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ORDERED TOPOLOGICAL SPACES AND THE COPRODUCT OF BOUNDED DISTRIBUTIVE LATTICES

BY

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We show that the coproduct of two bounded distributive lattices L_1 and L_2 is isomorphic to the lattice of all continuous monotonic functions from the Priestley space of L_1 to the discrete space L_2. Though this is an asymmetrical representation of the coproduct, it leads to simple new proofs of the characterizations of the centre and minimal Boolean extension of a coproduct. The representation generalizes Speed's description of the coproduct of a Boolean lattice and a bounded distributive lattice. It also gives an exact topological description of Quackenbush's representation $L_1[L_2]$ of the coproduct of bounded distributive lattices L_1 and L_2.

1. Preliminaries. An *ordered topological space* is a triple $(X, \leqslant, T)$ such that $(X, \leqslant)$ is a partially ordered set and (X, T) is a topological space. When there is no ambiguity, we speak of the underlying set X as the ordered (topological) space. A subset U of such a space X is called *increasing* if $y \in U$ whenever $x \leqslant y$ for some x in U. $\mathscr{D}(X)$ denotes the set of all clopen increasing subsets of X; under set-theoretic operations $\mathscr{D}(X)$ is a bounded distributive lattice. An ordered space X is said to be *totally ordered-disconnected* if x non $\leqslant y$ $(x, y \in X)$ implies $x \in U$ and $y \in X \setminus U$ for some $U \in \mathscr{D}(X)$. In a totally order-disconnected compact ordered space X, the topology is that in which the members of $\mathscr{D}(X)$ and $\mathscr{D}(X)' = \{X \setminus U : U \in \mathscr{D}(X)\}$ form a subbase for the open sets (cf. [13]).

If X and Y are ordered spaces, then $\mathscr{C}(X, Y)$ denotes the set of all continuous functions from X into Y; $\overline{\mathscr{C}}(X, Y)$ denotes the subset of $\mathscr{C}(X, Y)$ consisting of all functions whose ranges are finite; $\mathscr{C}_m(X, Y)$ denotes the subset of all monotonic functions in $\mathscr{C}(X, Y)$, where $f: X \to Y$ is *monotonic* if $x_1 \leqslant x_2$ $(x_1, x_2 \in X)$ implies $f(x_1) \leqslant f(x_2)$; $\overline{\mathscr{C}}_m(X, Y)$ is the set $\mathscr{C}_m(X, Y) \cap \overline{\mathscr{C}}(X, Y)$. A bounded distributive lattice L can be converted into an ordered space by endowing it with the discrete topology. Thus, if X is an ordered topological space and L is a bounded distributive lattice, then we can meaningfully consider $\mathscr{C}(X, L)$, $\overline{\mathscr{C}}_m(X, L)$ etc. If

$U \in \mathscr{D}(X)$ and $d \in L$, then we use the following notation: $[d) = \{a \in L: d \leqslant a\}$; $\chi(U)$ is the characteristic function of U, i.e. $\chi(U)(x) = 1$ if $x \in U$ and $\chi(U)(x) = 0$ if $x \in X \setminus U$; $c(d)$ is the constant function which assumes the value d at each point of X. Clearly, both $\chi(U)$ and $c(d)$ are members of $\overline{\mathscr{C}}_m(X, L)$. It readily follows that $\overline{\mathscr{C}}_m(X, L)$ is a bounded distributive lattice under pointwise-defined operations; for $f \in \mathscr{C}_m(X, L)$, we have $f = \bigvee \{c(d) \wedge \chi(f^{\leftarrow}([d))): d \in f(X)\}$.

$\mathbf{Dist}_{0,1}$ denotes the category of bounded distributive lattices and $(0, 1)$-homomorphisms, i.e. lattice homomorphisms f such that $f(0) = 0$ and $f(1) = 1$; **Spec** denotes the category of spectral spaces and strongly continuous functions; **Todc** denotes the category of totally order-disconnected compact ordered spaces and monotonic continuous functions. If L is an object in $\mathbf{Dist}_{0,1}$ and $\Sigma(L)$ denotes the set of all prime ideals of L endowed with the spectral (= hull-kernel = Stone) topology, while $\Delta(L)$ denotes the set of all prime filters of L endowed with the dual spectral (= dual hull-kernel = dual Stone) topology, then we obtain naturally equivalent equivalences $\Sigma, \Delta: \mathbf{Dist}_{0,1} \to \mathbf{Spec}^{\mathrm{op}}$. This is the classical duality of M. H. Stone (cf. [9], Section 11). Recently, Priestley [13] has shown that **Todc** is the dual or opposite category of $\mathbf{Dist}_{0,1}$. Thus, Priestley represents a bounded distributive lattice L as the lattice $\mathscr{D}(X)$ of a certain object X in **Todc**, which is unique up to an order-isomorphism and homeomorphism — the ordered space is denoted by $\mathrm{Pr}(L)$ and is called the *Priestley space* of L. In [7], the author showed that there is an isomorphism Ω: **Spec** $\to$ **Todc**. For an object X in **Spec**, the order on $\Omega(X)$ is the topological order: $x \leqslant y$ ($x, y \in X = \Omega X$) if and only if x is in the closure (with respect to the spectral topology) of $\{y\}$, while the topology is that which has the compact-open subsets of X, together with their complements, as a subbase. This means that Priestley's duality follows by composing the dual Ω^{op} of Ω with either of the classical functors Σ and Δ. Most importantly, it means for the purposes of this paper that we can take the space $\mathrm{Pr}(L)$ as either $\Omega\Sigma(L)$ or $\Omega\Delta(L)$. As partially ordered sets, the first is the set of prime ideals with a partial order which is the converse of set-theoretic inclusion, while the second is the set of prime filters ordered by set-inclusion.

2. Coproducts. Because the free bounded distributive lattice with one free generator exists, monics in $\mathbf{Dist}_{0,1}$ are one-to-one maps ([9], Lemma 3, p. 141). Hence, the $\mathbf{Dist}_{0,1}$-coproduct $L_1 \amalg L_2$ of two bounded distributive lattices L_1 and L_2 can be regarded as the bounded distributive lattice which is generated by the sublattices L_1 and L_2 and universal with respect to simultaneously extending $(0, 1)$-homomorphisms of L_1 and L_2 into another bounded distributive lattice. We need the following fundamental characterization of $L_1 \amalg L_2$:

A bounded distributive lattice L containing L_1 and L_2 as $(0, 1)$*-sublattice is isomorphic to $L_1 \amalg L_2$ if and only if it is generated by L_1 and L_2 and for any $l_1, m_1 \in L_1$ and $l_2, m_2 \in L_2$, $l_1 \wedge l_2 \leqslant m_1 \vee m_2$ implies that either $l_1 \leqslant m_1$ or $l_2 \leqslant m_2$.*

This characterization was stated without proof in [17]; Rousseau attributed it to W. Holsztyński. An extension to arbitrary $\mathbf{Dist}_{0,1}$ coproducts is proved, without the Axiom of Choice in Grätzer and Lakser's paper [10]; in his book ([9], Theorem 5, p. 131) Grätzer presents a proof of their result using prime ideals. It is perhaps worth noting that all proofs assume the existence of the coproduct.

Before proceeding to our results it should be mentioned that the existence and representation of $\mathbf{Dist}_{0,1}$-coproducts have been much studied. See Nerode [12], Theorem 2.3, and Grätzer [9], Section 12 (especially Lemma 3), for discussions dependent on the Axiom of Choice, and Chen [5] and Grätzer and Lakser [10] for choice-free constructions. Coproducts were first studied by Sikorski [16]. Coproducts of distributive lattices have also been studied because of their connection with Post algebras. The connection was first noticed by Rousseau [15] and subsequently exploited by others, e.g. Balbes and Dwinger [2], and Speed's paper [17] contains a comprehensive bibliography on Post algebras.

2.1. THEOREM. *Let X be an ordered topological space and L be a bounded distributive lattice. Then $\overline{\mathscr{C}}_m(X, L)$ is isomorphic to $\mathscr{D}(X) \amalg L$.*

Proof. Clearly, $\mathscr{D}(X)$ is isomorphic to the $(0, 1)$-sublattice $\{\chi(U): U \in \mathscr{D}(X)\}$ of $\overline{\mathscr{C}}_m(X, L)$, and L is isomorphic to the $(0, 1)$-sublattice $\{c(d): d \in L\}$. Let $f \in \overline{\mathscr{C}}_m(X, L)$. Then the range $f(X)$ of f is finite and for each $d \in f(X)$, $f^{\leftarrow}([d]) = \{x \in X: f(x) \in [d)\}$ is a non-empty clopen increasing subset of X. It is evident that

$$f = \bigvee \{c(d) \wedge \chi(f^{\leftarrow}([d))): d \in f(X)\}.$$

Hence, these sublattices generate $\overline{\mathscr{C}}_m(X, L)$.

Let us assume that $d_1, d_2 \in L$ and $U_1, U_2 \in \mathscr{D}(X)$. Suppose that $c(d_1) \wedge \chi(U_1) \leqslant c(d_2) \vee \chi(U_2)$. If $U_1 \nsubseteq U_2$, there is a point $y \in U_1 \setminus U_2$ and so

$$d_1 = d_1 \wedge 1 = (c(d_1) \wedge \chi(U_1))(y) \leqslant (c(d_2) \vee \chi(U_2))(y) = d_2 \vee 0 = d_2.$$

Hence $\overline{\mathscr{C}}_m(X, L)$ is isomorphic to $\mathscr{D}(X) \amalg L$.

2.2. COROLLARY. *Let X be a topological space and let $\mathscr{B}(X)$ denote its Boolean lattice of clopen subsets. Then, for any distributive lattice L, $\overline{\mathscr{C}}(X, L)$ is isomorphic to $\mathscr{B}(X) \amalg L$.*

Proof. Partially order X by using the equality relation. X can then be considered as an ordered topological space.

2.3. COROLLARY. *Let L_1 and L_2 be two bounded distributive lattices. Then $L_1 \amalg L_2$ is isomorphic to $\mathscr{C}_m(\Pr(L_1), L_2)$.*

Proof. $\Pr(L_1)$ is compact and L_1 is isomorphic to $\mathscr{D}(\Pr(L_1))$.

2.4. COROLLARY. *Let L_1 and L_2 be two bounded distributive lattices. Then $L_1 \amalg L_2$ is isomorphic to each of $\mathscr{C}_m(\Omega\Sigma(L_1), L_2)$ and $\mathscr{C}_m(\Omega\Delta(L_1), L_2)$.*

Proof. These are specializations of 2.3.

2.5. COROLLARY. *For any two bounded distributive lattices L_1 and L_2, $\mathscr{C}_m(\Pr(L_1), L_2)$ and $\mathscr{C}_m(\Pr(L_2), L_1)$ are isomorphic.*

Proof. $L_1 \amalg L_2$ is isomorphic to $L_2 \amalg L_1$.

2.6. COROLLARY. *Let L be a bounded distributive lattice and X be an ordered topological space. Then $\overline{\mathscr{C}}_m(X, L)$ is isomorphic to $\mathscr{C}_m(\Pr(\mathscr{D}(X)), L)$ and $\overline{\mathscr{C}}(X, L)$ is isomorphic to $\mathscr{C}(\Sigma(\mathscr{B}(X)), L)$.*

Proof. For a Boolean lattice B, $\Omega\Sigma(B) = \Sigma(B)$ since $\Sigma(B)$ is compact and totally disconnected and all primes in a Boolean algebra are maximal and so the topological order on $\Sigma(B)$ is the equality relation. The corollary now follows from 2.1, 2.2 and 2.3.

In [17], Speed proved that the coproduct of a Boolean lattice B and a bounded distributive lattice L is isomorphic to $\mathscr{C}(\Sigma(B), L)$. Theorem 2.1 and its corollaries can be regarded as generalization of this.

Let L_1 and L_2 be bounded distributive lattices. $\mathscr{F}(L_2)$ is the set of all prime filters of L_2, partially ordered by set-inclusion. Let $L_1[L_2]$ denote the $(0, 1)$-sublattice of the distributive lattice of all monotonic functions mapping $\mathscr{F}(L_2)$ into L_1 which is generated by $\{c(a): a \in L_1\} \cup \cup \{\chi(h(b)): b \in L_2\}$, where $h(b) = \{P \in \mathscr{F}(L_2): b \in P\}$. Quackenbush [14] showed that $L_1[L_2]$ is isomorphic to $L_1 \amalg L_2$. Corollary 2.4 can be viewed as a supplement to this. For, it is clear that $L_1[L_2]$ is actually equal to $\mathscr{C}_m(\Omega\Delta(L_2), L_1)$ and hence we not only have a precise description of $L_1[L_2]$, but also an explanation of Quackenbush's results.

If B_1 and B_2 are Boolean lattices, then their coproduct is isomorphic to $\mathscr{C}(\Sigma(B_1), B_2)$ and is thus a Boolean lattice. Thus, we obtain the known fact that two Boolean lattices B_1 and B_2 have the same coproduct in the subcategory **Bool** of Boolean algebras and Boolean homomorphisms as in $\mathbf{Dist}_{0,1}$. We now consider two known results concerning Boolean algebras in the setting of Theorem 2.1.

Let L be a bounded distributive lattice. $\mathscr{Z}(L)$ denotes the centre of L; $\mathscr{Z}(L)$ is the sublattice of L consisting of all complemented elements. Of course, it is a Boolean lattice. $B(L)$ denotes the minimal Boolean extension of L. For an ordered topological space X we use $\mathscr{E}(X)$ to denote $\{U: U \subseteq X;\ U, X \setminus U \in \mathscr{D}(X)\}$. Of course, $\mathscr{E}(X)$ is a Boolean sublattice of $\mathscr{D}(X)$ and, in fact, $\mathscr{E}(X) = \mathscr{Z}(\mathscr{D}(X))$.

2.7. THEOREM. *Let L be a bounded distributive lattice and X be an ordered topological space. Then $\mathscr{Z}(\overline{\mathscr{C}}_m(X, L))$ is isomorphic to $\mathscr{E}(X) \amalg \mathscr{Z}(L)$.*

Proof. Let $f \in \mathscr{Z}(\overline{\mathscr{C}}_m(X, L))$ and let g be the complement of f. Suppose $d \in L$ is in the range of f. Then $d = f(x)$ for some $x \in X$ and so $d' = g(x)$ is the complement of d in L. In addition, $f^{\leftarrow}([d)) = g^{\leftarrow}((d'])$, where $(d'] = \{a \in L\colon\ a \leqslant d'\}$. Hence $f^{\leftarrow}([d)) \in \mathscr{E}(X)$. Since $f = \bigvee \{c(d) \wedge \chi(f^{\leftarrow}[d))\colon d \in f(X)\}$, we can infer that $\mathscr{Z}(\overline{\mathscr{C}}_m(X, L))$ is contained within the $(0, 1)$-sublattice of $\overline{\mathscr{C}}_m(X, L)$ which is generated by $\chi(\mathscr{E}(X)) \cup c(\mathscr{Z}(L))$. But the reverse inclusion is obvious and so $\mathscr{Z}(\overline{\mathscr{C}}_m(X, L))$ is the $(0, 1)$-sublattice generated by $\chi(\mathscr{E}(X)) \cup c(\mathscr{Z}(L))$. The fundamental characterization of $\mathbf{Dist}_{0,1}$-coproducts implies that $\mathscr{Z}(\overline{\mathscr{C}}_m(X, L))$ is isomorphic to $\mathscr{E}(X) \amalg \mathscr{Z}(L)$.

2.8. COROLLARY (Balbes). *For any two bounded distributive lattices L_1 and L_2, $\mathscr{Z}(L_1 \underline{\amalg} L_2)$ is isomorphic to $\mathscr{Z}(L_1) \amalg \mathscr{Z}(L_2)$.*

Corollary 2.8 was established without using the Axiom of Choice by Balbes in [1]; his proof is highly computational. For another proof see Blok [4].

2.9. THEOREM. *Let L be a bounded distributive lattice and X be a totally order-disconnected compact ordered space. Then $B(\mathscr{C}_m(X, L))$ is isomorphic to $\mathscr{C}(X, B(L))$.*

Proof. Identify L with a $(0, 1)$-sublattice of $B(L)$. Then $\mathscr{C}_m(X, L)$ is a $(0, 1)$-sublattice of the Boolean algebra $\mathscr{C}(X, B(L))$. To establish the theorem it is sufficient to show that each element of $\mathscr{C}(X, B(L))$ is a finite supremum of elements or the complements of elements in $\mathscr{C}_m(X, L)$.

Let $f \in \mathscr{C}(X, B(L))$. As X is compact, $f(X)$ is finite and

$$f = \bigvee \{c(a) \wedge \chi(f^{\leftarrow}(\{a\}))\colon\ a \in f(X)\}.$$

Now, for any $a \in f(X)$, there are $d_1, \ldots, d_n, e_1, \ldots, e_n \in L$ such that

$$a = \bigvee_{i=1}^{n} d_i \wedge e_i'.$$

Clearly,

$$c(a) = \bigvee_{i=1}^{n} c(d_i) \wedge c(e_i)'.$$

Since $f^{\leftarrow}(\{a\})$ is a clopen subset of totally order-disconnected compact X, there exist $U_1, U_2, \ldots, U_m, V_1, V_2, \ldots, V_m \in \mathscr{D}(X)$ such that

$$f^{\leftarrow}(\{a\}) = \bigcup_{j=1}^{m} U_j \cap (X \setminus V_j).$$

Hence,

$$\chi(f^{\leftarrow}(\{a\})) = \bigvee_{j=1}^{m} \chi(U_j) \wedge \chi(V_j)'.$$

The conclusion is now immediate.

2.10. COROLLARY (Nerode). *For bounded distributive lattices L_1 and L_2, $B(L_1 \amalg L_2)$ is isomorphic to $B(L_1) \amalg B(L_2)$.*

Proof. $\Pr(L)$ is homeomorphic to $\Sigma(B(L))$.

From the duality proof of the existence of $L_1 \amalg L_2$, it is obvious that $\Pr(L_1 \amalg L_2)$ is order-isomorphic and homeomorphic to $\Pr(L_1) \times \Pr(L_2)$. However, this does not show how the prime ideals of $\mathscr{C}_m(\Pr(L_1), L_2)$ are "co-ordinatized" in $\mathscr{C}_m(\Pr(L_1), L_2)$. We now consider this question. The proof of our answer is longer than the previous proofs, for we are paying the price for our asymmetrical representation of $L_1 \amalg L_2$. Our method is along the lines of a procedure used by Kaplansky [11] in a different situation.

Let X be a totally order-disconnected compact ordered space and L be a bounded distributive lattice. A prime ideal P in $\mathscr{C}_m(X, L)$ is said to be *associated with point* $x \in L$ if whenever $f \in P$ and $g \in \mathscr{C}_m(X, L)$, $g(x) \leqslant f(x)$ implies $g \in P$.

A prime P can be associated with at most one point in X. For, suppose P is associated with two distinct points x and y. Choose $f \in P$ and $g \in \mathscr{C}_m(X, L) \setminus P$. Without loss of generality we may assume that $x \leqslant y$. Since X is totally order-disconnected, there exists $U \in \mathscr{D}(X)$ such that $x \in U$ and $y \in X \setminus U$. Define $h\colon X \to L$ by $h(z) = f(x)$ for each $z \in U$ and $h(z) = g(y)$ for each $z \in X \setminus U$. Then $h \in \mathscr{C}_m(X, L)$, $h(x) \leqslant f(x)$, so $h \in P$ and yet $g(y) \leqslant h(y)$, while $g \notin P$. This contradiction proves our assertion.

A prime P is associated with at least one point in X. Otherwise, for each $x \in X$ there must exist $f_x \in P$, $g_x \in \mathscr{C}_m(X, L) \setminus P$ such that $g_x(x) \leqslant f_x(x)$. Define $\langle f_x, g_x \rangle$ by

$$\langle f_x, g_x \rangle = \{y \in X\colon\ g_x(y) \leqslant f_x(y)\}.$$

Then $\langle f_x, g_x \rangle = \bigcup \{g_x^{\leftarrow}(\{a\}) \cap (f_x \wedge g_x)^{\leftarrow}(\{a\})\colon a \in L\}$ and so $\{\langle f_x, g_x \rangle\colon x \in X\}$ is an open cover of the compact space X. It follows that there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} \langle f_{x_i}, g_{x_i} \rangle = \Big\langle \bigvee_{i=1}^{n} f_{x_i}, \bigwedge_{i=1}^{n} g_{x_i} \Big\rangle.$$

Hence,

$$\bigwedge_{i=1}^{n} g_{x_i} \leqslant \bigvee_{i=1}^{n} f_{x_i}, \qquad \bigvee_{i=1}^{n} f_{x_i} \in P$$

and yet none of $g_{x_1}, \ldots, g_{x_n}$ are in prime ideal P. This contradiction yields the claim.

Thus a prime P is associated with one and only one point of X. Denote this point by $X(P)$ and define $L(P)$ by $L(P) = \{a \in L\colon f(X(P)) = a$ for some $f \in P\}$. It is easy to see that $L(P) = \{a \in L\colon c(a) \in P\} = c^{\leftarrow}(P \cap c(L))$ and so $L(P)$ is a prime ideal in L.

2.11. THEOREM. *Let X be a totally order-disconnected compact ordered space and L be a bounded distributive lattice. Then the map $f: P \to \big(X(P), L(P)\big)$ is an order-isomorphism and homeomorphism of $\Omega\Sigma\big(\mathscr{C}_m(X, L)\big)$ onto $X \times \Omega\Sigma(L)$.*

Proof. Suppose P_1 and P_2 are prime ideals in $\mathscr{C}_m(X, L)$. Assume that $P_1 \leqslant P_2$ in the order of $\Omega\Sigma\big(\mathscr{C}_m(X, L)\big)$. Hence $P_2 \subseteq P_1$. If $X(P_1) \nleqslant X(P_2)$, then there exists $U \in \mathscr{D}(X)$ such that $X(P_1) \in U$ and $X(P_2) \in X \setminus U$. Then $\chi(U)\big(X(P_2)\big) = 0 \leqslant g\big(X(P_2)\big)$ for any $g \in P_2$. Hence, $\chi(U) \in P_2$, and so $\chi(U) \in P_1$. But $\chi(U)\big(X(P_1)\big) = 1$, which implies $P_1 = \mathscr{C}_m(X, L)$. This is impossible. Hence $X(P_1) \leqslant X(P_2)$. Moreover, it is clear that $L(P_2) \subseteq L(P_1)$, whence $L(P_1) \leqslant L(P_2)$ in the order of $\Omega\Sigma(L)$.

Conversely, suppose $X(P_1) \leqslant X(P_2)$ and $L(P_1) \leqslant L(P_2)$. Now for any prime P in $\mathscr{C}_m(X, L)$ it is easy to see that $P = \{g \in \mathscr{C}_m(X, L): g\big(X(P)\big) \in L(P)\}$. Thus, if $f \in P_2$, $f\big(X(P_1)\big) \leqslant f\big(X(P_2)\big) \in L(P_2) \geqslant L(P_1)$, and so $f \in P_1$. Hence, $P_1 \leqslant P_2$. Thus, the map $P \to \big(X(P), L(P)\big)$ is, indeed, an order isomorphism.

Consider $f_1: \Omega\Sigma\big(\mathscr{C}_m(X, L)\big) \to X$, defined by $f_1(P) \in X(P)$ for each $P \in \Omega\Sigma\big(\mathscr{C}_m(X, L)\big)$. Let W be a basic clopen neighbourhood of $X(P)$ in X. Then $W = U \cap (X \setminus V)$ for suitable $U, V \in \mathscr{D}(X)$. Then $P \in g\big(\chi(U)\big) \cap h\big(\chi(V)\big)$. For $P \notin g\big(\chi(U)\big)$ implies $\chi(U) \in P$, yet $\chi(U)\big(X(P)\big) = 1$. This implies the impossibility that $P = \mathscr{C}_m(X, L)$. While $X(P) \in X \setminus V$ implies $\chi(V)\big(X(P)\big) = 0$ and so $\chi(V) \in P$, i.e. $P \in h\big(\chi(V)\big)$. Thus, $g\big(\chi(U)\big) \cap h\big(\chi(V)\big)$ is a clopen neighbourhood of P in $\Omega\Sigma\big(\mathscr{C}_m(X, L)\big)$. Since it is clear that f_1 maps this neighbourhood into W, f_1 is continuous.

Consider $f_2: \Omega\Sigma\big(\mathscr{C}_m(X, L)\big) \to \Omega\Sigma(L)$, defined by $f_2(P) = L(P)$ for each P in $\Omega\Sigma\big(\mathscr{C}_m(X, L)\big)$. For prime P, a basic clopen neighbourhood of $L(P)$ in $\Omega\Sigma(L)$ is of the form $g(a_1) \cap h(a_2)$ for some $a_1, a_2 \in L$. It is easy to see that $f_2\Big(g\big(c(a_1)\big) \cap h\big(c(a_2)\big)\Big) \subseteq g(a_1) \cap h(a_2)$. It follows that f_2 is continuous. Hence f is continuous. But f is a bijection and each of $\Omega\Sigma\big(\mathscr{C}_m(X, L)\big)$ and $X \times \Omega\Sigma(L)$ are compact and Hausdorff. Thus, f is an homeomorphism.

2.12. COROLLARY. *For two bounded distributive lattices L_1 and L_2, $\mathrm{Pr}(L_1 \amalg L_2)$ is order-isomorphic and homeomorphic to $\mathrm{Pr}(L_1) \times \mathrm{Pr}(L_2)$.*

Actually, the proof of 2.11 has nothing to do with the distributivity of the lattice L. Moreover, for any bounded lattice L we can legitimately consider $\Sigma(L)$ and $\Omega\Sigma(L)$. Here $\Sigma(L)$ is the set of all prime ideals of L, endowed with the spectral topology. Because of [6] (Theorem 2.5, Proposition 3.51), $\Sigma(L)$ is a spectral space and so $\Omega\Sigma(L)$ is totally order-disconnected and compact. Hence, 2.11 *is true for any bounded lattice L.* However, when one of L_1 and L_2 is not distributive, neither $\mathscr{C}_m\big(\Omega\Sigma(L_1), L_2\big)$ nor $\mathscr{C}_m\big(\Omega\Sigma(L_2), L_1\big)$ may be isomorphic to the coproduct of L_1 and L_2 in the category of bounded lattices and $(0, 1)$-homeomorphisms; this is demonstrated by an example due to H. Lakser in [14].

We close this paper by looking at some examples. Each of the results in the examples is known, but we feel it is instructive to look at them in light of our previous results.

2.13. For an integer $n \geqslant 2$ let $\boldsymbol{n}$ denote the n-element chain. If L is a bounded distributive lattice, then $L \amalg \boldsymbol{n}$ is isomorphic to $L^{(n-1)} = \{(a_1, a_2, \ldots, a_{n-1}) : a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{n-1}\}$, considered as a sublattice of the $(n-1)$-fold direct power of L. For, as a poset we can identify $\Omega\Sigma(\boldsymbol{n})$ with $\boldsymbol{n}-1$, and since $\boldsymbol{n}$ is finite, $\Omega\Sigma(\boldsymbol{n})$ has discrete topology. This example was first considered by Rousseau [15].

2.14. For L as in 2.13, $L^{(n-1)}$ is isomorphic to $\mathscr{C}_m(\mathrm{Pr}\, L, \boldsymbol{n})$. Hence, if L is Boolean, then $L^{(n-1)}$ is isomorphic to $\mathscr{C}(\Sigma(L), \boldsymbol{n})$. This follows from 2.5. It was first stated in a slightly different manner by Epstein [8] (Theorem 15).

2.15. Let B_n denote the Boolean algebra with $n \geqslant 2$ atoms and let $\bar{B}_n$ be the lattice obtained from B_n by adjoining a new largest element. For a Boolean algebra B, let $B^{[n+1]} = \{(a_1, a_2, \ldots, a_{n+1}) : a_1 \leqslant a_2 \wedge \ldots \wedge a_{n+1}\}$ considered as a sublattice of the $(n+1)^{\mathrm{st}}$ power of B. Then, $B^{[n+1]}$ is isomorphic to $B \amalg \bar{B}_n$. This example is due to Quackenbush [14]. Because of 2.3 it is immediately clear. Indeed, $\Omega\Delta(\bar{B}_n)$ is discrete as a topological space and as a poset it looks like $\{a_1, a_2, \ldots, a_n, e : e \leqslant a_i \text{ for each } i = 1, \ldots, n\}$, where the a_i may be thought of as the atoms of B_n, and e as the largest element of B_n (the dual atom in $\bar{B}_n$).

2.16. For $n \geqslant 1$, let $B(n)$ denote the free Boolean algebra on n free generators. Of course, $B(n+1)$ is isomorphic to $B(n) \amalg B(1)$, and $B(1)$ is isomorphic to B_2. Hence, $B(n+1)$ is isomorphic to $B(n)^2$ — all the (monotonic continuous) functions from the ordered space 2 with the trivial order and the discrete topology to $B(n)$. Thus, $B(2) \cong 2^2$ and induction proves the well-known result: $B(n)$ is isomorphic to 2^{2^n}.

2.17. For $n \geqslant 1$, let $L(n)$ denote the free bounded distributive lattice on n free generators. Of course, $L(1) \cong \boldsymbol{3}$ and $L(n+1) \cong L(n) \amalg L(1)$. Hence, $L(n+1) \cong \mathscr{C}_m(\mathrm{Pr}\,\boldsymbol{3}, L(n)) =$ the cardinal power $L(n)^{\boldsymbol{2}}$. Then, $L(2) \cong \boldsymbol{2}^{\boldsymbol{2}}$, induction, and the properties of cardinal powers ([3], third law of (4), Theorem 2, p. 57) allow us to assert $L(n) \cong \boldsymbol{2}^{\boldsymbol{2}^n}$, a result which is established by other means in [3] (Section 3.4, p. 59).

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