COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 1

AN APPROXIMATIVE GENERALIZATION OF ZAHORSKI'S THEOREM ON DERIVATIVE

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1. Zahorski proved in [10] the following theorem:

If a continuous function f has a derivative f' (finite or infinite) everywhere on I_0, then the set

$$E(\alpha, \beta) = \{x \in I_0 \colon \alpha < f'(x) < \beta\}$$

is in the class M_3 (see below) for each pair of numbers α, β, $-\infty \leqslant \alpha < \beta \leqslant +\infty$.

Marcus posed the following problem in [5]:

Is the above theorem still true if the ordinary derivative f' and $E(\alpha, \beta)$ are replaced by the approximate derivative f'_{ap} and

$$E_{\mathrm{ap}}(\alpha, \beta) = \{x \in I_0 \colon \alpha < f'_{\mathrm{ap}}(x) < \beta\}?$$

The purpose for this paper* is to prove a general theorem which yields an affirmative answer to this problem. Throughout this paper, f is a real-valued function defined on a fixed interval I_0. For each subset E of I_0, $\bar{E}$ and E' stand for the closure and the derived set of E, respectively, and $|E|$ denotes the Lebesgue measure of E. We write $f \in B_1$ if f is of Baire type one; $f \in (B_1, D)$ if f is of Baire type one and has the Darboux property.

We first state definitions.

Let $E \neq \emptyset$ be a linear set of type F_σ. $E \in M_2$ if every open interval (x_1, x_2) with at least one endpoint belonging to E contains a subset of E of positive measure. $E \in M_3$ if, for each $x \in E$ and any number $c > 0$, there is a number $\varepsilon(x, c) > 0$ such that, for any pair of numbers h, h_1,

$$hh_1 > 0, \quad h/h_1 < c, \quad \text{and} \quad |h + h_1| < \varepsilon(x, c)$$

* The authors wish to express their gratitude to the referee for his suggestions.

imply

$$|E\cap(x+h, x+h+h_1)| > 0 \quad \text{and} \quad |E\cap(x+h+h_1, x+h)| > 0$$

for $h > 0$ and $h < 0$, respectively. $f \in \mathcal{M}_i$ if, for any real number a, $\{x \in I_0: f(x) > a\}$ and $\{x \in I_0: f(x) < a\}$ are in M_i $(i = 2, 3)$. All these definitions were founded in [10].

Also, in [9], Weil gave the following definition:

$f \in B$ if, for every open interval (α, β), $x \in f^{-1}(\alpha, \beta)$ and a sequence of closed intervals I_n converging to x with $|f^{-1}(\alpha, \beta)\cap I_n| = 0$ for every n imply

$$\lim_{n\to\infty} |I_n|/d(x, I_n) = 0,$$

where $d(x, I_n) = \inf\{|x-y|: y \in I_n\}$.

2. We state known results on which this work is based.

THEOREM 1. *If $f \in (B_1, D)$, f'_{ap} (finite or infinite) exists except perhaps on a denumerable set, and $f'_{ap} \geq 0$ almost everywhere, then f is continuous and non-decreasing on I_0.*

THEOREM 2. *If $f \in (B_1, D)$, f'_{ap} (finite or infinite) exists everywhere on I_0, and $f'_{ap} \in B_1$, then $f'_{ap} \in \mathcal{M}_2$ and hence f'_{ap} has the Darboux property.*

These two theorems appear in [1]. However, it should be noted that the condition $f'_{ap} \in B_1$ in Theorem 2 is in fact a consequence of the remaining hypotheses because of Preiss' Theorem 3 in [8].

The following theorem can be obtained from Bruckner's result ([2], p. 76), but we present a direct and easy proof here.

THEOREM 3. *If $f \in (B_1, D)$ and g is continuous on I_0, then $f+g \in (B_1, D)$.*

Proof. With the aid of Neugebauer's work [7], we need only to show that for any real a

$$E = \{x \in I_0: f(x)+g(x) \geq a\} \quad \text{and} \quad F = \{x \in I_0: f(x)+g(x) \leq a\}$$

have compact components. Let Q be any component of E; then we may suppose that Q is a non-degenerate interval with endpoints $c < d$. If $f(c)+g(c) < a$, then we choose r with $g(c) < r < a-f(c)$. Since g is continuous, there is a $\delta > 0$ with $c+\delta < d$ such that

$$g(x) < r < a-f(c) \quad \text{if } x \in (c, c+\delta).$$

This implies that

$$f(x) > a-r > f(c) \quad \text{if } x \in (c, c+\delta).$$

Then this leads to a contradiction with the hypothesis that f has the Darboux property. Hence $f(c)+g(c) \geq a$ and, similarly, $f(d)+g(d) \geq a$. By the same reason, F also has compact components.

THEOREM 4. *If $f'_{\rm ap}$ (finite or infinite) exists on I_0, then $f'_{\rm ap}$ is finite almost everywhere.*

This can be obtained from Burkill and Haslam-Jones' results ([3], p. 355).

3. In this section, we assume that $f \in (B_1, D)$ has an approximate derivative $f'_{\rm ap}$ (finite or infinite) everywhere on I_0. It follows from Section 2 that, for any subinterval I of I_0, $f'_{\rm ap} \geqslant \lambda$ (or $f'_{\rm ap} \leqslant \lambda$) almost everywhere on I implies $f'_{\rm ap} \geqslant \lambda$ (or $f'_{\rm ap} \leqslant \lambda$) on I and that $f'_{\rm ap}$ has the Darboux property. Also, $\mu x - f(x), f(x) - \mu x, f(-x) + \mu x, -\mu x - f(-x)$ are in (B_1, D) for every real number μ and

$$|\{x \in I_0\colon\ f'_{\rm ap}(x) = \pm\infty\}| = 0.$$

With the aid of this remark we now prove

LEMMA 1. *$E_{\rm ap}(\alpha, \beta)$ is empty or of positive measure.*

Proof. The proof presented here parallels Clarkson's [4]. For simplicity, let $E = E_{\rm ap}(\alpha, \beta)$,

$$E_\alpha = \{x \in I_0\colon\ f'_{\rm ap}(x) \leqslant \alpha\} \quad \text{and} \quad E_\beta = \{x \in I_0\colon\ f'_{\rm ap}(x) \geqslant \beta\};$$

then $I_0 = E_\alpha \cup E \cup E_\beta$. Suppose that $E \neq \emptyset$ and $|E| = 0$. It suffices to prove this lemma for the case where α and β are finite. The proof is divided in two steps.

Step I. $E \subset E'_\alpha \cap E'_\beta$. Suppose that $x_0 \in E \setminus E'_\alpha$; then there is a closed interval I containing x_0 such that $f'_{\rm ap}(x) > \alpha$ for every $x \in I$. Since $|E| = 0$, $f'_{\rm ap} \geqslant \beta$ almost everywhere on I. By the remark, $f'_{\rm ap} \geqslant \beta$ on I and, in particular, $f'_{\rm ap}(x_0) \geqslant \beta$. This is contradictory to the fact that $x_0 \in E$. Hence $E \subset E'_\alpha$ and, similarly, $E \subset E'_\beta$.

Step II. $\bar{E}$ contains no point of continuity of $f'_{\rm ap}|\bar{E}$. Let $x_0 \in \bar{E}$ and let I be any open interval of I_0 containing x_0; then there is $x_1 \in I \cap E$. Since $E \subset E'_\alpha \cap E'_\beta$, we have

$$\inf\{f'_{\rm ap}(x)\colon\ x \in I\} \leqslant \alpha \quad \text{and} \quad \sup\{f'_{\rm ap}(x)\colon\ x \in I\} \geqslant \beta.$$

By the Darboux property of $f'_{\rm ap}$, we can conclude that

$$\inf\{f'_{\rm ap}(x)\colon\ x \in I \cap E\} = \alpha \quad \text{and} \quad \sup\{f'_{\rm ap}(x)\colon\ x \in I \cap E\} = \beta.$$

Now we have

$$\sup\{f'_{\rm ap}(x)\colon\ x \in I \cap \bar{E}\} - \inf\{f'_{\rm ap}(x)\colon\ x \in I \cap \bar{E}\} \geqslant \beta - \alpha.$$

This implies that $f'_{\rm ap}|\bar{E}$ is discontinuous at x_0, which contradicts the fact that $f'_{\rm ap} \in B_1$ (see [6], p. 143). Hence the proof is completed.

LEMMA 2. $f'_{\rm ap} \in B$.

Proof. We assume that $0 < f'_{ap}(0) < +\infty$, $f(0) = 0$, and $I_n = [a_n, b_n]$ is a sequence of closed intervals with $0 < a_n < b_n$ converging to 0 such that

$$|E_{ap}(0, +\infty) \cap I_n| = 0 \quad \text{for every } n.$$

By Lemma 1, $E_{ap}(0, +\infty) \cap I_n = \emptyset$, that is, $f'_{ap}(x) \leqslant 0$ or $f'_{ap}(x) = +\infty$ whenever $x \in I_n$ $(n = 1, 2, \ldots)$. Since

$$|\{x \in I_0 \colon f'_{ap}(x) = \pm\infty\}| = 0,$$

for every n, $f'_{ap} \leqslant 0$ almost everywhere on I_n. By Theorem 1, f is monotone non-increasing, continuous, and $f'_{ap} \leqslant 0$ on I_n $(n = 1, 2, \ldots)$.

Now the proof of this lemma can be completed in the same manner as Weil proved his Theorem 2 in [9].

THEOREM 5. $E_{ap}(\alpha, \beta) \in M_3$.

Proof. Suppose that $E_{ap}(\alpha, \beta) \notin M_3$. Then there are $x_0 \in E_{ap}(\alpha, \beta)$ and $c_0 > 0$ such that, for every n, there are h_n and h_{1n} with

$$h_n h_{1n} > 0, \quad h_n/h_{1n} < c_0, \quad |h_n + h_{1n}| < 1/n$$

but

$$|E_{ap}(\alpha, \beta) \cap (x_0 + h_n, x_0 + h_n + h_{1n})| = 0$$

or

$$|E_{ap}(\alpha, \beta) \cap (x_0 + h_n + h_{1n}, x_0 + h_n)| = 0.$$

We need only to show that this leads to a contradiction for the case where $h_n > 0$ for every n. Let

$$I_n = [x_0 + h_n, x_0 + h_n + h_{1n}];$$

then $\{I_n\}$ is a sequence of closed intervals converging to x_0 and

$$|I_n|/d(x_0, I_n) = h_{1n}/h_n > 1/c_0 \quad \text{for every } n.$$

By Lemma 2, we can conclude that

$$|E_{ap}(\alpha, \beta) \cap (x_0 + h_n, x_0 + h_n + h_{1n})| > 0 \quad \text{for some } n.$$

This is contradictory to the assumption. Hence $E_{ap}(\alpha, \beta) \in M_3$.

COROLLARY. *If f'_{ap} is real valued on I_0, then $f'_{ap} \in \mathscr{M}_3$.*

This is a direct consequence of the theorem.

It may be interesting to note that there is a function f satisfying the conditions of the theorem without being approximately continuous. Let

$$f(x) = \begin{cases} (1 - x^{1/3}) \sin x^{-1} & \text{if } x > 0, \\ -1 & \text{if } x = 0, \\ x^{1/3} - 1 & \text{if } x < 0. \end{cases}$$

Clearly, $f \in (B_1, D)$, $f'_{ap}(x) = f'(x)$ is finite for $x \neq 0$, and $f'_{ap}(0) = f'(0) = +\infty$. We show that

$$\overline{\lim_{x\to 0}} \operatorname{ap} f(x) \geqslant 0 > f(0),$$

and hence f is not approximately continuous at $x = 0$. To do this, we need only to check that $x = 0$ is not a point of dispersion of the set $\{x: f(x) > 0\}$. Suppose the contrary, that is, there is a $\delta > 0$ such that

$$|\{x: f(x) > 0,\, 0 < x \leqslant h\}|/h < \tfrac{1}{2} \quad \text{if } h \in (0, \delta).$$

We choose n_0 with $1/2n_0\pi < \delta$ and let $h_0 = 1/2n_0\pi$. Then we have

$$\begin{aligned} |E| &= |\{x:\, f(x) > 0,\, 0 < x \leqslant h_0\}| \\ &= \left|\bigcup_{n=n_0}^{\infty} \left(\frac{1}{(2n+1)\pi}, \frac{1}{2n\pi}\right)\right| = \frac{1}{\pi} \sum_{n=n_0}^{\infty} \frac{1}{2n(2n+1)}, \end{aligned}$$

$$\begin{aligned} |F| &= |\{x:\, f(x) \leqslant 0,\, 0 < x \leqslant h_0\}| \\ &= \left|\bigcup_{n=n_0}^{\infty} \left[\frac{1}{(2n+2)\pi}, \frac{1}{(2n+1)\pi}\right]\right| \\ &= \frac{1}{\pi} \sum_{n=n_0}^{\infty} \frac{1}{(2n+1)(2n+2)} \leqslant |E|. \end{aligned}$$

By the choice of h_0,

$$1 = \frac{|E| + |F|}{h_0} \leqslant \frac{2|E|}{h_0} < 1.$$

This is impossible. Hence

$$\overline{\lim_{x\to 0}} \operatorname{ap} f(x) \geqslant 0.$$

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Reçu par la Rédaction le 8. 6. 1976;
en version modifiée le 2. 12. 1976