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QUASI-MONOTONE AND CONFLUENT IMAGES OF IRREDUCIBLE CONTINUA

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In [4] Lelek posed the following question:

Suppose that M is an irreducible continuum and f is a local homeomorphism of M onto the continuum N. Does it follow that N is irreducible?

In [5] Mohler provided affirmative answers to two special cases of this question. In this note we show that the answer to Lelek's question is "yes". This follows from our principal result, Theorem 3, which says that the quasi-monotone image of an irreducible continuum is irreducible. (If a map from one continuum onto another is either monotone or a local homeomorphism, then it is *quasi-monotone*; if it is either monotone or open, then it is *confluent*.)

A *continuum* is a compact connected metric space. If p and q are points of the continuum M, then *M is irreducible from p to q* provided that no proper subcontinuum of M contains both p and q; a continuum is *irreducible* provided it is irreducible between some pair of its points.

A finite collection $\mathcal{S}$ of subsets of a topological space is a *chain* provided $\mathcal{S}$ can be counted, $\mathcal{S} = \{S_1, \ldots, S_n\}$ so that $S_i \cap S_j \neq \emptyset$ iff $|i-j| \leqslant 1$. If x belongs only to S_1, and y belongs only to S_n, then $\mathcal{S}$ is a *chain from x to y*. The union of the members of $\mathcal{S}$ is denoted by $\mathcal{S}^*$.

If M is a continuum, an *essential sum decomposition* of M is a finite collection $\mathcal{D}$ of subcontinua of M such that

(i) $M = \mathcal{D}^*$,

(ii) if $D \in \mathcal{D}$, then D contains a point not in the union of the other members of $\mathcal{D}$.

THEOREM 1. *The continuum M is irreducible from p to q if and only if each essential sum decomposition of M is a chain from p to q.*

Proof. Suppose that $\mathcal{D} = \{D_1, \ldots, D_n\}$ is an essential sum decomposition of M. Using normality and the fact that each member of $\mathcal{D}$ contains a point not in the union of the other members, we may obtain a collection $\mathcal{U} = \{U_1, \ldots, U_n\}$ of open subsets of M such that

(i) for each i, $D_i \subset U_i$,

(ii) $U_i \cap U_j \neq \emptyset$ if and only if $D_i \cap D_j \neq \emptyset$.

Since $\mathscr{U}$ is an open cover of M, it follows from Theorem 8 of [3], p. 136, that there is a chain from p to q whose elements are members of $\mathscr{U}$. We may assume that this chain is $\{U_1, \ldots, U_k\}$ and that $p \in U_1$, $q \in U_k$. From (ii) we see that the corresponding subcollection $\mathscr{E}$ of $\mathscr{D}$, $\mathscr{E} = \{D_1, \ldots, D_k\}$, must also be a chain, though perhaps $p \notin \mathscr{E}^*$. However, we may find a continuum $D_q \in \mathscr{D}$ such that $p \in D_q$. Since $p \in U_q \cap U_1$, $D_q \cap D_1 \neq \emptyset$. Letting D_r be the last element of $\mathscr{E}$ which D_q meets, we obtain a chain $\{D_q, D_r, \ldots, D_k\}$. Thus we can assume that $p \in \mathscr{E}^*$ and, by a similar argument, that $q \in \mathscr{E}^*$. Moreover, $\mathscr{E}$ must be a chain from p to q and $\mathscr{E} = \mathscr{D}$, otherwise there is a subcollection $\mathscr{F}$ of $\mathscr{E}$ such that $\mathscr{F}$ is a proper subcollection of $\mathscr{D}$ and a chain from p to q. Then $\mathscr{F}^*$ is a proper subcontinuum of M containing p and q, which contradicts the irreducibility of M.

Conversely, if each essential sum decomposition of M is a chain, then an essential sum decomposition into three subcontinua of M has the property that some two of these continua are disjoint. Applying a theorem of Sorgenfrey [6], we conclude that M is irreducible.

If $\mathscr{C}$ is a finite collection of subsets covering a topological space, then one may obtain an abstract simplicial complex, called the *nerve of* $\mathscr{C}$, as in [2], p. 68. There is a geometric simplicial complex, in Euclidean space, which is abstractly isomorphic to the nerve of $\mathscr{C}$. The polyhedron of this geometric simplicial complex is called the *geometric realisation* of the nerve of $\mathscr{C}$. Using this language, we may rephrase Theorem 1.

THEOREM 2. *A continuum M is irreducible if and only if for each essential sum decomposition $\mathscr{D}$ of M, the geometric realisation of the nerve of $\mathscr{D}$ is an arc or a point.*

If X and Y are topological spaces, a mapping $f: X \to Y$ is *quasi-monotone* provided that for each subcontinuum K of Y with non-void interior, $f^{-1}[K]$ has finitely many components, each of which is mapped onto K.

THEOREM 3. *If M is an irreducible continuum, N is a continuum and f is a quasi-monotone map of M onto N, then N is irreducible.*

Proof. Suppose that M is irreducible from p to q. We wish to show that N is irreducible from $f(p)$ to some point. If this fails, then certainly N is decomposable. Moreover, according to Theorem 4 of [3], p. 192, there are proper subcontinua A and B of N such that $N = A \cup B$ and $f(p) \in A \cap B$. Since A has non-void interior and f is quasi-monotone, $f^{-1}[A]$ has only finitely many components, $A_1, \ldots, A_j$. Denote the components of $f^{-1}[B]$ by $B_1, \ldots, B_k$. Since each component of $f^{-1}[A]$ is mapped onto A, no component of $f^{-1}[A]$ is contained in $f^{-1}[B]$. Similarly, no component of $f^{-1}[B]$ is contained in $f^{-1}[A]$. It follows that $\{A_1, \ldots, A_j, B_1, \ldots, B_k\}$

is an essential sum decomposition of M; denote this collection by $\mathscr{D}$. We now apply Theorem 1 and conclude that $\mathscr{D}$ is a chain from p to q. But this is impossible, since p belongs to distinct elements of this chain, namely, some component of $f^{-1}[A]$ and some component of $f^{-1}[B]$. Thus N is irreducible from $f(p)$ to some point.

Another generalisation of the concept of monotone mapping is given by the following definition. A mapping $f\colon X\to Y$ is *confluent* provided that for each subcontinuum K of Y, each component of $f^{-1}[K]$ is mapped onto K. The following theorem is a corollary of a theorem of Whyburn ([7], Theorem 7.5, p. 148):

THEOREM 4. *If f is an open mapping of a continuum M onto a continuum N, then f is confluent.*

Unlike quasi-monotone mappings, confluent mappings do not preserve irreducibility. Indeed, even open images of irreducible continua need not be irreducible, as is shown by the following example of Charatonik [1], p. 216.

Example. The irreducible continuum M is the sum of two concentric circles in the plane, together with a topological line limiting on the outer circle from the inside and on the inner circle from the outside. The continuum N is the outer circle and an open map of M onto N is given by radial projection.

As we see from the corollary to the next theorem, the reason that the function in the preceding example fails to preserve irreducibility is that point-inverses have too many components.

THEOREM 5. *Suppose that M and N are continua, $f\colon M\to N$ is a confluent mapping onto N such that for each $p\in N$, $f^{-1}[p]$ has finitely many components. Then f is quasi-monotone.*

Proof. Suppose that A is a subcontinuum of N and $p\in A$. Then $f^{-1}[p]$ has finitely many components. Since each component of $f^{-1}[A]$ contains a component of $f^{-1}[p]$, $f^{-1}[A]$ has at most as many components as $f^{-1}[p]$. Since f is confluent, each component of $f^{-1}[A]$ is mapped onto A, thus f is quasi-monotone.

COROLLARY 6. *Suppose that M and N are continua, and $f\colon M\to N$ is a confluent mapping onto N such that for each $p\in N$, $f^{-1}[p]$ has finitely many components. If M is irreducible, then so is N.*

Note that any mapping of a continuum M onto an indecomposable continuum N must be quasi-monotone. From this it follows that the converse to Theorem 5 is false. For instance, let M be an indecomposable continuum and f the mapping induced by identifying two points of M. Then f is quasi-monotone but not confluent.

A function f from a topological space X to a topological space Y is a *local homeomorphism* provided that each $x\in X$ has an open neighbourhood U_x such that $f[U_x]$ is open in Y and f restricted to U_x is a homeomorphism of U_x onto $f[U_x]$. Clearly, local homeomorphisms are open maps, hence, if X and Y are continua, confluent maps.

The following theorem follows immediately from Lemma 1 of [5], p. 69:

THEOREM 7. *If f is a local homeomorphism of the continuum M onto the continuum N, then for each $p \in N$, $f^{-1}[p]$ has finitely many components.*

An immediate consequence of Corollary 6 and Theorem 7 is an affirmative answer to Lelek's question.

THEOREM 8. *If f is a local homeomorphism of the irreducible continuum M onto the continuum N, then N is irreducible.*

REFERENCES

[1] J. J. Charatonik, *Confluent mappings and the unicoherence of continua*, Fundamenta Mathematicae 56 (1964), p. 213-220.

[2] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1948.

[3] K. Kuratowski, *Topology II*, 1968.

[4] A. Lelek, *Problem 200*, Colloquium Mathematicum 5 (1957), p. 118.

[5] L. Mohler, *On locally homeomorphic images of irreducible continua*, ibidem 22 (1970), p. 69-73.

[6] R. H. Sorgenfrey, *Concerning continua irreducible about n points*, American Journal of Mathematics 68 (1946), p. 667-671.

[7] G. T. Whyburn, *Analytic topology*, American Mathematical Society Colloquium Publications 28 (1942).

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