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ON OPERATORS ON L_0

BY

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In [4] the author showed that operators on the spaces L_p (0, 1) for $0 \leqslant p < 1$ cannot be too small. Precisely, if $T\colon L_p \to X$ is a non-zero operator where X is an F-space, then there is a closed subspace V of L_p isomorphic to l_2 such that $T|V$ is an embedding (isomorphism into X). This note* represents an attempt to push these results a little further in the case $p = 0$.

It was originally the author's belief that if $T\colon L_0 \to X$ is a non-zero operator, then there is a Borel subset A of (0, 1) such that the restriction of T to $L_0(A)$ (functions supported on A) is an isomorphism. This is, however, not the case although it is true when $X = L_0$. An example in [5] shows that it is possible to find a proper closed subspace N of L_0 such that the quotient mapping does not have this property.

We therefore introduce the notion of a weakly singular operator on L_0, which is defined to be an operator which fails to be an isomorphism on any subspace $L_0(A)$, where A has positive measure. In Theorem 1 we characterize weakly singular operators and we use this characterization to show that if X is an ordered F-space whose positive cone is closed and $T\colon L_0 \to X$ is positive and weakly singular, then $T = 0$. This result is related to recent work of Aliprantis and Burkinshaw [1] who show that the topology of L_0 is minimal with respect to locally solid topologies on L_0 considered as a vector lattice. Stated in terms of operators the result of Aliprantis and Burkinshaw shows that a lattice isomorphism $T\colon L_0 \to X$, where X is a locally solid topological vector lattice, is a topological isomorphism.

We conclude by mentioning that the existence of non-zero weakly singular operators implies that certain subspaces of L_0 and L_∞ cannot be ultrabarrelled.

Our notation is as follows. $\mathscr{B}$ will denote the collection of Borel subsets of (0, 1) and $\mathscr{D}$ the sub-algebra of $\mathscr{B}$ generated by the dyadic intervals

$$[(k-1)\cdot 2^{-n}, k\cdot 2^{-n})\cap(0, 1).$$

λ will denote the usual Lebesgue measure on $\mathscr{B}$.

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The space $L_0 = L_0(0, 1)$ consists of all real Borel functions on (0, 1), where, as usual, functions differing only on a set of measure zero will be identified. L_0 is an F-space (complete metrizable topological vector space) when equipped with the topology of convergence in measure; this topology is given by the F-norm

$$\|f\| = \int_0^1 \frac{|f(t)|}{1+|f(t)|}\,dt.$$

Λ will denote the subspace of L_0 of all simple functions and Γ the subspace of all countably simple functions. L_0 is a lattice and, for $f \in L_0$, we write, as usual, $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$. If $A \in \mathscr{B}$, then $L_0(A)$ is the subset of L_0 of all f supported in A and, for $g \in L_0$, $R_A g$ is defined by

$$R_A g(t) = \begin{cases} g(t), & t \in A, \\ 0, & t \notin A. \end{cases}$$

Thus R_A is a projection of L_0 onto its subspace $L_0(A)$.

If $\mathscr{N}$ denotes the ideal of sets in $\mathscr{B}$ of measure zero, then $\mathscr{B}/\mathscr{N}$ is a complete Boolean algebra. We shall refer to the lattice infimum of a collection of sets $(A_i)_{i\in I}$ in $\mathscr{B}$, and by this we mean the set A in $\mathscr{B}$ unique up to sets of measure zero such that

$$\lambda(A\backslash A_i) = 0, \qquad i \in I,$$

and if $\lambda(B\backslash A_i) = 0$ $(i \in I)$, then $\lambda(B\backslash A) = 0$.

The term operator will mean a continuous linear map and all spaces under consideration will be F-spaces.

Definition. An operator $T\colon L_0 \to X$ (where X is an F-space) is *weakly singular* if, for every Borel subset A of (0, 1) with $\lambda(A) > 0$, $T|L_0(A)$ fails to be an isomorphism.

Remarks. (1) If $T\colon L_0 \to X$ is weakly singular and $S\colon X \to Y$ is any linear operator, then $ST\colon L_0 \to Y$ is also weakly singular. However, other "ideal" properties of weak singularity are not clear. In particular, if $S\colon L_0 \to X$ and $T\colon L_0 \to X$ are weakly singular, is $S+T$ weakly singular? **(P 1273)**

(2) If $T\colon L_0 \to L_0$ is weakly singular, then $T = 0$ (see [5]).

(3) There is a closed subspace M of L_0 such that the quotient map $q\colon L_0 \to L_0/M$ is a non-zero weakly singular operator [5].

Lemma 1. *Suppose $T\colon L_0 \to X$ is an operator, $\varepsilon > 0$, and*

$$B_\varepsilon = \{f \in L_0\colon \lambda(|f| > 1) \leqslant \varepsilon\}.$$

Suppose 0 is not in the closure of $T(L_0\backslash B_\varepsilon)$. Then there is a Borel subset A of (0, 1) such that $T|L_0(A)$ is an isomorphism and $\lambda(A) \geqslant 1-\varepsilon$.

Proof. We may suppose X is F-normed in such a way that if $f \in L_0$ and $\|Tf\| \leqslant 1$, then $f \in B_\varepsilon$.

Let $\mathscr{F}$ be the set of sequences $\Phi = (\varphi_n)$ in L_0 such that $\sum \|T\varphi_n\| < \infty$. For each $\Phi \in \mathscr{F}$, let $A(\Phi) = \{t : \varphi_n(t) \to 0\}$. Let A be the lattice infimum of $(A(\Phi) : \Phi \in \mathscr{F})$ so that $\lambda(A \setminus A(\Phi)) = 0$ for each $\Phi \in \mathscr{F}$. Then, if $f_n \in L_0(A)$ and $\sum \|Tf_n\| < \infty$, we have $f_n \to 0$ a.e. Thus it follows easily that $f_n \in L_0(A)$ and $Tf_n \to 0$. Then $f_n \to 0$ in measure.

The proof will therefore be complete if we can show that $\lambda(A) \geqslant 1 - \varepsilon$. Now

$$\lambda(A) = \inf \{\lambda(\bigcap_{i=1}^{k} A(\Phi^i)) : \Phi^i \in \mathscr{F},\ i = 1, 2, \ldots, k\}.$$

Fix k and $(\Phi^i : i = 1, \ldots, k)$ in $\mathscr{F}$. For $(s_1, \ldots, s_k) \in (0, 1)^k$ we have $(s_1 \varphi_n^1 + \ldots + s_k \varphi_n^k)_{n=1}^{\infty} \in \mathscr{F}$. Let Δ be the set of $(t, s_1, \ldots, s_k) \in (0, 1)^{k+1}$ such that

$$\lim_{n \to \infty} \sum_{i=1}^{k} s_i \varphi_n^i(t) = 0.$$

For fixed t, let

$$\Delta_t = \{(s_1, \ldots, s_k) \in (0, 1)^k : (t, s_1, \ldots, s_k) \in \Delta\}.$$

Then $\lambda_k(\Delta_t) = 0$ unless $t \in \bigcap_{i=1}^{k} A(\Phi^i)$, and in this case $\lambda_k(\Delta_t) = 1$. (Here λ_k is product Lebesgue measure on $(0, 1)^k$.) Hence

$$\lambda_{k+1}(\Delta) = \lambda(\bigcap_{i=1}^{k} A(\Phi^i)).$$

It follows that there exists $(s_1, \ldots, s_k) \in (0, 1)^k$ such that if

$$\Psi = (s_1 \varphi_n^1 + \ldots + s_k \varphi_n^k),$$

then

$$\lambda(A(\Psi)) \leqslant \lambda(\bigcap_{i=1}^{k} A(\Phi^i)).$$

Thus

$$\lambda(A) = \inf_{\Phi \in \mathscr{F}} \lambda(A(\Phi)).$$

We complete the proof by showing $\lambda(A(\Phi)) \geqslant 1 - \varepsilon$.

To do this let η be a normally distributed random variable with mean zero and variance one, defined on some probability space (Ω, Σ, P); let $(\eta_n)_{n=1}^{\infty}$ be a sequence of independent copies of η defined also in (Ω, Σ, P). Then, for $\omega \in \Omega$,

$$\|\eta_n(\omega)\, T\varphi_n\| \leqslant (|\eta_n(\omega)| + 1) \|T\varphi_n\|,$$

and hence

$$\int_\Omega \sum_{n=1}^{\infty} \|\eta_n(\omega)\, T\varphi_n\|\, dP(\omega) \leqslant (\mathrm{E}(|\eta|)+1) \sum_{n=1}^{\infty} \|T\varphi_n\|.$$

In particular,

$$\sum_{n=1}^{\infty} \|\eta_n(\omega)\, T\varphi_n\| < \infty \quad P\text{-a.e.}$$

Thus, if

$$\Omega_m = \{\omega\colon \sum_{n=1}^{\infty} \|m^{-1}\eta_n(\omega)\, T\varphi_n\| \leqslant 1\},$$

then $P(\Omega_m) \to 1$ as $m \to \infty$.

For fixed m, N, define $g_{m,N}\colon (0, 1)\times\Omega \to \boldsymbol{R}$ by

$$g_{m,N}(t, \omega) = \sum_{n=1}^{N} m^{-1}\eta_n(\omega)\varphi_n(t)$$

and let $Q_{m,N} = \{(t, \omega)\colon |g_{m,N}(t, \omega)| > 1\}$. Then

$$(\lambda\times P)(Q_{m,N}) = \int_0^1 \int_\Omega \chi_{Q_{m,N}}(t, \omega)\, dP(\omega)\, dt = \int_0^1 P\Big(|\eta| > m\big(\sum_{n=1}^{N} |\varphi_n(t)|^2\big)^{-1/2}\Big) dt$$

while, on the other hand,

$$(\lambda\times P)(Q_{m,N}) = \int_\Omega \int_0^1 \chi_{Q_{m,N}}(t, \omega)\, dt\, dP(\omega)$$

$$\leqslant P(\Omega-\Omega_m) + \int_{\Omega_m} \int_0^1 \chi_{Q_{m,N}}(t, \omega)\, dt\, dP(\omega) \leqslant P(\Omega-\Omega_m)+\varepsilon P(\Omega_m).$$

Letting $N \to \infty$, by the dominated convergence theorem, we obtain

$$\int_0^1 P\Big(|\eta| > m\big(\sum_{n=1}^{\infty} |\varphi_n(t)|^2\big)^{-1/2}\Big) dt \leqslant P(\Omega-\Omega_m)+\varepsilon P(\Omega_m),$$

and hence

$$\lambda\Big(t\colon \sum_{n=1}^{\infty} |\varphi_n(t)|^2 = \infty\Big) \leqslant P(\Omega-\Omega_m)+\varepsilon P(\Omega_m).$$

Let $m \to \infty$. We have $\sum |\varphi_n(t)|^2 < \infty$ except on a set of measure at most ε. Thus $\lambda(A(\Phi)) \geqslant 1-\varepsilon$ and the result follows.

We derive two immediate consequences from Lemma 1.

LEMMA 2. *Suppose $T\colon L_0 \to X$ is weakly singular. Then for every $m \in N$ and $\varepsilon > 0$ there exists $f \in L_0$ such that $|f(t)| \geqslant m$ a.e. and $\|Tf\| \leqslant \varepsilon$.*

Proof. If the lemma were false, then for some ε $(0<\varepsilon<1)$ and $m \in N$ the inequality $|f| \geqslant m$ a.e. implied $\|Tf\| > \varepsilon$. Pick $\delta > 0$ so that $\|f\| \leqslant \delta$ implies $\|Tf\| < \frac{1}{2}\varepsilon$. Then if $f \notin B_{1-\delta}$, there exists $g \in L_0$ with $\|g\| \leqslant \delta$ and $|mf+g| \geqslant m$ a.e. Hence

$$\|T(mf+g)\| > \varepsilon, \quad \|T(mf)\| > \varepsilon/2, \quad \|Tf\| > \varepsilon/2m.$$

Thus $T(L_0 \backslash B_{1-\delta})$ does not have 0 as a closure point, and so there is a set A of measure δ such that $T|L_0(A)$ is an isomorphism. This contradiction proves the lemma.

LEMMA 3. *Suppose $(X_n)_{n=1}^{\infty}$ is a sequence of F-spaces and $\omega(X_n)$ their countable product. Suppose $T\colon L_0 \to \omega(X_n)$ is a linear operator given by $Tf = (T_n f)_{n=1}^{\infty}$, where $T_n\colon L_0 \to X_n$. Suppose for every n the map*

$$S_n\colon L_0 \to X_1 \oplus \ldots \oplus X_n$$

given by $S_n = T_1 \oplus \ldots \oplus T_n$ is weakly singular. Then T is also weakly singular.

Proof. If T is not weakly singular, there exists a Borel set A of positive measure such that $T|L_0(A)$ is an isomorphism. Thus, it suffices to consider the case where T is an isomorphism. Then there is a neighborhood of zero in $\omega(X_n)$ of the form

$$V = \{(x_n)_{n=1}^{\infty}\colon \|x_i\| \leqslant \varepsilon_i,\ i = 1, \ldots, N\}$$

such that $V \cap T(L_0 \backslash B_{1/2}) = \emptyset$. Hence, if $V' \subset X_1 \oplus \ldots \oplus X_N$ is the set

$$V' = \{(x_n)_{n=1}^{N}\colon \|x_i\| \leqslant \varepsilon_i,\ i = 1, \ldots, N\},$$

then $V' \cap S_N(L_0 \backslash B_{1/2}) = \emptyset$. By Lemma 1, S_N fails to be weakly singular and this proves Lemma 3.

THEOREM 1. *Suppose $T\colon L_0 \to X$ is a linear operator. Then the following conditions are equivalent:*

(i) *T is weakly singular.*

(ii) *There is a sequence u_n with $u_n \in A$, $|u_n| \geqslant n$, and $TR_A u_n \to 0$ for every $A \in \mathscr{B}$.*

(iii) *There is a sequence v_n in L_0 with $|v_n| \to \infty$ uniformly and $TR_A v_n \to 0$ for every $A \in \mathscr{B}$.*

(iv) *There is a sequence $v_n \in L_0$ with $v_n \geqslant 0$ and $v_n \to \infty$ uniformly so that $TR_A v_n \to 0$ for every $A \in \mathscr{B}$.*

Proof. (i)$\Rightarrow$(ii). Let $(D(n)\colon n \in N)$ be an enumeration of the sets of $\mathscr{G}$ and consider the map $\tilde{T}\colon L_0 \to \omega(X)$ (countable product of copies of X) given by

$$\tilde{T}f = (TR_{D(n)} f)_{n=1}^{\infty}.$$

If $\tilde{T}$ is not weakly singular, then there exist $N \in N$ and $A \in \mathscr{B}$ such that $S_N | L_0(A)$ is an isomorphism where

$$S_N f = (TR_{D(1)} f, \ldots, TR_{D(N)} f)$$

(S_N: $L_0 \to X_1 \oplus \ldots \oplus X_N$). Let A_0 be an atom of the algebra of sets generated by $(A, D(1), \ldots, D(N))$, which is contained in A. Consequently, TR_{A_0} is an embedding and T is not weakly singular. We conclude that $\tilde{T}$ is weakly singular.

Now, by Lemma 2 we may find a sequence $f_n \in L_0$ with $|f_n| \geqslant n$ and $\tilde{T}f_n \to 0$. Choose $|u_n| \geqslant n$ so that $u_n \in \Lambda$, $u_n - f_n \to 0$, and $|u_n| \geqslant n$. Then $\tilde{T}u_n \to 0$.

If $A \in \mathscr{B}$ and $\varepsilon > 0$, then we may choose $\delta > 0$ so that if $\|f\| \leqslant \delta$, then $\|Tf\| \leqslant \varepsilon$. Choose $D \in \mathscr{D}$ so that $\lambda(A \Delta D) \leqslant \delta$. Then $\|R_A u_n - R_D u_n\| \leqslant \delta$, and so $\|TR_A u_n - TR_D u_n\| \leqslant \varepsilon$. Thus

$$\limsup_{n \to \infty} \|TR_A u_n\| \leqslant \varepsilon$$

since $\tilde{T}u_n \to 0$ implies $TR_D u_n \to 0$. As $\varepsilon > 0$ is arbitrary, the implication (i) $\Rightarrow$ (ii) is established.

(ii) $\Rightarrow$ (iv). First, observe that by the preceding argument it suffices to show that $TR_D v_n \to 0$ for $D \in \mathscr{D}$, i.e. that $\tilde{T}v_n \to 0$. Next, observe that if s is a simple function in L_0 (i.e. a finite linear combination of characteristic functions), then $\tilde{T}(su_n) \to 0$.

For each $\varepsilon > 0$ let $K(\varepsilon) = \{f \in \Lambda : \lambda(\operatorname{supp} f^-) \leqslant \varepsilon\}$. Suppose $f \in K(\varepsilon)$. Then for $n \geqslant 2$

$$\lambda(f^-(1+u_n) > 0) + \lambda(f^-(1-u_n) > 0) \leqslant \varepsilon,$$

and hence, for some $\theta_n = \pm 1$, if $g_n = f^+ - f^-(1+\theta_n u_n)$, then $g_n \in K(\frac{1}{2}\varepsilon)$. Now

$$\tilde{T}g_n = \tilde{T}f - \tilde{T}(\theta_n u_n f^-) \to \tilde{T}f \quad \text{as } n \to \infty.$$

Hence $\tilde{T}f \in \overline{\tilde{T}K(\frac{1}{2}\varepsilon)}$ so that $\tilde{T}K(\varepsilon) \subset \overline{\tilde{T}(K(\frac{1}{2}\varepsilon))}$. It now follows that $\tilde{T}\Lambda = \tilde{T}K(1) \subset \overline{\tilde{T}K(\delta)}$ for every $\delta > 0$.

Thus there exist $f_n \in K(1/n)$ such that $\tilde{T}(f_n + n) \to 0$. Since $f_n \in K(1/n)$, we have $f_n - f_n^+ \to 0$, and so $\tilde{T}(f_n^+ + n) \to 0$. Writing $v_n = f_n^+ + n$ we complete the proof.

(iv) $\Rightarrow$ (iii) is immediate.

(iii) $\Rightarrow$ (i). For $A \in \mathscr{B}$ with $\lambda(A) > 0$, the sequence $(R_A u_n)_{n=1}^{\infty}$ does not converge to zero in $L_0(A)$ but $TR_A u_n \to 0$.

For the next theorem we shall use the term "ordered F-space" to denote an F-space X equipped with a partial ordering $\leqslant$ satisfying

(1) $x+u \leqslant y+u$ whenever $x \leqslant y$ and $u \in X$;

(2) $\lambda x \leqslant \lambda y$ whenever $x \leqslant y$ and $\lambda \geqslant 0$;

(3) $x \leqslant y$ and $y \leqslant x$ if and only if $y = x$;

(4) the positive cone $\{x : x \geqslant 0\}$ is closed.

THEOREM 2. *Suppose X is an ordered F-space and T: $L_0 \to X$ is positive (i.e. $x \geqslant 0$ implies $Tx \geqslant 0$) and weakly singular. Then $T = 0$.*

Proof. Suppose $u \in L_\infty$. Choose, by Theorem 1 (iii), a sequence $v_n \in L_0$ with $v_n \to \infty$ uniformly, $v_n \geqslant 0$, and $Tv_n \to 0$. Then, eventually, $u \leqslant v_n$, and so $Tu \leqslant Tv_n$. Since the positive cone is closed, $Tu \leqslant 0$ and, analogously, $T(-u) \leqslant 0$. Thus, as L_∞ is dense in L_0, we have $T = 0$.

COROLLARY. *Suppose X is an F-space and T: $L_0 \to X$ is a weakly singular operator. Let P denote the positive cone of L_0. Then $T(P)$ is dense in $T(X)$.*

Proof. Let $M = \overline{T(X)}$ and $Q = \overline{T(P)}$. Then Q is a wedge (see [3]). Let $N = Q \cap (-Q)$; N is a closed subspace of M.

Consider the quotient space M/N and let π: $M \to M/N$ be the quotient map. Then $\pi(Q)$ is a cone in M/N (i.e. $\pi(Q) \cap -\pi(Q) = \{0\}$) and may easily be checked to be closed. Consider the space M/N ordered by the cone $\pi(Q)$ and consider $\pi \circ T$: $L_0 \to M/N$. Then $\pi \circ T$ is weakly singular, and hence $\pi \circ T = 0$. Consequently, $M = N$ and the corollary follows.

Finally, we remark that the existence of non-zero weakly singular operators implies the failure of certain spaces to be ultrabarrelled (see [2] and [6]). If T: $L_0 \to X$ is non-zero and weakly singular, we may find $v_n \geqslant 0$, $v_n \in \Lambda$, $v_n \to \infty$ uniformly so that $TR_A v_n \to 0$ for all $A \in \mathscr{B}$. In fact, it is easy to see that $T(v_n s) \to 0$ for all countably simple functions s. Let Γ be the space of countably simple functions in L_0. If Γ were an ultrabarrelled subspace of L_0, then we could conclude from the Banach-Steinhaus theorem that the operators $f \to T(v_n f)$ are equicontinuous. Hence, if $f \in L_0$, then

$$T(v_n v_n^{-1} f) \to 0,$$

i.e. $Tf = 0$. Thus $T = 0$ and we have a contradiction.

A similar argument shows that $\Gamma \cap L_\infty$ cannot be ultrabarrelled in L_∞. Of course, Γ is barrelled in L_∞; see [6] for further information.

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