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A METRIZATION THEOREM FOR THE PRODUCT OF ORDERED CONTINUA

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A *continuum* is a compact, connected Hausdorff space. An *ordered continuum* [3] is a continuum having exactly two non-separating points. It is well known that an ordered continuum can be totally ordered in such a way that the order topology and the original topology coincide. An ordered continuum with non-separating points a and b will be denoted by $[a, b]$. The arc (denoted by $[0, 1]$) is the only ordered continuum which is separable [2].

Let D denote the cartesian product of two ordered continua, i.e., $D = [a, b]\times[c, d]$. The continuum D is a natural generalization of the euclidean 2-cell E ($E = [0, 1]\times[0, 1]$) and is metrizable if and only if it is homeomorphic to E. The set

$$\partial D = \{\{a\}\times[c, d]\}\cup\{\{b\}\times[c, d]\}\cup\{[a, b]\times\{c\}\}\cup\{[a, b]\times\{d\}\}$$

is called the *boundary* of D.

The euclidean 2-cell E has a natural geometry. Indeed, it is clear that the collection $\mathscr{L}$ of maximal straight line segments in E has the following properties:

(1) $(\{x\}\times[0, 1])\in\mathscr{L}$ for each $x\in[0, 1]$.

(2) $([0, 1]\times\{y\})\in\mathscr{L}$ for each $y\in[0, 1]$.

(3) If p and q are points of E, then there exists a unique $L\in\mathscr{L}$ containing p and q.

(4) If $L\in\mathscr{L}$, then $\operatorname{card}(L\cap\partial E)\geqslant 2$.

It is the purpose of this paper* to show that if the product D contains a collection $\mathscr{L}$ of subcontinua satisfying properties (1) through (4) above, then D is metrizable.

The proof of the theorem depends on the following result of Ball [1]:

(*) If S is a connected ordered space and $\{f_n\}$ is a sequence of continuous functions of S into itself such that for each point $p\in S$, $\{f_n(p)\}\to p$ and, for infinitely many integers n, $f_n(p)\neq p$, then S is separable.

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THEOREM. *Let $D = [a, b]\times[c, d]$. Suppose that $\mathscr{L}$ is a collection of subcontinua of D with the following properties:*

(1) $(\{x\}\times[c, d])\in\mathscr{L}$ *for each* $x\in[a, b]$.

(2) $([a, b]\times\{y\})\in\mathscr{L}$ *for each* $y\in[c, d]$.

(3) *If p and q are points of D, then there exists a unique $L\in\mathscr{L}$ containing p and q.*

(4) *If $L\in\mathscr{L}$, then* $\operatorname{card}(L\cap\partial D)\geqslant 2$.

Then D is metrizable.

Proof. By applying properties (1) through (3) and the closed graph theorem it follows that the continuum L which is associated with the points (a, c) and (b, d) of D is the graph of an order-preserving homeomorphism of $[a, b]$ onto $[c, d]$. Hence we can assume that $D = [a, b]\times\times[a, b]$ and that the diagonal Δ of D is contained in $\mathscr{L}$.

More generally, if (r, s) and (t, u) are in D such that $r < t$ and $s < u$, and if $L\in\mathscr{L}$ is the continuum associated with (r, s) and (t, u), then $L\cap([r, t]\times[s, u])$ is the graph of an order-preserving homeomorphism of $[r, t]$ onto $[s, u]$. By applying (4) it follows that if $L\in\mathscr{L}$ is associated with the points (a, a) and (x, y) where $y < x$, then $L\leqslant\Delta$ (in the natural sense) and L is the graph of a homeomorphism of $[a, b]$ into $[a, b]$.

Choose an increasing sequence $z_1 < z_2 < \ldots < z_n\ldots$ in $[a, b]$ and let z denote the limit of this sequence. Let $L_n\in\mathscr{L}$ be the unique continuum associated with the points (a, a) and (z, z_n), and let f_n denote the homeomorphism associated with L_n. Observe that $f_1\leqslant f_2\leqslant\ldots\leqslant i$ (where i is the identity map associated with $\Delta\in\mathscr{L}$).

Now $\{f_n(x)\}\to x$ for each $x\in[a, b]$. Otherwise, suppose that $\{f_n(x)\}\to x_0 < x$, for some $x\in[a, b]$. Let $L_0\in\mathscr{L}$ be associated with the points (a, a) and (x, x_0) and let f_0 denote the homeomorphism associated with L_0. Then $f_n\leqslant f_0\leqslant i$ for each n; hence, $f_n(z) = z_n\leqslant f_0(z)\leqslant z$. Since $\{z_n\}\to z$, it follows that $f_0(z) = z$. Thus L_0 and Δ contain the points (a, a) and (z, z). This contradicts the uniqueness in (3).

Thus for each $x\in[a, b]\setminus\{a\}$, $\{f_n(x)\}\to x$ and $f_n(x)\neq x$. According to (*), $[a, b]\setminus\{a\}$ is separable. Thus $[a, b]$ is homeomorphic to $[0, 1]$ and D is metrizable.

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