_t-\mathrm{ad}\big(\mathrm{ad}(a)a_t\big)X\big)\big|_{t=0}\\ &= \big(\mathrm{ad}(a)Y, \mathrm{ad}(a)Z-[X, \mathrm{ad}(a)Y]\big).\end{aligned}$$

Hereafter, we identify the Lie algebra $\tilde{g}$ of $\tilde{G}$ with the product $g\times g$. From the above formula it follows that if $[h, g]\subset h$, then

$$\mathrm{ad}(\tilde{a})(\overline{Y}, \overline{Z}) = \big(\overline{\mathrm{ad}(a)Y}, \overline{\mathrm{ad}(a)Z}\big)$$

for any $\tilde{a} = (a, X)$ of $\tilde{H}$ and Y, Z of g, where, as previously, $\overline{Y}, \overline{Z}$, etc. are elements of g/h represented by Y, Z, etc. Thus, (a) follows immediately from this equality.

Now, let us assume that $\tilde{B}$ is an $\mathrm{ad}(\tilde{H})$-invariant positive-definite symmetric bilinear form on $g/h\times g/h$. This form defines an $\mathrm{ad}(\tilde{H})$-invariant symmetric bilinear form on $g\times g$ by putting

$$B\big((Y, Y'), (Z, Z')\big) = \tilde{B}\big((\overline{Y}, \overline{Y}'), (\overline{Z}, \overline{Z}')\big).$$

Because of the equalities

$$\begin{aligned}&B\big(\mathrm{ad}(\tilde{a})(Y, Y'), \mathrm{ad}(\tilde{a})(Z, Z')\big)\\ &= B\big((\mathrm{ad}(a)Y, \mathrm{ad}(a)Y'), (\mathrm{ad}(a)Z, \mathrm{ad}(a)Z')\big)-\\ &\quad -B\big((0, [X, \mathrm{ad}(a)Y]), (\mathrm{ad}(a)Z, \mathrm{ad}(a)Z')\big)-\\ &\quad -B\big((\mathrm{ad}(a)Y, \mathrm{ad}(a)Y'), (0, [X, \mathrm{ad}(a)Z])\big)+\\ &\quad +B\big((0, [X, \mathrm{ad}(a)Y]), (0, [X, \mathrm{ad}(a)Z])\big) = B\big((Y, Y'), (Z, Z')\big),\end{aligned}$$

where $\tilde{a} = (a, X)\in\tilde{H}$, we see (putting $X = 0$) that

$$B\big((\mathrm{ad}(a)Y, \mathrm{ad}(a)Y'), (\mathrm{ad}(a)Z, \mathrm{ad}(a)Z')\big) = B\big((Y, Y'), (Z, Z')\big)$$

and, consequently, that

$$\begin{aligned}B\big((0, [X, Y]), (0, [X, Z])\big)-B\big((0, [X, Y]), (Z, Z')\big)-&\\ -B\big((Y, Y'), (0, [X, Z])\big) &= 0\end{aligned}$$

for any X of h and Y, Y', Z, Z' of g. Taking $Z = 0$ and $Z' = [X, Y]$ we obtain

$$B\big((0, [X, Y]), (0, [X, Y])\big) = 0.$$

Thus $[X, Y] \in h$ for any $X \in h$ and $Y \in g$, which completes the proof of Theorem 1.

We recall that a positive-definite Riemannian metric g on an arbitrary manifold N determines the Riemannian metric $\tilde{g}$ on the tangent bundle TN which is defined by the formula

$$\tilde{g}(v, w) = g\big(d\pi(v), d\pi(w)\big) + g\big(K(v), K(w)\big),$$

where v and w are vectors tangent to TN, π is the natural projection $TN \to N$, and $K\colon TTN \to TN$ is the connection map corresponding to the Levi-Civitá connection ∇ on the Riemannian manifold (N, g) (see [2]). The mapping K is completely determined by the equality

$$K\big(dY(v)\big) = \nabla_v Y$$

for any vector field Y on N and any vector v of TN. The metric g was defined and investigated by Sasaki [6].

THEOREM 2. *If g is a G-invariant positive-definite Riemannian metric on a connected homogeneous space $M = G/H$, and the metric $\tilde{g}$ is $\tilde{G}$-invariant, then the Levi-Civitá connection ∇ on (M, g) is flat.*

Proof. Let us take an arbitrary element $\tilde{a} = (a, X)$ of $\tilde{G}$ and denote by X' the vector field on M defined by

$$X'_x = d\sigma_x(X),$$

where σ_x is the mapping $G \ni b \mapsto bx$. We have the equalities

$$(*) \qquad K \circ \tilde{a} = a \circ K + \nabla X' \circ a \circ d\pi$$

and

$$(**) \qquad d\pi \circ \tilde{a} = a \circ d\pi.$$

Hereafter, G (respectively, $\tilde{G}$) is considered as a group of diffeomorphisms of M or TM (respectively, TM or TTM). The reader can distinguish without difficulties these different meanings of symbols a, $\tilde{a}$, etc.

In order to prove the first of the equalities above let us take a vector field Y on M and a vector u of TM, and put $v = dY(u)$. Then

$$\begin{aligned} K \circ \tilde{a}(v) &= K\big(d(\tilde{a}Y)(u)\big) = K\big(d(aY)(au) + dX'(au)\big) \\ &= \nabla_{au} aY + \nabla_{au} X' = aK(v) + \nabla_{au} X'. \end{aligned}$$

This yields $(*)$. The proof of $(**)$ is similar.

It follows from $(**)$ that

$$g\big(d\pi \circ \tilde{a}(v), d\pi \circ \tilde{a}(w)\big) = g\big(d\pi(v), d\pi(w)\big)$$

for any $v, w, \tilde{a}$. Therefore, the metric $\tilde{g}$ is $\tilde{G}$-invariant if and only if

$$g\big(K \circ \tilde{a}(v), K \circ \tilde{a}(w)\big) = g\big(K(v), K(w)\big)$$

for any v, w, $\tilde{a}$. Using relation $(*)$ we obtain

$$g\big(K\circ\tilde{a}(v), K\circ\tilde{a}(w)\big)-g\big(K(v), K(w)\big)$$
$$= g\big(aK(v), \nabla_{\mathrm{ad}\pi(w)}X'\big)+g\big(aK(w), \nabla_{\mathrm{ad}\pi(v)}X'\big)+g\big(\nabla_{\mathrm{ad}\pi(v)}X', \nabla_{\mathrm{ad}\pi(w)}X'\big).$$

This shows that $\tilde{g}$ is a $\tilde{G}$-invariant metric if and only if $\nabla X' = 0$ for any X of g.

The vector fields X', $X\in g$ generate the module of vector fields on an open neighbourhood of the origin of M. Thus, if the metric $\tilde{g}$ is invariant, then the connection ∇ is flat.

The following fact is an immediate consequence of Theorem 2 and the result of Kowalski [5].

COROLLARY. *If the metric $\tilde{g}$ on TM is $\tilde{G}$-invariant, then the Riemannian manifold $(TM, \tilde{g})$ is flat.*

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