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ON INTERNAL MOVABILITY; INTERNAL SHAPE AND INTERNAL MANR-SPACES

BY

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0. Introduction. Our purpose in this paper is to extend both the concepts of internal movability and internal shape to arbitrary metrizable spaces and to study the relations between them.

Internal movability for compact metric spaces was introduced by Bogatyi in [2] and studied, among others, by Dydak [4], Laguna and Sanjurjo [13]. In [13] the authors outlined an internal shape theory for compact metrizable spaces. In this paper we extend such an internal shape theory to arbitrary metrizable spaces and we prove that if we limit ourselves to the class of internally movable metrizable spaces, then this internal shape is the same as the shape in the sense of Fox [6]. On the other hand, we introduce the concept of internal MANR-space and we prove that in the compact case this concept is the same as that of internal FANR-space introduced by Laguna and Sanjurjo in [13]. Some properties of internal MANR-spaces are established; in particular, we prove that every homotopy class of mutations between a metrizable space and an internal MANR-space is generated by a continuous function. This fact allows us to prove a result which is an extension of a theorem due to Borsuk and Olędzki in [3].

In this paper ANR-spaces are those for metrizable spaces and we denote by $\simeq$ the homotopy relation. The relation $f\colon U \to V$, $f \in \boldsymbol{f}$, means that the triple $f\colon U \to V$ belongs to the mutation $\boldsymbol{f}$. We denote by $U(X, P)$, $V(Y, Q)$,... the families of all open neighborhoods of X in P, of Y in Q,..., where $X \subset P$, $Y \subset Q$,... as closed subsets. Finally, we denote by $\boldsymbol{u}_X$ the identity mutation on $U(X, P)$ if there is no confusion, otherwise we denote it by $\boldsymbol{u}_{X,P}$.

1. On internal movability in metric spaces.

1.1. DEFINITION (compare with Definition 1 in [2], p. 94). Let P be an ANR-space and $X \subset P$ a closed subset. We say that X is *internally movable* in P if for every $U \in U(X, P)$ there exist $U_0 \in U(X, P)$ and a homotopy $H\colon U_0 \times I \to U$ such that

$$H(x, 0) = x, \quad H(x, 1) \in X \quad \text{for every } x \in U_0.$$

Let us prove now a characterization of internal movability of closed subsets of ANR-spaces, involving mutations.

1.2. PROPOSITION. *Let P be an ANR-space and $X \subset P$ a closed subset. Then the following statements are equivalent:*

(I) *X is internally movable in P.*

(II) *There exists a mutation $\boldsymbol{f}: U(X, P) \to U(X, P)$ homotopic to the identity mutation on X and such that for every $f: U \to V$, $f \in \boldsymbol{f}$, the relation $f(U) \subset X$ holds.*

Proof. (I)$\Rightarrow$(II). Let $U \in U(X, P)$; then there exist $U_0 \in U(X, P)$ and a homotopy $F: U_0 \times I \to U$ such that $F(x, 0) = x$ and $F(x, 1) \in X$ for every $x \in U_0$. Let now

$$\boldsymbol{f}_U = \{g: U'_0 \to V \text{ with } U'_0 \in U(X, P),\ U'_0 \subset U_0,\ U \subset V \text{ and } g(x) = F(x, 1) \text{ for } x \in U'_0\}$$

and

$$\boldsymbol{f} = \bigcup_{U \in U(X,P)} \boldsymbol{f}_U.$$

In order to prove that $\boldsymbol{f}$ is a mutation it is enough to see that if $f_1, f_2: U \to V$, $f_1, f_2 \in \boldsymbol{f}$, then there exists $U' \in U(X, P)$ such that $f_1|U' \simeq f_2|U'$ in V. Let us suppose that

$$f_1, f_2: U \to V, \quad f_1, f_2 \in \boldsymbol{f}.$$

Then, by the definition of $\boldsymbol{f}$, there exist $U_1, U_2, U_{01}, U_{02} \in U(X, P)$ with $U_{0i} \subset U_i (i = 1, 2)$ and $F_i: U_{0i} \times I \to U_i$ such that $F_i(x, 0) = x$, $F_i(x, 1) \in X$ for every $x \in U_{0i}$ and $i = 1, 2$. Moreover,

$$U_1 \cup U_2 \subset V, \quad U \subset U_{01} \cap U_{02}$$

and

$$f_1(x) = F_1(x, 1), \quad f_2(x) = F_2(x, 1).$$

Then we have

$$f_1 = F_1(\cdot, 1)|U \simeq i_{V,U} \simeq F_2(\cdot, 1)|U = f_2,$$

where $i_{V,U}: U \to V$ is the inclusion map. Consequently, we have proved that $\boldsymbol{f}$ is a mutation homotopic to the identity mutation on X. It is obvious that $\boldsymbol{f}$ satisfies all the conditions in (II).

(II)$\Rightarrow$(I). Let $\boldsymbol{f}: U(X, P) \to U(X, P)$ be a mutation satisfying all the conditions in (II). Let $U \in U(X, P)$; then there exists $f: U' \to U$, $f \in \boldsymbol{f}$. Since $\boldsymbol{f} \simeq \boldsymbol{u}_X$, there exist $U_0 \in U(X, P)$, with $U_0 \subset U'$, and a homotopy $F: U_0 \times I \to U$ such that $F(x, 0) = x$ and $F(x, 1) = f(x)$ for every $x \in U_0$. On the other hand, $f(x) \in X$ for every $x \in U_0$, and then X is internally movable in P.

This characterization allows us to prove, in a short way, the following fact:

1.3. PROPOSITION. *Let P and Q be ANR-spaces and $X \subset P$, $Y \subset Q$ as closed subsets. Let us suppose that Y is weakly homotopy dominated by X. Then if X is internally movable in P, Y is also internally movable in Q.*

Proof. Let $f\colon X \to Y$ and $g\colon Y \to X$ be continuous maps such that $f \cdot g$ is weakly homotopic to the identity map on Y. Let

$$\boldsymbol{h}\colon U(X, P) \to U(X, P)$$

be a mutation satisfying all the conditions in (II) of Proposition 1.2. Let us suppose that

$$\hat{\boldsymbol{f}}\colon U(X, P) \to V(Y, Q), \qquad \hat{\boldsymbol{g}}\colon V(Y, Q) \to U(X, P)$$

are mutations generated by the maps f and g, respectively. Then $\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{g}} \simeq \boldsymbol{u}_Y$. It is obvious that the mutation $\boldsymbol{k} = \hat{\boldsymbol{f}} \cdot \boldsymbol{h} \cdot \hat{\boldsymbol{g}}$ satisfies all the conditions of (II) in Proposition 1.2, and then Y is internally movable in Q.

1.4. Remark. (I) Proposition 1.3 was proved by Bogatyi (see [2], Theorem 3) in the compact case.

(II) As a consequence of Proposition 1.3 we infer that a retract of an internally movable subset of an ANR-space is internally movable.

By the Kuratowski–Wojdysławski Theorem (see [10], p. 81) and Proposition 1.3 we can define the following topological invariant:

1.5. DEFINITION. Let X be a metrizable space. We say that X is *internally movable* if X is homeomorphic to a closed internally movable subset of an ANR-space.

1.6. EXAMPLE. (I) Every ANR-space is internally movable.

(II) From Theorem 2-4 in [14] it follows that every MAR-space is internally movable.

(III) There exists a compact MANR-space which is not internally movable; e.g., the Warsaw circle.

Since the Warsaw circle has the shape of the 1-dimensional sphere S^1, we can assure that the internal movability is not a shape (in the sense of Borsuk) invariant.

(IV) Every retract of an internally movable space is internally movable while, using the Warsaw circle, not every fundamental retract of an internally movable space is internally movable.

Using the same arguments as in Theorem 3.1 in [1] we have

1.7. THEOREM. *Let X be a metrizable space and D a partition of X into closed subsets such that the following two conditions are satisfied:*

(I) *$p\colon X \to X/D$ is closed, where p is the natural projection onto the decomposition space X/D.*

(II) (*covering*) $\dim(X/D) = 0$.

If, in addition, every member of D is internally movable, then X is internally movable.

As in [1] we can derive the following facts from Theorem 1.7:

1.8. COROLLARY. *Let $X \in S_0$* (see Definition 2.1 in [1]) *be a metrizable space. If every component of X is internally movable, then X is internally movable.*

1.9. Remark. (I) Let us note that Corollary 1.8 is a generalization of Theorem 5 due to Bogatyi in [2].

(II) As a consequence of Corollary 1.8 we infer that every metrizable space with (covering) dimension $= 0$ is internally movable.

Using Proposition 1.3 we see that if an internally movable space X is a product of a family of spaces, then every factor of such a product is internally movable. We do not know if, in the absence of compactness, a metrizable product of a family of internally movable spaces is internally movable. In this sense, as a consequence of Theorem 1.7, we have the following

1.10. COROLLARY (compare with Theorem 3.4 in [1]). *Let X be a locally compact, metrizable space with compact components, and Y a metrizable space with (covering)* $\dim(Y) = 0$. *Then $X \times Y$ is internally movable if and only if X is internally movable.*

1.11. Remark. As we have pointed out before, there exist movable spaces which are not internally movable. On the other hand, if a metrizable space is internally movable, then it is movable in the sense of [14].

2. On internal mutations and internal shape.

2.1. DEFINITION. Let X and Y be metrizable spaces and P, Q be ANR-spaces such that $X \subset P$, $Y \subset Q$ as closed subsets. A mutation

$$\boldsymbol{f}\colon U(X, P) \to V(Y, Q)$$

is said to be *internal* provided $f(X) \subset Y$ for every $f\colon U \to V$, $f \in \boldsymbol{f}$.

2.2. EXAMPLE. (I) The identity mutation $\boldsymbol{u}_X$ is internal.

(II) If $f\colon X \to Y$ is a continuous map and $\hat{\boldsymbol{f}}\colon U(X, P) \to V(Y, Q)$ is a mutation generated by f, then $\hat{\boldsymbol{f}}$ is internal.

(III) If $\boldsymbol{f}\colon U(X, P) \to V(Y, Q)$ and $\boldsymbol{g}\colon V(Y, Q) \to W(Z, R)$ are internal mutations, then $\boldsymbol{g} \cdot \boldsymbol{f}$ is an internal mutation.

2.3. PROPOSITION (compare with Proposition 1.1 in [13]). *If a metrizable space Y is internally movable, then every mutation $\boldsymbol{f}\colon U(X, P) \to V(Y, Q)$ is homotopic to an internal mutation.*

Proof. Let

$$\boldsymbol{f}\colon U(X, P) \to V(Y, Q)$$

be an arbitrary mutation and

$$g\colon V(Y, Q) \to V(Y, Q)$$

a mutation satisfying all the conditions of (II) in Proposition 1.2. It is easy to see that $g \cdot f$ is an internal mutation homotopic to f.

Let us define now an internal shape theory for arbitrary metrizable spaces:

2.4. DEFINITION. Let X and Y be metrizable spaces. We say that X and Y *have the same internal shape* (and write $\mathrm{Ish}(X) = \mathrm{Ish}(Y)$) if there exist ANR-spaces P and Q, with $X \subset P$, $Y \subset Q$ as closed subsets, and two internal mutations

$$f\colon U(X, P) \to V(Y, Q) \quad \text{and} \quad g\colon V(Y, Q) \to U(X, P)$$

such that

$$g \cdot f \simeq u_X \quad \text{and} \quad f \cdot g \simeq u_Y.$$

If we assume only that the relation $g \cdot f \simeq u_X$ holds, then we say that X is *internally shape dominated* by Y (and write $\mathrm{Ish}(Y) \geqslant \mathrm{Ish}(X)$).

Remark. It is obvious that $\mathrm{Ish}(X) = \mathrm{Ish}(Y)$ (resp. $\mathrm{Ish}(X) \geqslant \mathrm{Ish}(Y)$) implies that

$$\mathrm{Sh}(X) = \mathrm{Sh}(Y) \quad (\text{resp. } \mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y))$$

but the converse is not true, e.g., S^1 and the Warsaw circle.

Using the same arguments as in Proposition 1.3 we have

2.5. PROPOSITION. *Let X and Y be metrizable spaces such that X is internally movable. Let us suppose that*

$$f\colon U(X, P) \to V(Y, Q), \quad g\colon V(Y, Q) \to U(X, P)$$

are mutations with $f \cdot g \simeq u_Y$. If, in addition, f is an internal mutation, then Y is internally movable.

Proposition 2.5 is in fact a generalization of Proposition 1.3. As a consequence of Proposition 2.5 we have

2.6. COROLLARY. *Let X and Y be metrizable spaces such that* $\mathrm{Ish}(X) \geqslant \mathrm{Ish}(Y)$. *If X is internally movable, then Y is also internally movable.*

2.7. THEOREM. *Let X and Y be internally movable spaces. Then the following two statements are equivalent*:

(I) $\mathrm{Ish}(X) = \mathrm{Ish}(Y)$ (*resp.* $\mathrm{Ish}(X) \geqslant \mathrm{Ish}(Y)$);

(II) $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ (*resp.* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$).

Proof. (I)$\Rightarrow$(II) is always true.

(II)$\Rightarrow$(I). Let us suppose that

$$f\colon U(X, P) \to V(Y, Q) \quad \text{and} \quad g\colon V(Y, Q) \to U(X, P)$$

are mutations such that $\boldsymbol{g}\cdot\boldsymbol{f}\simeq\boldsymbol{u}_X$ and $\boldsymbol{f}\cdot\boldsymbol{g}\simeq\boldsymbol{u}_Y$. From Proposition 2.5 it follows that there exist two internal mutations $\boldsymbol{f}'$ and $\boldsymbol{g}'$ such that $\boldsymbol{f}'\simeq\boldsymbol{f}$ and $\boldsymbol{g}'\simeq\boldsymbol{g}$. Then

$$\boldsymbol{g}'\cdot\boldsymbol{f}'\simeq\boldsymbol{u}_X \quad\text{and}\quad \boldsymbol{f}'\cdot\boldsymbol{g}'\simeq\boldsymbol{u}_Y$$

and, consequently, $\mathrm{Ish}(X)=\mathrm{Ish}(Y)$.

From Theorem 2.7 it follows that the internal shape theory introduced here is the same as shape theory in the sense of Fox if we restrict ourselves to the class of all internally movable spaces.

As we have pointed out in the Introduction, an internal shape theory for compact metric spaces has been outlined in [13]. In order to end this section, we are going to prove that the internal shape theory introduced here is just an extension to the arbitrary metrizable case of such an internal shape theory for compact metric spaces.

Let us suppose that X and Y are compact metric spaces embedded in the Hilbert cube Q and that $\boldsymbol{F}=\{f_k, X, Y\}_{Q,Q}$ is an internal fundamental sequence in the sense of [13]. Let us denote by $U(X,Q)$ and $V(Y,Q)$ the families of all open neighborhoods of X in Q and of Y in Q, respectively. Let $\boldsymbol{f}$ be the collection of all triples $f\colon U\to V$, where $U\in U(X,P)$, $V\in V(Y,Q)$ and f is a continuous map satisfying the following two conditions:

(I) $f(X)\subset Y$;

(II) $f\simeq f_k|U$ in V for almost all k.

It is easy to see that $\boldsymbol{f}$ is an internal mutation. Let us define the following:

2.8. Definition. Let X and Y be compact metric spaces embedded in the Hilbert cube Q, and $\boldsymbol{F}=\{f_k, X, Y\}_{Q,Q}$ an internal fundamental sequence. We say that an internal mutation $\boldsymbol{h}\colon U(X,P)\to V(Y,Q)$ is *associated* with the internal fundamental sequence $\boldsymbol{F}$ if for every $h\colon U\to V$, $h\in\boldsymbol{h}$, the relation $h\simeq f_k|U$ in V holds for almost all k.

As we have shown before, with every internal fundamental sequence there is associated an internal mutation.

2.9. Proposition. (I) *Let us suppose that* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *are internal mutations associated with the internal fundamental sequences* $\boldsymbol{F}$ *and* $\boldsymbol{G}$. *Then* $\boldsymbol{g}\cdot\boldsymbol{f}$ *is associated with* $\boldsymbol{G}\cdot\boldsymbol{F}$ *(when the compositions make sense).*

(II) *The identity mutation* $\boldsymbol{u}_X$ *is associated with the identity fundamental sequence* $\boldsymbol{I}_X$.

(III) *Let* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *be internal mutations associated with the internal fundamental sequences* $\boldsymbol{F}$ *and* $\boldsymbol{G}$, *respectively. Then* $\boldsymbol{f}\simeq\boldsymbol{g}$ *if and only if* $\boldsymbol{F}\simeq\boldsymbol{G}$.

(IV) *Given any internal mutation* $\boldsymbol{f}\colon U(X,Q)\to U(X,Q)$ *(where* Q *is the Hilbert cube), there is an internal fundamental sequence* $\boldsymbol{F}^*$ *from* X *to* Y *such that* $\boldsymbol{f}^*\simeq\boldsymbol{f}$, *where* $\boldsymbol{f}^*$ *is any internal mutation associated with* $\boldsymbol{F}^*$.

(I) and (II) are obvious, (III) and (IV) can be proved following step by step the proof of (4.6) and (4.7) in [6].

As a consequence of Proposition 2.9 we have

2.10. COROLLARY. *Let X and Y be compact metric spaces. Then the following two statements are equivalent:*

(I) $\mathrm{Ish}(X) = \mathrm{Ish}(Y)$ (*resp.* $\mathrm{Ish}(X) \geqslant \mathrm{Ish}(Y)$);

(II) *X is internally shape equivalent to Y in the sense of* [13] (*resp. Y is internally shape dominated by X in the sense of* [13]).

3. On internal MANR-spaces. In this section we introduce and study the notion of internal MANR-spaces.

3.1. DEFINITION. A metrizable space is said to be an *internal MANR-space* (denoted by IMANR-space) provided that for every metrizable space X' containing X as a closed subset and every ANR-space P containing X' as a closed subset there exists an internal mutation $\boldsymbol{f}\colon U(A, P) \to U(X, P)$ such that $\boldsymbol{f} \cdot \hat{\boldsymbol{j}} \simeq \boldsymbol{u}_X$, where A is a closed neighborhood of X in X' and $\hat{\boldsymbol{j}}\colon U(X, P) \to U(A, P)$ is a mutation generated by the inclusion map $j\colon X \to A$.

In order to obtain some results, we are going to give the following characterization of IMANR-spaces:

3.2. THEOREM (compare with Proposition 3.1 in [13]). *Let X be a metrizable space. Then the following two statements are equivalent:*

(I) *X is an IMANR-space;*

(II) *X is an internally movable MANR-space.*

Proof. (I)$\Rightarrow$(II). Let X be an IMANR-space and P an ANR-space containing X as a closed subset. Then there exist a closed neighborhood A of X in P and an internal mutation $\boldsymbol{f}\colon U(X, P) \to U(A, P)$ such that $\boldsymbol{f} \cdot \hat{\boldsymbol{j}} \simeq \boldsymbol{u}_X$, where $\hat{\boldsymbol{j}}$ is a mutation generated by the inclusion map $j\colon X \to A$. Let us denote by $\mathring{A}$ the interior of A in P. Then $\mathring{A}$ is an ANR-space and $X \subset \mathring{A}$ as a closed subset. Let

$$\boldsymbol{f}' = \{f|\mathring{A}\colon \mathring{A} \to V \text{ with } f \in \boldsymbol{f} \text{ and } V \in U(X, \mathring{A})\}.$$

It is clear that $\boldsymbol{f}'\colon U(\mathring{A}, \mathring{A}) \to U(X, \mathring{A})$ is an internal mutation such that $\boldsymbol{f}' \cdot \hat{\boldsymbol{j}}' \simeq \boldsymbol{u}_X$, where $\hat{\boldsymbol{j}}'$ is a mutation generated by the inclusion map $j'\colon X \to \mathring{A}$ and, consequently, $\mathrm{Ish}(\mathring{A}) \geqslant \mathrm{Ish}(X)$, and hence X is internally movable. On the other hand, let

$$V \in U(X, \mathring{A}) \quad \text{and} \quad f\colon \mathring{A} \to V,\ f \in \boldsymbol{f};$$

it is obvious that $f|X \simeq 1_X$ (the identity map) in V. From the homotopy extension theorem it follows that there exists a continuous map $r_f\colon \mathring{A} \to V$ such that $r_f(x) = x$ for every $x \in X$ and $r_f \simeq f$ in V. It is easy to see that the family

$$\boldsymbol{r} = \{r_f\colon \mathring{A} \to V, \text{ where } f\colon \mathring{A} \to V,\ f \in \boldsymbol{f} \text{ and } V \in U(X, \mathring{A})\}$$

is a mutational retraction, and then (see Theorem (4.11) in [8]) X is an MANR-space.

(II)$\Rightarrow$(I). Let X be an internally movable MANR-space and X' a metrizable space containing X as a closed subset. Let us suppose that P is an ANR-space with $X' \subset P$ as a closed subset. Since X is an MANR-space, there exist a closed neighborhood A of X in X' and a mutational retraction $\boldsymbol{r}$: $U(A, P) \to U(X, P)$. On the other hand, from the internal movability of X it follows (from Proposition 2.3) that there exists an internal mutation $\boldsymbol{f}$: $U(A, P) \to U(X, P)$ such that $\boldsymbol{f} \simeq \boldsymbol{r}$. Since $\boldsymbol{r}\cdot\hat{\boldsymbol{j}} \simeq \boldsymbol{u}_X$ (where $\hat{\boldsymbol{j}}$ is a mutation generated by the inclusion map j: $X \to A$), it follows that $\boldsymbol{f}\cdot\hat{\boldsymbol{j}} \simeq \boldsymbol{u}_X$ and, consequently, X is an IMANR-space.

As consequences of the last theorem we have

3.3. COROLLARY. *Every metrizable space internally shape dominated by an IMANR-space is an IMANR-space.*

The proof of Corollary 3.3 follows from Theorem 3.2, Corollary 2.6 in this paper and from the fact, proved by Godlewski in [9], that a space shape dominated by an MANR-space is an MANR-space.

3.4. COROLLARY. *Let X be a compact metrizable space. Then the following two statements are equivalent:*

(I) *X is an IMANR-space;*

(II) *X is an internal FANR-space in the sense of* [13].

Corollary 3.4 is a consequence of Theorem 3.2 in this paper, Proposition 3.1 in [13] and of the fact that the compact MANR-spaces are just the FANR-spaces (see [8]).

As we know, every homotopy class of mutations from a metrizable space to an ANR-space is generated by a continuous map. We are going to prove now that the IMANR-spaces have also such a property.

3.5. PROPOSITION. *Let $\boldsymbol{f}$: $U(X, P) \to V(Y, Q)$ be a mutation. If Y is an IMANR-space, then there exists a continuous map f: $X \to Y$ such that $\boldsymbol{f} \simeq \hat{\boldsymbol{f}}$, where $\hat{\boldsymbol{f}}$ is a mutation generated by the map f.*

Proof. Since Y is an internally movable space, there exists an internal mutation $\boldsymbol{h}$: $U(X, P) \to V(Y, Q)$ such that $\boldsymbol{f} \simeq \boldsymbol{h}$. On the other hand, there exist $W_0 \in V(Y, Q)$ and a mutational retraction

$$\boldsymbol{r}\colon\ V(W_0, W_0) \to V(Y, W_0).$$

Let now h_0: $U_0 \to W_0$, $h_0 \in \boldsymbol{h}$ and $V \in V(Y, Q)$. There exist

$$W \in V(Y, W_0) \text{ with } W \subset W_0 \cap V \quad \text{and} \quad r\colon W_0 \to W,\ r \in \boldsymbol{r}.$$

Let

$$\boldsymbol{g} = \{r\cdot h_0 \cdot i_{U_0,U'}\colon\ U' \to V, \text{ where } V \in V(Y, Q),\ r\colon\ W_0 \to W,\ r \in \boldsymbol{r},$$
$$\text{with } W \subset W_0 \cap V \text{ and } i_{U_0,U'}\colon\ U' \to U_0 \text{ is the inclusion map}\}.$$

It is clear that $\boldsymbol{g}$ is a mutation; moreover, for every $g\colon U \to V$, $g \in \boldsymbol{g}$, we have $g(x) = h_0(x)$ for every $x \in X$. Let us denote by $\hat{\boldsymbol{h}}_0$ a mutation generated by the continuous map $h_0|X\colon X \to Y$. It is obvious that $\boldsymbol{g} \simeq \hat{\boldsymbol{h}}_0$. In order to complete the proof we are going to see that $\boldsymbol{g} \simeq \boldsymbol{h}$.

Let $g\colon U \to V$, $g \in \boldsymbol{g}$. Then, by the construction of $\boldsymbol{g}$, we have $U \subset U_0$ and there exist

$$W \in V(Y, W_0) \text{ with } W \subset W_0 \cap V \quad \text{and} \quad r\colon W_0 \to W,\ r \in \boldsymbol{r},$$

such that $g = r \cdot h_0$. Let now $h\colon U_1 \to W$, $h_1 \in \boldsymbol{h}$. Then $h\colon U_1 \to V$ belongs to $\boldsymbol{h}$. If $U_2 = U_1 \cap U$, then

$$g|U_2\colon U_2 \to V \in \boldsymbol{g} \quad \text{and} \quad h|U_2\colon U_2 \to V \in \boldsymbol{h}.$$

On the other hand,

$$h_0|U_2\colon U_2 \to W_0 \in \boldsymbol{h} \quad \text{and} \quad h|U_2\colon U_2 \to W_0 \in \boldsymbol{h},$$

and then there exist $U_3 \in U(X, P)$ with $U_3 \subset U_2$ and a homotopy $H\colon U_3 \times I \to W_0$ such that $H(x, 0) = h_0(x)$ and $H(x, 1) = h(x)$ for every $x \in U_3$. Consequently,

$$r \cdot H\colon U_3 \times I \to V$$

is a homotopy with

$$r \cdot H(x, 0) = r \cdot h_0(x) = g(x), \quad r \cdot H(x, 1) = r \cdot h(x) \quad \text{for every } x \in U_3.$$

On the other hand, $r \cdot h(x) = h(x)$ for every $x \in X$; then $g|X \simeq h|X$ in V and, consequently, there exists $U_4 \in U(X, P)$ with $U_4 \subset U_3$ such that $g|U_4 \simeq h|U_4$ in V. This fact clearly implies that $\boldsymbol{g} \simeq \boldsymbol{h}$ and the proof is complete.

The last result is implicitly contained in [4] (see also Proposition 3.3 in [13] for the compact case and fundamental sequences).

We are going to deal now with relations between the concept of IMANR-space and the shape domination. More precisely, we shall give a shape representation theorem for spaces shape dominated by an IMANR-space. Such a result says almost that a metrizable space shape dominated by an IMANR-space Y can be (topologically) embedded as a mutational retract of a product $Y \times P$, where P is an AR-space. But there is a hedging clause arising from the unprovability in ZFC of the existence of measurable cardinals. In particular, we have

3.6. THEOREM. *Let Y be an internal MANR-space and X a real-compact metrizable space such that* $\mathrm{Sh}(Y) \geqslant \mathrm{Sh}(X)$. *Then there exist an AR-space P and a homeomorphism $m\colon X \to Y \times P$ onto a closed subset such that $m(X)$ is a mutational retract of $Y \times P$.*

Proof. By the Kuratowski–Wojdysławski theorem, there exist an AR-space P and a homeomorphism $h\colon X \to P$ onto a closed subset of P. Since $Y \times P$ and Y have the same homotopy type, we have $\mathrm{Sh}(Y \times P) \geqslant \mathrm{Sh}(X)$ and

$Y \times P$ is an internally movable MANR-space. Let S and T be two ANR-spaces such that $Y \subset S$ and $X \subset T$ as closed subsets. Let $\hat{S} = S \times P$. Then there exist two mutations

$$\boldsymbol{f}\colon U(X, T) \to V(Y \times P, \hat{S}) \quad \text{and} \quad \boldsymbol{g}\colon V(Y \times P, \hat{S}) \to U(X, T)$$

such that $\boldsymbol{g} \cdot \boldsymbol{f} \simeq \boldsymbol{u}_{X,T}$. From Proposition 3.5 it follows that there exists a continuous map $f'\colon X \to Y \times P$ such that $\hat{\boldsymbol{f}}' \simeq \boldsymbol{f}$ (where $\hat{\boldsymbol{f}}'$ is a mutation generated by f'). Let

$$p_1\colon Y \times P \to Y \quad \text{and} \quad p_2\colon Y \times P \to P$$

be the projections and $f'_1 = p_1 \cdot f'$, $f'_2 = p_2 \cdot f'$. Let us define now $m\colon X \to Y \times P$ as follows:

$$m(x) = (f'_1(x), h(x)).$$

Obviously, m is a continuous map, and since P is an AR-space, m is homotopic to f'. Let

$$\hat{\boldsymbol{M}}\colon U(X, T) \to V(Y \times P, \hat{S})$$

be a mutation generated by m; then $\hat{\boldsymbol{M}} \simeq \hat{\boldsymbol{f}}' \simeq \boldsymbol{f}$.

Let $Z = m(X)$. We are going to prove that $m\colon X \to Z$ is a homeomorphism and that Z is a closed subset of $Y \times P$. Since $h\colon X \to P$ is a homeomorphism onto a closed subset of P, it follows that $p_2|Z\colon Z \to h(X)$ is a homeomorphism because $p_2|Z$ is a continuous bijective open map. Let

$$m^{-1} = h^{-1} \cdot (p_2|Z);$$

$m^{-1}\colon Z \to X$ is the inverse map of m.

In order to prove that Z is closed in $Y \times P$ we need the real-compactness of X. As usual, R is the real line. Let $k\colon Z \to R$ be a continuous map; then

$$k \cdot (p_2|Z)^{-1}\colon h(X) \to R$$

is continuous. Since $h(X)$ is closed in P, there exists a continuous extension $\bar{k}\colon P \to R$ of $k \cdot (p_2|Z)^{-1}$. Finally, let $\hat{k}\colon Y \times P \to R$ be the map $\hat{k} = \bar{k} \cdot p_2$. If $z \in Z$, then

$$\hat{k}(z) = \bar{k}(p_2(z)) = k \cdot (p_2|Z)^{-1}(p_2(z)) = k(z).$$

Then $\hat{k}$ is a continuous extension of k to $Y \times P$. In particular, we have proved that every continuous map from Z to R can be continuously extended to the closure $\bar{Z}$ of Z in $Y \times P$, and since Z is real-compact (because X is real-compact), we have (see [5], pp. 271–272) $\bar{Z} = Z$, and hence Z is closed in $Y \times P$.

Let us denote by

$$\hat{\boldsymbol{m}}\colon U(X, T) \to V(Z, \hat{S})$$

a mutation generated by m, and by

$$\hat{\boldsymbol{m}}^{-1}\colon V(Z, \hat{S}) \to U(X, T)$$

a mutation generated by m^{-1}. Let us recall that $\hat{\boldsymbol{M}} \simeq \hat{\boldsymbol{f}}' \simeq \boldsymbol{f}$, and then

$$\boldsymbol{g}\cdot\hat{\boldsymbol{M}} \simeq \boldsymbol{u}_{X,T}.$$

On the other hand, $\hat{\boldsymbol{m}}\cdot\hat{\boldsymbol{m}}^{-1} \simeq \boldsymbol{u}_{Z,\hat{S}}$. Let us denote by $\boldsymbol{g}'$ the mutation $\boldsymbol{g}$ considered as a mutation from $V(Z, \hat{S})$ to $U(X, T)$; $\boldsymbol{g}$ is then an extension of $\boldsymbol{g}'$ (see [8], p. 50). Since $\boldsymbol{g}\cdot\hat{\boldsymbol{M}} \simeq \boldsymbol{u}_{X,T}$, it follows that

$$\boldsymbol{u}_{Z,\hat{S}} \simeq \hat{\boldsymbol{m}}\cdot\hat{\boldsymbol{m}}^{-1} \simeq \hat{\boldsymbol{m}}\cdot\boldsymbol{g}\cdot\hat{\boldsymbol{M}}\cdot\hat{\boldsymbol{m}}^{-1}.$$

Let now $m_1 \in \hat{\boldsymbol{m}}$, $g \in \boldsymbol{g}$, $M_1 \in \hat{\boldsymbol{M}}$ and $m'_1 \in \hat{\boldsymbol{m}}^{-1}$ be such that the composition $m_1\cdot g\cdot M_1\cdot m'_1$ makes sense and let $z \in Z$. We have

$$m_1\cdot g\cdot M_1\cdot m'_1(z) = m_1(g(z)),$$

and then

$$\hat{\boldsymbol{m}}\cdot\boldsymbol{g}\cdot\hat{\boldsymbol{M}}\cdot\hat{\boldsymbol{m}}^{-1} \simeq \hat{\boldsymbol{m}}\cdot\boldsymbol{g}';$$

consequently, $\boldsymbol{u}_{Z,\hat{S}} \simeq \hat{\boldsymbol{m}}\cdot\boldsymbol{g}'$. On the other hand, since $\boldsymbol{g}$ is an extension of $\boldsymbol{g}'$, we infer that

$$\hat{\boldsymbol{m}}\cdot\boldsymbol{g}\colon V(Y\times P, \hat{S}) \to V(Z, \hat{S})$$

is an extension of $\hat{\boldsymbol{m}}\cdot\boldsymbol{g}'$, and from the homotopy extension theorem for mutations (due to Godlewski in [9]) it follows that there exists a mutation

$$\boldsymbol{r}\colon V(Y\times P, \hat{S}) \to V(Z, \hat{S})$$

which is an extension of $\boldsymbol{u}_{Z,\hat{S}}$. Consequently, $\boldsymbol{r}$ is a mutational retraction and the proof is complete.

3.7. Remark. (I) As we know, a metrizable space with nonmeasurable cardinal is real-compact (see [7], p. 232). On the other hand, the problem of the existence of measurable cardinals is unprovable in ZFC.

(II) Theorem 3.6 is not true if we assume only that Y is an MANR-space, even in the compact case. For example, let Y be the Warsaw circle and X the 1-dimensional sphere. As we know, $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. Let us suppose that P is an AR-space such that X can be (topologically) embedded as a mutational retract of $Y\times P$. Let us consider the projection $p_1\colon Y\times P \to Y$. As we know, $p_1(X)$ must be an arc in Y and, consequently, $p_1(X)\times P$ has a trivial shape. On the other hand, $X \subset p_1(X)\times P$ must be a mutational retract of $p_1(X)\times P$, and this is not possible because X does not have trivial shape.

As we have pointed out in the proof of Theorem 3.6, the AR-space P can be chosen with the only condition that X can be embedded as a closed subset of P. We derive the following consequences:

3.8. COROLLARY. *Let Y be an internally movable MANR-space. Then a compact metrizable space X is shape dominated by Y if and only if X can be (topologically) embedded as a mutational retract of $Y\times Q$, where Q is the Hilbert cube.*

3.9. Remark. If in Corollary 3.8 we suppose that Y is compact, then we obtain the main result of Borsuk and Olędzki in [3].

3.10. COROLLARY. *Let Y be an internally movable $MANR$-space. Then a locally compact separable metrizable space X with (covering)* $\dim(X) = n$ *is shape dominated by Y if and only if X can be embedded as a mutational retract of* $Y \times R^{2n+1}$.

The proof of Corollary 3.10 is directly obtained from Theorem 3.6 in this paper and Corollary 21 and Lemma 22 in [12].

Following Hyman's notation in [11], if $X \subset Y$, then

$$Z(X, Y) = Y \times [0, 1] - ((Y - X) \times \{0\}) = X \times \{0\} \cup Y \times (0, 1].$$

Hyman has proved in [11], Lemma 4, that if Y is an ANR-space and $X \subset Y$, then $Z(X, Y)$ is an ANR-space and X is homeomorphic to a closed subset of $Z(X, Y)$. Using the same arguments as Hyman we have

3.11. COROLLARY. *Let Y be an internally movable $MANR$-space. Then a separable metrizable space X is shape dominated by Y if and only if there exists $A \subset Q$ (where Q is the Hilbert cube) such that X is homeomorphic to a mutational retract of $Y \times Z(A, Q)$.*

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