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*ON EXTENDING OF PARTIAL BOOLEAN ALGEBRAS TO PARTIAL *-ALGEBRAS*

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Kochen and Specker introduced and examined in [3] and [4] the concepts of a partial Boolean algebra (PBA) and of a partial algebra over an arbitrary field K. As to the second name, perhaps the term "partial (commutative) linear algebra" would be more suitable. A very general concept of "partial algebra" has been presented for years in algebra. In [4] it is noticed that the set of all idempotents of a partial algebra (in the sense of Kochen and Specker) forms a PBA.

In this note the concept of a partial *-algebra is examined. This notion is a slight modification of that of a partial algebra; namely, by a *partial *-algebra* we mean any partial algebra equipped with an operation of involution. We prove that any PBA can be extended to a partial *-algebra, which is equivalent to the proposition that the converse of the result by Kochen and Specker holds.

Definition 1. A system $\mathscr{A} = \langle A; ⫰; +, \cdot, {}^*; \mathbf{1}\rangle$ is said to be a *partial *-algebra* over the field of complex numbers C if the following conditions are satisfied:

(1) $⫰ \subseteq A\times A$ is a non-empty symmetric and reflexive relation. $⫰$ is called the *relation of commeasurability*.

(2) $+$ and $\cdot$ are partial binary operations whose domains and ranges satisfy the connections $\mathrm{Dom}(+) = \mathrm{Dom}(\cdot) = ⫰$, $\mathrm{Rg}(+) = \mathrm{Rg}(\cdot) = A$.

(3) ${}^*\colon A \mapsto A$.

(4) $\mathbf{1}$ is a distinguished element in A. $\mathbf{1}$ is the unit of the partial *-algebra $\mathscr{A}$.

(5) $a ⫰ \mathbf{1}$ for every $a \in A$.

(6) If $a ⫰ b$, then $a ⫰ b^*$ and $\lambda a ⫰ b$ for any complex number λ.

(7) If a, b, c are pairwise in the relation $⫰$, then $a+b ⫰ c$ and $a\cdot b ⫰ c$.

(8) If a, b, c are pairwise in the relation $\between$, then the set $\{a, b, c\}$ generates in $\mathscr{A}$ a commutative linear algebra with the involution * and the unit $\mathbf{1}$.

Condition (8) may be replaced by the following system of axioms:

(8′) If a, b, c are mutally in the relation $\between$, then:

(L1) $a+b = b+a$,
(L2) $(a+b)+c = a+(b+c)$,
(L3) if $a+c = b+c$, then $a = b$,
(L4) $\lambda(a+b) = \lambda a+\lambda b$,
(L5) $(\lambda_1+\lambda_2)a = \lambda_1 a+\lambda_2 a$,
(L6) $(\lambda_1\lambda_2)a = \lambda_1(\lambda_2 a)$,
(L7) $1a = a$, where $1 \in C$;

(A1) $a\cdot b = b\cdot a$,
(A2) $(a\cdot b)\cdot c = a\cdot(b\cdot c)$,
(A3) $(a+b)\cdot c = a\cdot c+b\cdot c$,
(A4) $(\lambda a)\cdot b = \lambda(a\cdot b) = a\cdot(\lambda b)$ for any complex number λ,
(A5) $\mathbf{1}\cdot a = a = a\cdot\mathbf{1}$, where $\mathbf{1}$ is the unit of $\mathscr{A}$;

(A*1) $(a+b)^* = a^*+b^*$,
(A*2) $(\lambda a)^* = \bar{\lambda}a^*$,
(A*3) $(a\cdot b)^* = b^*\cdot a^*$,
(A*4) $(a^*)^* = a$.

Recall that (L1)-(L7), (A2)-(A5), and (A*1)-(A*4) are axioms of linear algebras with involution.

Notice that every commutative linear algebra $\mathscr{A} = \langle A; +, \cdot, ^*; \mathbf{1}\rangle$ with involution becomes a partial *-algebra provided $\between = A\times A$.

Here we write some simple properties of partial *-algebras:

1.1. $0a = 0\mathbf{1}$ for any $a \in A$ $(0 \in C)$.

The element $0a$ (a arbitrary) is the zero of a partial *-algebra and is denoted by $\mathbf{0}$.

1.2. For any $a \in A$ and $\lambda \in C$,

$$a \between \mathbf{0}, \quad a+\mathbf{0} = a, \quad 0a = \mathbf{0}, \quad \lambda\mathbf{0} = \mathbf{0}.$$

Let $-a = (-1)a$ and $a-b = a+(-1)b$, where $a\between b$.

1.3. Let $a\between b$. Then the equation $a+x = b$ has exactly one solution x such that $x\between b$, namely $x = b-a$.

A *partial Boolean algebra* $\mathscr{B} = \langle B; \between; \vee, \neg, \mathbf{1}\rangle$ is given by a non-empty set B, a binary relation $\between \subseteq B\times B$, a unary function $\neg$ from B into B, a partial binary function $\vee$ from $B\times B$ into B, and an element $\mathbf{1} \in B$ called the unit of $\mathscr{B}$. The domain of $\vee$ is a subset of $B\times B$. The properties of $\mathscr{B}$ are the following:

1. The relation $\between$ (called also the relation of commeasurability) is symmetric and reflexive.

2. For all $a \in B$, $a \between \mathbf{1}$.

3. The partial function $\vee$ is defined exactly on $\between$.

4. If $a \between b$, then $a \between \neg b$.

5. If a, b, c are mutually commeasurable, then $a \vee b \between c$.

6. If a, b, c are mutually commeasurable, then the Boolean polynomials in a, b, c form a Boolean algebra.

Let PBA denote the class of all partial Boolean algebras.

Basic properties of partial Boolean algebras can be found in [1], [3], and [4].

Let $\mathscr{A} = \langle A; \between; +, \cdot, ^*; \mathbf{1}\rangle$ be a partial *-algebra. An element $a \in A$ is said to be a *Hermitian idempotent* if $a = a \cdot a = a^*$. Let B be the set of all Hermitian idempotents in $\mathscr{A}$. Notice that $\mathbf{1} \in B$. Let $a, b \in B$. We put

$$\neg a = \mathbf{1} - a,$$
$$a \vee b = (a+b) - (a \cdot b) \quad \text{ if } a \between b.$$

The system $\mathscr{B} = \langle B; \between, \vee, \neg; \mathbf{1}\rangle$ is a partial Boolean algebra (here $\between$ is a restriction of the relation of commeasurability to the set $B \times B$).

In what follows we admit the following convention: algebraic objects are denoted by capital script letters (eventually, with subscripts) and their underlying sets are denoted by the same italic letters (with the same subscripts).

Let $\mathscr{B} \in$ PBA and let $\{\mathscr{B}_\lambda\}_{\lambda \in \Lambda}$ be the indexed family of all maximal Boolean subalgebras contained in $\mathscr{B}$. Let X_λ denote the Stone space of all maximal filters in $\mathscr{B}_\lambda$ and let $\mathscr{F}_\lambda$ be the Stone field of all simultaneously open and closed subsets of X_λ. Let i_λ be the Stone isomorphism of $\mathscr{B}_\lambda$ onto $\mathscr{F}_\lambda$.

Let

$$X = \prod_{\lambda \in \Lambda} X_\lambda.$$

For every $A \subseteq X_\lambda$ let A^* be the set of all functions $x \in X$ such that $x(\lambda) \in A$, and let $\mathscr{F}_\lambda^*$ be the field of subsets of X formed from all sets A^*, where $A \in F_\lambda$.

Let $\mathscr{F}^*$ be the least field of subsets of X containing all algebras $\mathscr{F}_\lambda^*$. The fields $\{\mathscr{F}_\lambda^*\}_{\lambda \in \Lambda}$ are independent in $\mathscr{F}^*$.

A function $h_\lambda \colon \mathscr{B}_\lambda \mapsto \mathscr{F}_\lambda^*$, where $h_\lambda a = (i_\lambda a)^*$, maps $\mathscr{B}_\lambda$ isomorphically onto $\mathscr{F}_\lambda^*$.

It is easy to show that the set

$$B^* = \bigcup_{\lambda \in \Lambda} F_\lambda^* \subseteq 2^X,$$

equipped with the relation of commeasurability $\between$, where $A_1^* \between A_2^*$ iff there exists a $\lambda \in \Lambda$ such that $A_1^*, A_2^* \in F_\lambda^*$ ($A_1^*, A_2^* \in B^*$), and with the usual set-theoretical operations of join restricted to $\between$ and complementation, is a partial Boolean algebra. This partial Boolean algebra is denoted by $\mathscr{B}^*$.

We define the following equivalence relation $\sim$ in $\mathscr{B}^*$:

$$A_1^* \sim A_2^* \quad \text{iff} \quad A_1^* = h_{\lambda_1} a \text{ and } A_2^* = h_{\lambda_2} a$$

for a certain (and unique) $a \in B$.

Let $|A_1^*|, |A_2^*| \in B^*/_\sim$. Then we define

$$|A_1^*| \between |A_2^*| \quad \text{iff} \quad A_1^* = h_{\lambda_1} a_1,\ A_2^* = h_{\lambda_2} a_2, \text{ and } a_1 \between a_2.$$

The function $\neg$ is defined by

$$\neg |A^*| = |X - A^*|.$$

Notice that if $|A_1^*| \between |A_2^*|$, then there exists $\mathscr{B}_{\lambda_0}$ such that $a_1, a_2 \in B_{\lambda_0}$, where $A_1^* = h_{\lambda_1} a_1$, $A_2^* = h_{\lambda_2} a_2$ (B_{λ_0} is the carrier of $\mathscr{B}_{\lambda_0}$). Let $A_3^* = h_{\lambda_0}(a_1 \vee a_2)$. Then we define the function $\vee$ as follows:

$$|A_1^*| \vee |A_2^*| = |A_3^*|.$$

THEOREM 1 (see [3] and [4]). *Let $\mathscr{B} \in PBA$. Then*

(i) $\mathscr{B}^*/_\sim \in PBA$.

(ii) *A mapping $\varphi(A^*) = |A^*|$ maps $\mathscr{B}^*$ homomorphically onto $\mathscr{B}^*/_\sim$.*

(iii) *$\mathscr{B}^*/_\sim$ and B are isomorphic.*

Let X be a non-empty set and let $\mathscr{F}$ be any fixed field of subsets of X. Then $f\colon X \mapsto C$ is a *simple function* (over $\mathscr{F}$) if f equals a finite linear combination of characteristic functions of sets from $\mathscr{F}$. Let $\mathscr{W}_{\mathscr{F}}$ be a commutative linear algebra of all simple functions over $\mathscr{F}$ with the usual addition and multiplication of complex functions and with conjugation as an involution.

LEMMA 1. *Let $\mathscr{F}$ and $\mathscr{W}_{\mathscr{F}}$ be as above. Then each element $w \in W_{\mathscr{F}}$ ($w \neq 0$) has a unique representation of the form*

$$(*) \qquad w = \sum_{i=1}^{n} a_i \chi_{A_i} \quad (A_i \in F,\ n < \omega)$$

up to a permutation of the numbers $\{1, 2, \ldots, n\}$, where $a_i \neq 0$, $a_i \neq a_j$ for $i \neq j$, and $A_i \neq \emptyset$, $A_i \cap A_j = \emptyset$ for $i \neq j$.

The proof is straightforward. $(*)$ is called the *canonical representation* of w.

Our aim is to prove the following theorem:

THEOREM 2. *Every partial Boolean algebra is isomorphic to the partial Boolean algebra of all Hermitian idempotents of a certain partial *-algebra.*

In a shortened form this theorem can be expressed as follows:

Every partial Boolean algebra is embeddable into a certain partial *-algebra.

Proof. Let $\mathscr{B} = \langle B; \updownarrow; \vee, \neg; 1\rangle \in \mathrm{PBA}$. Let X_λ, $\mathscr{F}_\lambda$, i_λ, X, $\mathscr{F}^*_\lambda$, h_λ, $\mathscr{F}^*$, $\mathscr{B}^*$, $\mathscr{B}^*/_\sim$ be defined as above. Let $\mathscr{W}^*_\lambda$ denote a commutative *-algebra of simple functions, spanned over characteristic functions of sets from F^*_λ. Let

$$W^* = \bigcup_{\lambda\in\Lambda} W^*_\lambda.$$

Let w_1, w_2 be in W^* and let

$$w_1 = \sum_{i=1}^{m} a_i\chi_{A_i^*} \quad \text{and} \quad w_2 = \sum_{j=1}^{n} \beta_j\chi_{B_j^*}$$

be canonical representations of w_1 and w_2.

Let $\approx$ be defined in W^* as follows:

$w_1 \approx w_2$ iff $m = n$ and there exists a permutation σ of the numbers $\{1, 2, \ldots, m\}$ such that $A_i^* \sim B^*_{\sigma(i)}$ for every $1 \leqslant i \leqslant m$ and $a_i = \beta_{\sigma(i)}$ for every $1 \leqslant i \leqslant m$.

$\approx$ is an equivalence relation in W^*. Let $\langle w\rangle, \langle u\rangle \in W^*/_\approx$. Define $\langle w\rangle \hat{\updownarrow} \langle u\rangle$ iff for the canonical representations of the elements w and u,

$$w = \sum_{i=1}^{m} a_i\chi_{A_i^*} \quad \text{and} \quad u = \sum_{j=1}^{n} \beta_j\chi_{B_j^*},$$

we have $|A_i^*| \updownarrow |B_j^*|$ for each i $(1 \leqslant i \leqslant m)$ and for each j $(1 \leqslant j \leqslant n)$.

The definition of $\hat{\updownarrow}$ does not depend on the choice of representatives of $\langle w\rangle$ and $\langle u\rangle$. Notice that $\langle w\rangle \hat{\updownarrow} \langle u\rangle$ iff there exist a $\lambda_0 \in \Lambda$ and elements $w_1 \in \langle w\rangle$, $u_1 \in \langle u\rangle$ such that $w_1, u_1 \in W^*_{\lambda_0}$.

Let $\langle w\rangle \hat{\updownarrow} \langle u\rangle$. Then there are a $\lambda_0 \in \Lambda$ and elements $w_1 \in \langle w\rangle$, $u_1 \in \langle u\rangle$ such that $w_1, u_1 \in W^*_{\lambda_0}$. Then $w_1 + u_1 \in W^*_{\lambda_0}$ and $w_1 \cdot u_1 \in W^*_{\lambda_0}$. We put

$$\langle w\rangle + \langle u\rangle = \langle w_1 + u_1\rangle \quad \text{and} \quad \langle w\rangle \cdot \langle u\rangle = \langle w_1 \cdot u_1\rangle.$$

The definitions of $+$ and $\cdot$ are correct. Notice that classes $\langle u\rangle$, $\langle w\rangle$, $\langle v\rangle$ are mutually in the relation $\hat{\updownarrow}$ iff there are a $\lambda_0 \in \Lambda$ and elements $u_1 \in \langle u\rangle$, $w_1 \in \langle w\rangle$, $v_1 \in \langle v\rangle$ such that $u_1, w_1, v_1 \in W^*_{\lambda_0}$.

An involution $*$ is defined in $W^*/_\approx$ by

$$\langle w\rangle^* = \langle \overline{w}\rangle,$$

where $\overline{w}$ is conjugate to w.

The unit element in $W^*/_\approx$ is an equivalence class determined by the characteristic function of the whole set

$$X = \prod_{\lambda\in\Lambda} X_\lambda.$$

It is easy to check that the system

$$\mathscr{W}^*/_{\approx} = \langle W^*/_{\approx}; \mathring{\delta}; +, \cdot, {}^*; \mathbf{1}\rangle$$

satisfies all the axioms of partial $*$-algebras.

Let

$$A_1^*, A_2^* \in B^* = \bigcup_{\lambda \in \Lambda} F_\lambda^*.$$

Notice that

$$A_1^* \sim A_2^* \quad \text{iff} \quad \chi_{A_1^*} \approx \chi_{A_2^*}.$$

Each Hermitian idempotent in $\mathscr{W}^*/_{\approx}$ is of the form $\langle\chi_{A^*}\rangle$, where $A^* \in B^*$. Let $\mathscr{L}$ be a partial Boolean algebra of all Hermitian idempotents in $\mathscr{W}^*/_{\approx}$. A function ψ: $\mathscr{B}^*/_{\sim} \mapsto \mathscr{L}$ defined by $\psi(|A^*|) = \langle\chi_{A^*}\rangle$ maps $\mathscr{B}^*/_{\sim}$ isomorphically onto $\mathscr{L}$. By Theorem 1, the partial Boolean algebra $\mathscr{B}$ is isomorphic to $\mathscr{B}^*/_{\sim}$. Hence $\mathscr{B}$ and $\mathscr{L}$ are isomorphic. Thus the proof is completed.

Thus, with any partial Boolean algebra $\mathscr{B}$ we can associate, in a unique way, a partial $*$-algebra $\mathscr{W}^*/_{\approx}$ constructed as above. This particular partial $*$-algebra will be denoted by $\mathscr{W}_{\mathscr{B}}$.

Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be partial $*$-algebras. A mapping h: $\mathscr{A}_1 \mapsto \mathscr{A}_2$ is said to be a ***homomorphism*** if the following conditions are fulfilled:

(i) if $a \mathring{\delta} b$, then $ha \mathring{\delta} hb$ $(a, b \in A_1)$;

(ii) if $a \mathring{\delta} b$, then $h(a+b) = ha+hb$ and $h(a\cdot b) = ha\cdot hb$;

(iii) $h(\lambda a) = \lambda ha$;

(iv) $h(a^*) = (ha)^*$;

(v) $h\mathbf{1} = \mathbf{1}$.

A one-to-one homomorphism is called a ***monomorphism***.

THEOREM 3. *Let $\mathscr{B}_1, \mathscr{B}_2 \in PBA$. Let h_0: $\mathscr{B}_1 \mapsto \mathscr{B}_2$ be a homomorphism* (see [3]). *Then h_0 can be extended to a homomorphism h: $\mathscr{W}_{\mathscr{B}_1} \mapsto \mathscr{W}_{\mathscr{B}_2}$. If h_0 is an epimorphism* (*a monomorphism*), *then h is an epimorphism* (*a monomorphism*).

Proof. It suffices to notice that any element $w \in W_{\mathscr{B}}$ $(w \neq \mathbf{0})$, where $\mathscr{B} \in$ PBA, has a unique representation of the form

$$(**) \qquad w = \sum_{i=1}^{n} a_i p_i$$

up to a permutation of the set $\{1, 2, \ldots, n\}$, where p_i is a Hermitian idempotent, $p_i \neq \mathbf{0}$ $(i = 1, 2, \ldots, n)$, $p_i \cdot p_j = \mathbf{0}$ for $i \neq j$, and $a_i \neq 0$ $(i = 1, 2, \ldots, n)$, $a_i \neq a_j$ for $i \neq j$.

$(**)$ is called the *canonical representation* of w.

$\mathscr{B}_1$ and $\mathscr{B}_2$ can be identified with partial Boolean algebras of Hermitian idempotents in $\mathscr{W}_{\mathscr{B}_1}$ and $\mathscr{W}_{\mathscr{B}_2}$, respectively. We put

$$h\Big(\sum_{i=1}^{n} a_i p_i\Big) = \sum_{i=1}^{n} a_i h_0 p_i.$$

Thus we obtain a function h which maps $W_{\mathscr{B}_1}$ into $W_{\mathscr{B}_2}$. The function h is well defined, since representation (**) is unique. h is also an extension of h_0. Simple computations show that h is a homomorphism.

In case where h_0 is a monomorphism we need some comments. If (**) is the canonical representation of w, then the sum

$$\sum_{i=1}^{n} a_i h_0 p_i$$

is the canonical representation of hw. It follows that if $w_1 \neq w_2$, then $hw_1 \neq hw_2$.

COROLLARY 1. *A partial Boolean algebra $\mathscr{B}$ is embeddable into a Boolean algebra iff $\mathscr{W}_{\mathscr{B}}$ is embeddable into a commutative linear algebra with involution.*

Indeed, $\mathscr{W}_{\mathscr{B}'}$ is a commutative algebra with involution iff $\mathscr{B}'$ is a Boolean algebra.

There exist partial Boolean algebras which cannot be even homomorphically mapped into a Boolean algebra (see [2] and [3]). Hence there exist partial *-algebras which cannot be extended to commutative linear algebras with involution.

A partial Boolean algebra $\mathscr{B}$ is *transitive* if the relation $\subseteq$ defined in $\mathscr{B}$ by

$$a \subseteq b \quad \text{iff} \quad a \mathbin{\circ\!\!\!\!\!\!\smile} b \text{ and } a \vee b = b$$

is a partial order in $\mathscr{B}$.

If $\mathscr{A} = \langle A; +, \cdot, {}^*; \mathbf{1}\rangle$ is a linear algebra (not necessarily commutative) with the involution *, then the set $L_{\mathscr{A}}$ of all Hermitian idempotents in $\mathscr{A}$ forms a transitive partial Boolean algebra $\mathscr{L}_{\mathscr{A}}$, where

$$a \mathbin{\circ\!\!\!\!\!\!\smile} b \quad \text{iff} \quad a\cdot b = b\cdot a \quad (a, b \in L_{\mathscr{A}}).$$

The remaining operations in $L_{\mathscr{A}}$ are defined as in case of partial *-algebras.

Let $\mathscr{A}$ be a linear algebra with involution and with a unit $\mathbf{1}$. Let $\{\mathscr{A}_\lambda\}_{\lambda\in\Lambda}$ be the family of all maximal, with respect to the inclusion, commutative algebras with involution, contained in $\mathscr{A}$. Let

$$A_0 = \bigcup_{\lambda\in\Lambda} A_\lambda.$$

Let

$a \between b$ iff there is a $\lambda \in \Lambda$ such that $a, b \in A_\lambda$ $(a, b \in A_0)$.

It is easy to check that the system

$$\mathscr{A}_0 = \langle A_0; \between; +, \cdot, ^*; \mathbf{1}\rangle,$$

with operations inherited from $\mathscr{A}$, is a partial *-algebra.

COROLLARY 2. *A partial Boolean algebra $\mathscr{B}$ is embeddable into a partial Boolean algebra of Hermitian idempotents of a certain linear algebra with involution iff $\mathscr{W}_{\mathscr{B}}$ can be extended to a linear algebra with involution.*

Indeed, let $\mathscr{A}$ be a linear algebra with involution and let h_0 be an embedding of $\mathscr{B}$ into $\mathscr{L}_{\mathscr{A}}$. Let

$$w = \sum_{i=1}^{m} a_i p_i \quad \text{and} \quad u = \sum_{j=1}^{n} \beta_j q_j \quad (w \neq u)$$

be elements of $W_{\mathscr{B}}$ in their canonical representations. Easy computations show that

$$\sum_{i=1}^{m} a_i h_0 p_i \neq \sum_{j=1}^{n} \beta_j h_0 q_j.$$

A mapping h,

$$hw = \sum_{i=1}^{m} a_i h_0 p_i,$$

is well defined. Moreover, h is an embedding of $\mathscr{W}_{\mathscr{B}}$ into $\mathscr{A}$.

There exist partial Boolean algebras not embeddable into transitive ones (see [3]). Hence there exist partial *-algebras which cannot be extended to linear algebras with involution.

Let $\mathscr{B} \in$ PBA. We define a *finitely additive spectral measure* as a homomorphism of the algebra $\mathscr{B}(C)$ of Borel sets of complex numbers into $\mathscr{B}$. A spectral measure $E\colon \mathscr{B}(C) \mapsto \mathscr{B}$ has a *finite carrier* iff there exists a finite set $\Delta = \{a_1, a_2, \ldots, a_n\}$ $(\Delta \subset C)$ such that $E(\Delta) = \mathbf{1}$.

THEOREM 4. *Let $\mathscr{B} \in PBA$. Then there are a one-to-one correspondence between elements of a partial *-algebra $\mathscr{W}_{\mathscr{B}}$ and finitely additive spectral measures with finite carriers and values in $\mathscr{B}$.*

Proof. Let

$$w = \sum_{i=1}^{n} a_i p_i$$

be the canonical representation of w $(w \in W_{\mathscr{B}})$. We have to consider two cases.

$$\text{(I)} \qquad \sum_{i=1}^{n} p_i \neq \mathbf{1}.$$

Let

$$a_{n+1} = 0 \in C \quad \text{and} \quad p_{n+1} = 1 - \sum_{i=1}^{n} p_i.$$

We define a spectral measure E_w corresponding to w as follows. Let Δ be a Borel set ($\Delta \subseteq C$). Put

$$\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} = \Delta \cap \{a_1, a_2, \ldots, a_n, a_{n+1}\}.$$

Then

$$E_w(\Delta) = p_{i_1} + p_{i_2} + \ldots + p_{i_k}.$$

(II)
$$\sum_{i=1}^{n} p_i = 1.$$

Let Δ be a Borel set and put

$$\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} = \Delta \cap \{a_1, a_2, \ldots, a_n\}.$$

Then

$$E_w(\Delta) = p_{i_1} + p_{i_2} + \ldots + p_{i_k}.$$

It is clear that $w_1 \neq w_2$ implies $E_{w_1} \neq E_{w_2}$ ($w_1, w_2 \in W_{\mathscr{B}}$).

Now, let E be any finitely additive spectral measure with a finite carrier Δ_0 (E: $\mathscr{B}(C) \mapsto \mathscr{B}$). Let $\Delta_0 = \{a_1, a_2, \ldots, a_n\}$. We put

$$w = \sum_{i=1}^{n} a_i E(\{a_i\}).$$

Then $w \in W_{\mathscr{B}}$ and the formula for w is the canonical representation of w iff $0 \notin \Delta_0$. Moreover, $E = E_w$.

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