COLLOQUIUM MATHEMATICUM

VOL. LII 1987 FASC. 1

MEANS ON ADJUNCTION SPACES

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1. Introduction. Let X be a topological space and $X^n = X \times X \times \ldots \times X$ its n-fold Cartesian product. An *n-mean* on X is a map (= continuous function) $m\colon X^n \to X$ satisfying the two conditions below:

1. $m(x, x, \ldots, x) = x$, for each $x \in X$,

2. $m(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = m(x_1, \ldots, x_n)$, for each $(x_1, \ldots, x_n) \in X^n$ and $\sigma \in S_n$.

Here S_n is the group of permutations of the set $\{1, 2, \ldots, n\}$, also called the symmetric group on n elements.

A space admitting an n-mean is referred to as an *m_n-space*. It is an *m-space* if it is an m_n-space, for some $n \geq 2$.

For topological spaces, this concept was introduced and studied by G. Aumann in [1]. In this paper he proved that every retract of an m_n-space is an m_n-space. Clearly, this result extends to r-images. Since $m(x_1, x_2, \ldots, x_n) = (x_1 + x_2 + \ldots + x_n)/n$ is always an n-mean on a convex subset of a normed linear space and every AR (see [4], Theorem 2.1, p. 85) is the r-image of such a set, we see that every AR admits an n-mean, for $n \geq 2$. In this paper, we extend this result by establishing Theorem 1.1 below.

Here, and elsewhere in this paper, by a pair of spaces we shall mean a space X and a closed subspace A. Also, our spaces are Hausdorff.

THEOREM 1.1. *Let (X, A) be a pair of finite dimensional* AR's *(not necessarily compact). Let f be a closed mapping of A onto a closed, metrizable subspace of a space Y. Then, for $n \geq 2$, $X \cup_f Y$ admits an n-mean if and only if Y does.*

We remark that the mapping condition is always satisfied if X is compact.

In establishing this result, we are led to the consideration of a type of symmetric product X^n/T of a space X introduced by Eckmann in [11]. For $n \geq 2$, this product is closely related to, but different from the product $X(n)$ considered by Borsuk and Ulam in [6], and others in [5], [7], [8], [13], and [15]. Other results on m-spaces can be found in [2], [17], [18], and [20].

2. Definitions and notation. First, let us define the space X^n/T. For each $\sigma \in S_n$, the (twisting) homeomorphism

$$T_\sigma\colon X^n \to X^n$$

is given by $T_\sigma(\xi_1, \ldots, \xi_n) = (\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)})$. We shall say that $(\xi_1, \ldots, \xi_n) \overset{T}{\sim} (\eta_1, \ldots, \eta_n)$ if there exists $\sigma \in S_n$ with $(\eta_1, \ldots, \eta_n) = T_\sigma(\xi_1, \ldots, \xi_n)$. It is not difficult to see that this is an equivalence relation on X^n. The resulting quotient space is designated X^n/T and

$$\nu\colon X^n \to X^n/T$$

is the natural map. One should also observe that X^n/T is the orbit space of the action $S_n \times X^n \to X^n$ of the finite group S_n on X^n. If $S \subset X^n$, $\nu(S)$ will also be designated $[S]$. Similarly, we shall sometimes write $[\xi_1, \ldots, \xi_n]$ for $\nu((\xi_1, \ldots, \xi_n))$.

The product $X(n)$ is defined as follows: The points of $X(n)$ are those non-empty subsets of X having n or fewer elements. If

$$\pi\colon X^n \to X(n)$$

is given by $\pi(x_1, \ldots, x_n) = \{x_1, x_2, x_3, \ldots, x_n\}$, we assign the quotient topology to $X(n)$. If X is metric, $X(n)$ is a subspace of the hyperspace 2^X (with the Hausdorff metric) by [6].

By an ANR, we shall mean an ANR($\mathfrak{M}$)-space of Borsuk [4]. Thus, an AR is an AR($\mathfrak{M}$)-space of Borsuk. We remark that this is consistent with the notation of Hu [14], and that neither ANR's nor AR's need be compact.

Concerning diagram terminology, we are quite standard. In particular, h completes a diagram such as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

if h is a function from X to Z making it commute. To avoid cumbersome notation, functions appearing in diagrams that are obvious restrictions of other existing functions will usually be given the same name.

Let X and Y be disjoint spaces, A a closed subset of X, and $f\colon A \to Y$ a map. Let $\mathscr{D}$ be the decomposition of the topological sum $X + Y$ whose elements are: $\{x\}$, for $x \in X \sim A$; $\{y\}$, for $y \in Y \sim f(A)$; $\{y\} \cup f^{-1}(y)$, for $y \in f(A)$. The quotient space $(X+Y)/\mathscr{D}$ is denoted by $X \cup_f Y$ and we say that *X is attached to Y by the map f* and f is the *attaching map*. The map $p\colon X + Y \to X \cup_f Y$ is the *natural* (quotient) map.

3. Symmetric products. In [13], Ganea proved that the natural map $\pi\colon X^n \to X(n)$ is closed, but for $n > 3$ may not be open. Concerning

$v\colon X_n \to X^n/T$, however, several authors have observed the following theorem (and corollary).

THEOREM 3.1. *For each space X, the natural map $v\colon X^n \to X^n/T$ is both open and closed.*

Proof. For each subset S of X^n, it is easy to see that

$$v^{-1}v(S) = \bigcup_{\sigma \in S_n} T_\sigma(S).$$

If U is open, $v^{-1}v(U)$ is the union of the open sets $T_\sigma(U)$. (Recall that T_σ is a homeomorphism.) Thus, since v is a quotient map, $v(U)$ is open.

If F is closed, $v^{-1}v(F)$ is the finite union of the closed sets $T_\sigma(F)$. It follows that $v(F)$ is closed.

COROLLARY 3.2. *If X is Hausdorff, so is X^n/T. If X is metrizable, so is X^n/T.*

Proof. It is clear that X^n/T is always Hausdorff since v is closed, X is Hausdorff, and point-inverses finite.

By Stone [21] or Balachandran [3], the clopen (closed and open) image of a metrizable space is metrizable. Since X^n is metrizable and v clopen, the result follows.

There is clearly a map φ completing the diagram below.

$$\begin{tikzcd} & X^n \arrow[dl, "v"'] \arrow[dr, "\pi"] & \\ X^n/T \arrow[rr, dashed] & & X(n) \end{tikzcd}$$

For $n = 2$, φ is actually a homeomorphism. Since v and π are both closed, φ is always closed. Again from Theorem 3.1, we see that φ is open if and only if π is open. (Recall that Ganea [13] has shown that π need not be open, for $n > 2$.)

By Theorem 3.1, we see that $v|\Delta X$ embeds ΔX as a closed subset of X^n/T. Denoting the inverse of $v|\Delta X$ followed by the map $(\xi, \ldots, \xi) \to \xi$ by j, the following lemma was established by Eckmann in [11].

LEMMA 3.3. *For each n-mean $m\colon X^n \to X$, there is a unique map $\tilde{m}$ completing the diagram*

$$\begin{tikzcd} X^n \arrow[d, "v"'] \arrow[dr, "m"] & \\ X^n/T \arrow[r, dashed] & X \end{tikzcd}$$

Moreover, $\tilde{m}|[\Delta X] = j$. Conversely, if $\tilde{m}\colon X^n/T \to X$ is a map extending j, $m = \tilde{m}v$ is an n-mean on X.

Eckmann used this lemma to prove that a contractible, finite polyhedron admits an n-mean, $n \geqslant 2$. If X is an AR (as is a contractible, finite polyhedron) the extension is possible, clearly. This offers another proof that each AR admits an n-mean. This lemma is used throughout this paper. We extend the result on AR's slightly with the next theorem.

THEOREM 3.4. *Let X be an* AR *and A a closed subset of X. Then any n-mean on A can be extended to an n-mean on X.*

Proof. A^n and ΔX are closed subsets of X^n and $A^n \cap \Delta X = \Delta A$. Thus m can be extended to m' on $A^n \cup \Delta X$ by $m'(\xi, \ldots, \xi) = \xi$, for $\xi \in X$. Modifying the previous lemma slightly, we see that m' determines a map $\tilde{m}'$ completing the diagram

$$\begin{tikzcd}
A^n \cup \Delta X \arrow[d, "\nu"'] \arrow[dr, "m'"] & \\
{[A^n \cup \Delta X]} \arrow[r, dashed] & X
\end{tikzcd}$$

The map $\nu|(A^n \cup \Delta X)$ is closed and thus a quotient, so $\tilde{m}'$ is continuous. Also, $[A^n \cup \Delta X]$ is a closed subset of the metrizable space X^n/T, and X an AR. Thus, by [4], Theorem 4.2, p. 87, $\tilde{m}'$ admits an extension m'': $X^n/T \to X$. Clearly, $m''\nu$ is an n-mean for X extending m.

The converse of this theorem is not true even for cellular subsets of a manifold. For, let

$$X = \{(x, y)\colon\ 0 < x \leqslant 1,\ y = \sin(1/x)\} \cup \{(x, y)\colon\ -1 \leqslant y \leqslant 1\}.$$

Then X is cellular (i.e., $X = \bigcap_{i=1}^{\infty} D_i$, where each D_i is a topological disk in the plane and $D_{i+1} \subset \operatorname{Int} D_i$), but X admits no mean [2], whereas D_1 does, for $n \geqslant 2$ [1].

4. Invariance of AR and ANR under π and ν. In [6], Borsuk and Ulam asked if the property of being an AR were a π-invariant. In [13] Ganea partially answered this question, for a compact, finite dimensional (metric) space X, by proving that $X(n)$ is an ANR or AR accordingly as X is. Later Jaworowski [15] removed the hypothesis of finite dimensionality. Finally, Cauty [8] eliminated the compactness assumption and obtained the same result in [9] for X^n/T (which he denotes by $X^{[n]}$).

LEMMA 4.1. *Let (X, A) be a pair of finite dimensional* AR's, *$n \geqslant 2$, and $1 \leqslant j < n$. Then $\nu(A^j \times X^{n-j})$ is an* AR.

Before giving the proof, let us remark that should $n = 2$, we could apply Cauty's result [9] directly as follows: Considering the diagram $A \times X \supset A \times A \xrightarrow{\nu} \nu(A \times A)$, it is not difficult to see that $\nu(A \times X)$ is homeomorphic to the adjunction space $(A \times X) \cup_\nu \nu(A \times A)$. But $A \times X$, $A \times A$, and $\nu(A \times A)$ (by

Cauty's result) are AR's. Also, $\nu(A \times A)$ is metrizable, so by [14], Theorem 1.3, p. 181, the adjunction space is an AR.

This allows us to drop the hypothesis of finite dimensionality in the lemma and ultimately in Theorem 1.1 itself for the case $n = 2$. For $n > 2$, we present a relative version of Ganea's proof in [13] of contractibility and local contractibility of $\nu(A^j \times X^{n-j})$. But first a relative homotopy lemma is needed.

LEMMA 4.2. *The following are true.*

(1) *If (X, A) is an* AR *pair, there is a homotopy $h\colon X \times I \to X$ contracting X to a point so that $h(A \times I) \subset A$.*

(2) *If (X, A) is an* ANR *pair and U a neighborhood of $x_0 \in A$, there are a neighborhood U_0 of x_0 and a homotopy $h\colon U_0 \times I \to U$ contracting U_0 to a point in U such that $h((U_0 \cap A) \times I) \subset A$.*

Proof. For (1), let $h'\colon A \times I \to A$ contract A to a point a_0. Extend h' to $h''\colon (X \times \{0, 1\}) \cup (A \times I) \to X$ by $h''(x, 0) = x$ and $h''(x, 1) = a_0$. Since $(X \times \{0, 1\}) \cup (A \times I)$ is closed in $X \times I$ and X is an AR, h'' extends to h: $X \times I \to X$ by [4], Theorem 4.2, p. 87.

To prove (2), we use a rather technical proposition of S. T. Hu ([14], Proposition 3.4, p. 121). Let U be a neighborhood of $a_0 \in A$. Let G and V be neighborhoods of a_0 such that $V \subset \bar{V} \subset G \subset U$ and $G \cap A$ contracts to a point in $U \cap A$. Let α be the open covering $\{G, X \sim \bar{V}\}$ of X. By [14], Proposition 3.4, p. 121, there is a homotopy $h'\colon X \times I \to X$ satisfying (a) h'_0 = identity, (b) $h'(x, t) = x$ if $x \in A$, $t \in I$, and (c) for some neighborhood W of A, $h'_1(W) \subset A$. Additionally, this homotopy satisfies the particular condition $h'(V \times I) \subset G$.

Put $U_0 = V \cap W$. Let $h''\colon (G \cap A) \times [1, 2] \to U \cap A$ contract $G \cap A$ to a point in $U \cap A$. Define a homotopy $h\colon U_0 \times [0, 2] \to U$ by

$$h(x, t) = \begin{cases} h'(x, t), & \text{if} \quad 0 \leqslant t \leqslant 1, \\ h''(h'(x, 1), t), & \text{if} \quad 1 \leqslant t \leqslant 2. \end{cases}$$

It is easy to see that h is the desired homotopy.

Proof of Lemma 4.1. Since X is finite dimensional, it follows from Nagami [19] that X^n/T, thus $\nu(A^j \times X^{n-j})$ is also. It will suffice to show that $\nu(A^j \times X^{n-j})$ is contractible and locally contractible. For the contractibility let $h\colon X \times I \to X$ be the homotopy obtained in part one of Lemma 4.2. Define the mapping $g\colon X^n \times I \to X^n$ by the formula $g((\xi_1, \ldots, \xi_n), t) = (h(\xi_1, t), \ldots, h(\xi_n, t))$ and consider the diagram below (where $\imath$ denotes the identity map).

$$\begin{tikzcd} X^n \times I \arrow[r, "g"] \arrow[d, "\nu \times 1"'] & X^n \arrow[d, "\nu"] \\ X^n/T \times I \arrow[r, dashed] & X^n/T \end{tikzcd}$$

There is a function $\tilde{g}$ completing the diagram given by

$$\tilde{g}([\xi_1, \ldots, \xi_n], t) = vg((\xi_1, \ldots, \xi_n), t).$$

To see that $\tilde{g}$ is well-defined, assume

$$(\xi_1, \ldots, \xi_n) \overset{T}{\sim} (\eta_1, \ldots, \eta_n).$$

Then we have

$$(h(\xi_1, t), \ldots, h(\xi_n, t)) \overset{T}{\sim} (h(\eta_1, t), \ldots, h(\eta_n, t)),$$

clearly. As for continuity, observe that by a theorem of J. H. C. Whitehead ([10], Theorem 4.1, p. 262), $v \times 1$ is a quotient map. This, together with the continuity of vg and commutativity insure that $\tilde{g}$ is continuous.

It is easy to see that $\tilde{g}$ contracts X^n/T (in itself) to the point $[a_0, \ldots, a_0]$. Now suppose $[\xi_1, \ldots, \xi_n] \in v(A^j \times X^{n-j})$. It follows that $(\xi_1, \ldots, \xi_n)$ has at least j coordinates in A and further that $\tilde{g}([\xi_1, \ldots, \xi_n], t) \in v(A^j \times X^{n-j})$. Thus $v(A^j \times X^{n-j})$ is contractible (via $\tilde{g} \mid v(A^j \times X^{n-j}) \times I$).

The preservation of local contractibility is proved along the same lines, but is a bit more delicate.

Let U be a neighborhood of $[\xi_1^0, \ldots, \xi_n^0]$ in X^n/T. Let $\zeta_1, \ldots, \zeta_l$ (written in the same order as they occur) be the distinct terms of $(\xi_1^0, \ldots, \xi_n^0)$, with ζ_i occurring k_i times, $i = 1, \ldots, l$. Denote by $(\eta_1^0, \ldots, \eta_n^0)$ the sequence

$$(\underbrace{\zeta_1, \ldots, \zeta_1}_{k_1}, \ldots, \underbrace{\zeta_l, \ldots, \zeta_l}_{k_l}).$$

Clearly, we have $[\eta_1^0, \ldots, \eta_n^0] = [\xi_1^0, \ldots, \xi_n^0]$. Choose mutually exclusive neighborhoods $V_1, \ldots, V_l$ of $\zeta_1, \ldots, \zeta_l$ so that $v(V) \subset U$, where V is the product $(V_1)^{k_1} \times \ldots \times (V_l)^{k_l}$.

Since X is locally contractible, there are neighborhoods $V_1^0, \ldots, V_l^0$ of $\zeta_1, \ldots, \zeta_l$ contractible in $V_1, \ldots, V_l$ via the homotopies $h_1, \ldots, h_l$ of Lemma 4.2 (2) to the points $x_1, \ldots, x_l$. That is, if $\zeta_i \in A$, make sure that $h_i((V_i^0 \cap A) \times I) \subset A$.

Let $V_0 = (V_1^0)^{k_1} \times \ldots \times (V_l^0)^{k_l}$ and $U_0 = v(V_0)$. Since v is an open map, U_0 is an open subset of X^n/T. (Now consider the diagram below, where g is constructed from $h_1, \ldots, h_l$ in the obvious way.)

$$\begin{tikzcd} V_0 \times I \arrow[r, "g"] \arrow[d, "v \times 1"'] & V \arrow[d, "v"] \\ U_0 \times I \arrow[r, dashed] & U \end{tikzcd}$$

If $(\xi_1, \ldots, \xi_n) \overset{T}{\sim} (\eta_1, \ldots, \eta_n)$ and both are members of V_0, we have $(\xi_1, \ldots, \xi_{k_1}) \overset{T}{\sim} (\eta_1, \ldots, \eta_{k_1})$ whereupon

$$(h_1(\xi_1, t), \ldots, h_1(\xi_{k_1}, t)) \overset{T}{\sim} (h_1(\eta_1, t), \ldots, h_1(\eta_{k_1}, t)), \quad \text{etc.}$$

Thus, there is a function $\tilde{g}$ completing the diagram. The map $v | V_0$ is open and thus is a quotient. As before, $\tilde{g}$ is continuous and contracts U_0 to a point in U. If $[\xi_1^0, \ldots, \xi_n^0] \in v(A^j \times X^{n-j})$, then we have $\tilde{g}([\xi_1^0, \ldots, \xi_n^0], t) \in v(A^j \times X^{n-j})$. The result follows.

COROLLARY 4.3. *The closed subset* $v(A \times X^{n-1}) \cup v(\Delta X)$ *of* X^n/T *is an* AR.

Proof. By Lemma 4.1 $v(A \times X^{n-1})$ is an AR. Since $v(\Delta X)$ is homeomorphic to X, it is an AR. Moreover, they are both closed subsets of X^n/T and their intersection is $v(\Delta A)$ which is homeomorphic to A and thus an AR. By [4], Theorem 6.1, p. 90, the result follows.

The following lemma is standard.

LEMMA 4.4. *Each map* $f\colon X \to Y$ *induces a map* $\tilde{f}^n$ *completing the diagram below*, $n \geq 2$.

$$\begin{tikzcd} X^n \arrow[r, "f^n"] \arrow[d, "v"'] & Y^n \arrow[d, "v"] \\ X^n/T \arrow[r, dashed] & Y^n/T \end{tikzcd}$$

Proof. Immediate. (We used v to indicate both natural maps.)

5. A proof of Theorem 1.1. Let (X_0, A_0) be a pair of finite dimensional AR's and $f\colon A_0 \to B_0$ a closed mapping onto B_0, a closed, metrizable subspace of Y_0. Let $p\colon X_0 + Y_0 \to X_0 \cup_f Y_0$ be the natural map. Put $X = p(X_0)$, $A = p(A_0)$, $Y = p(Y_0)$, $B = p(B_0)$, and $Z = X_0 \cup_f Y_0$. Then X and Y are closed in Z (since B_0 and A_0 are closed in Y and X, resp.). Moreover, $Z = X \cup Y$ and $A = X \cap Y$. Thus A is closed. Let us establish a few more facts concerning the general nature of this decomposition.

First, let us show that p is a closed mapping. Hence, assume F is a closed subset of $X_0 + Y_0$. Then $F \cap X_0$ is closed. By [10], Proposition 6.2, p. 128, $f(F)$ is closed in $X_0 \cup_f Y_0$ if $(F \cap Y_0) \cup f(F \cap A_0)$ is closed in Y_0. Clearly, $F \cap Y_0$ is closed in Y_0. Since $F \cap A_0$ is a closed subset of A_0 and f is closed, $f(F \cap A_0)$ is closed in B_0, which is closed in Y_0. The result follows.

Next, we will prove that Z is Hausdorff. Let z_1 and z_2 be distinct points of Z. Then $p^{-1}(z_i)$ is the union of a closed subset of X_0 and a closed subset of Y_0, the latter being either a singleton or the empty set. Since X_0 is normal and Y_0 Hausdorff, $p^{-1}(z_1)$ and $p^{-1}(z_2)$ have disjoint open neighborhoods. The remainder of the argument is standard and relies on the closedness of p.

Finally, let us show that Y is a retract of Z. Clearly there is a retraction $r\colon X_0 \to A_0$. Consider the diagram below.

$$\begin{tikzcd} X_0 + Y_0 \arrow[r, "r \cup 1"] \arrow[d, "p"'] & A_0 \cup Y_0 \arrow[d, "p"] \\ Z \arrow[r, dashed] & Y \end{tikzcd}$$

It is easy to see that there is a retraction completing this diagram. We could also appeal to [14], Proposition 3.2, p. 16, since f can be extended over X_0.

Concerning the proof of Theorem 1.1 itself, if Z admits an n-mean, then Y does also, being a retract of Z.

To prove the converse, we proceed in a sequence of steps, each asserting that some set is a retract of another.

Step 1. The set $[A \times X^{n-1}] \cup [\Delta X]$ is a retract of X^n/T.

To establish this, recall that $[A_0 \times X_0^{n-1}] \cup [\Delta X_0]$ is an AR by Lemma 4.3. Thus, there is a retraction r_0: $X_0^n/T \to [A_0 \times X_0^{n-1}] \cup [\Delta X_0]$.

Consider the diagram below.

$$
\begin{tikzcd}
X_0^n \arrow[r, "p^n"] \arrow[d, "\nu"'] & X^n \arrow[r, "\nu"] & X^n/T \arrow[d, dashed] \\
X_0^n/T \arrow[r, "r_0"] & {[A_0 \times X_0^{n-1}] \cup [\Delta X_0]} \arrow[r, "\tilde{p}_n"] & {[A \times X^{n-1}] \cup [\Delta X]}
\end{tikzcd}
$$

Let us define a relation r as follows: If $[p\xi_1, \ldots, p\xi_n] \in X^n/T$, put $r[p\xi_1, \ldots, p\xi_n] = \tilde{p}^n r_0 \nu(\xi_1, \ldots, \xi_n)$. If r is a function, it clearly completes this diagram. Suppose that

$$[p\xi_1, \ldots, p\xi_n] = [p\eta_1, \ldots, p\eta_n].$$

If $(\xi_1, \ldots, \xi_n) \overset{T}{\sim} (\eta_1, \ldots, \eta_n)$, we are done. Otherwise, we may assume that $\xi_1, \eta_1 \in A_0$ and $p\xi_1 = p\eta_1$. Then $[\xi_1, \ldots, \xi_n]$, $[\eta_1, \ldots, \eta_n] \in [A_0 \times X^{n-1}]$, so

$$
\begin{aligned}
r[p\xi_1, \ldots, p\xi_n] &= \tilde{p}^n r_0 \nu(\xi_1, \ldots, \xi_n) = \tilde{p}^n r_0 [\xi_1, \ldots, \xi_n] \\
&= \tilde{p}^n [\xi_1, \ldots, \xi_n] = [p\xi_1, \ldots, p\xi_n].
\end{aligned}
$$

Similarly, using the representation $[p\eta_1, \ldots, p\eta_n]$ we get $[p\eta_1, \ldots, p\eta_n]$. Thus, r is a function and is the identity on $[A \times X^{n-1}]$. If $[p\xi_1, \ldots, p\xi_n] \in \Delta X$, either $\xi_1, \ldots, \xi_n \in A_0$ and $p(\xi_1) = \ldots = p(\xi_n)$ or $\xi_1 \in X_0 \sim A_0$ and $\xi_1 = \ldots = \xi_n$. In either case, we have $r[p\xi_1, \ldots, p\xi_n] = [p\xi_1, \ldots, p\xi_n]$.

To show that r is continuous, it will suffice to demonstrate that $p^n | X_0^n$: $X_0^n \to X^n$ is a quotient map. We shall do this by showing $p | X_0$: $X_0 \to X$ is a bi-quotient (see the definition in Michael [16]) map. Since the class of bi-quotient mappings is closed under the formation of (arbitrary) products [16], it will follow that $p^n | X_0^n = (p | X_0)^n$ is a bi-quotient. But bi-quotients are quotients, so $p^n | X_0^n$ is a quotient.

To effect this, let us first observe that $p | X_0$ is a closed mapping. Since p is closed and X_0 is closed in $X_0 + Y_0$, this is clear. Since f is closed and B_0 metrizable, the Hanai–Morita–Stone Theorem (see [21]) asserts that $\partial f^{-1}(y)$ is compact, for each $y \in B_0$. Clearly then, $\partial (p | X_0)^{-1}(y)$ is compact, for each $y \in X$. Since, as we noted above, $p | X_0$ is closed, again by [21], we have X

metrizable. By Michael ([16], Corollary 9.10), we see that $p|X_0$ is a biquotient map. This completes the proof of Step 1.

Before proceeding with the next step, let us observe that

$$Z^n/T = [X^n] \cup [X^{n-1} \times Y] \cup \ldots \cup [X \times Y^{n-1}] \cup [Y^n].$$

Step 2. The set $[X^{n-1} \times Y] \cup [X^{n-2} \times Y^2] \cup \ldots \cup [Y^n] \cup [\Delta Z]$ is a retract of Z^n/T.

To see this, observe that in Z^n/T, $[X^n]$ meets the above set in $[X^{n-1} \times A] \cup [\Delta X] = [A \times X^{n-1}] \cup [\Delta X]$. Thus we may extend the retraction r obtained in Step 1 by the identity outside $[X^n]$.

Step 3. The set $[Y^n] \cup [\Delta Z]$ is a retract of $[X^{n-1} \times Y] \cup [X^{n-2} \times Y^2] \cup \ldots \cup [Y^n] \cup [\Delta Z]$ (and thus of Z^n/T, by Step 2).

First let us define a retraction of Z^n/T onto $[Y^n]$. The mapping r mentioned in the diagram below is the retraction $Z \to Y$ obtained in the proof of the converse of this theorem.

$$\begin{tikzcd} Z^n \arrow[r, "r^n"] \arrow[d, "\nu"'] & Y^n \arrow[d, "\nu"] \\ Z^n/T \arrow[r, dashed] & {[Y^n]} \end{tikzcd}$$

By passage to the quotient, we obtain a retraction $r^{\tilde{n}}$: $Z^n/T \to [Y^n]$.

If 1: $[\Delta Z] \to [\Delta Z]$ denotes the identity, then

$$(r^{\tilde{n}}|([X^{n-1} \times Y] \cup \ldots \cup [Y^n])) \cup 1$$

is the retraction required in Step 3, since

$$([X^{n-1} \times Y] \cup \ldots \cup [Y^n]) \cap [\Delta Z] = [\Delta Y].$$

Now let m be an n-mean on Y. By Lemma 3.3, there is a map $\tilde{m}$: $[Y^n] \to Y$ with $\tilde{m}|[\Delta Y] = j|[\Delta Y]$. We can extend $\tilde{m}$ to $\tilde{m}'$: $[Y^n] \cup [\Delta Z] \to Z$ so that $\tilde{m}'|[\Delta Z] = j$.

Combining Step 2 and Step 3 of the foregoing, we obtain a retraction, say r: $Z^n/T \to [Y^n] \cup [\Delta Z]$. Thus $\tilde{m}'r$: $Z^n/T \to Z$ is an extension of $\tilde{m}'$. It is clear that $\tilde{m}'r\nu$ is an n-mean on Z extending m.

Corollary 5.1. *Let (X, A) be a finite dimensional* AR *pair with A compact. If f: $A \to Y$ is a map, then $X \cup_f Y$ admits an n-mean if and only if Y does, $n \geq 2$.*

Corollary 5.2. *Let (X, A) be a finite dimensional* AR *pair, (Y, A) a pair, and $Z = X \cup Y$. If $X \cap Y = A$, then Z admits an n-mean if and only if Y does, $n \geq 2$.*

Corollary 5.3. *Let $(X_1, A_1), \ldots, (X_k, A_k)$ be a sequence of k pairs of finite dimensional* AR's *and f_1: $A_1 \to Y$ a closed map onto a closed, metrizable subspace of Y. Let $Z_1 = X_1 \cup_{f_1} Y$, and for each i, $2 \leq i \leq k$, let $Z_i = X_i \cup_{f_i} Z_{i-1}$, where f_i: $A_i \to Z_{i-1}$ is a closed map onto a closed, metrizable subspace of Z_{i-1}. Then Z_k admits an n-mean if and only if y does, $n \geq 2$.*

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Reçu par la Rédaction le 05. 11. 1981