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COVERINGS OF STABLY PARALLELIZABLE MANIFOLDS

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Let M, $\tilde{M}$ be smooth, compact, stably parallelizable manifolds such that $\tilde{M} \to M$ is a finite covering. One may ask how framings on M, their invariants and properties are related to those of $\tilde{M}$. In this note we consider the property of being the boundary of a parallelizable manifold.

We investigate the following construction. Given a d-fold covering $\tilde{N} \to N$ with N stably parallelizable and a homotopy sphere Σ, we have the covering

$$\tilde{N} \# \underbrace{\Sigma \# \ldots \# \Sigma}_{d \text{ times}} \to N \# \Sigma.$$

Under some conditions on $\tilde{N}$ and Σ, the manifold $\tilde{N} \# \Sigma \# \ldots \# \Sigma$ bounds a parallelizable manifold but $N \# \Sigma$ does not. This provides a series of examples of coverings such that the covering manifold admits a framing which is a framed boundary, but the covered manifold does not. Our construction shows that, given a covering $\tilde{M} \to M$, a mere change of differential structure often leads to an example of this type.

The following problem is especially interesting for compact Lie groups:

(*) If $\tilde{M} \to M$ is a finite covering, is M the boundary of a parallelizable manifold provided that $\tilde{M}$ is such a boundary?

Since Guest and Pastor [3] proved that most of 1-connected Lie groups are boundaries of parallelizable manifolds, an affirmative answer to (*) for Lie groups would imply that all compact Lie groups are such boundaries. Note that [4] implies that for coverings of even degree of Lie groups the answer is "yes".

We are also motivated by a question investigated in [2], namely the relation between the e-invariant of M (which is a compact Lie group G) and of $\tilde{M}$.

We assume that all manifolds considered are compact, connected and oriented. First we construct a family of $(2m)$-fold coverings of stably parallelizable manifolds where $\tilde{M}$ admits a framing which gives the zero element in Ω^{fr} but no framing of M bounds a framed manifold.

To prove our first theorem we use the generalized arithmetic genus

$$\alpha\colon\ \Omega_n^{\mathrm{Spin}} \to \widetilde{\mathrm{KO}}(S^n)$$

(cf. [8]). For a framing φ on M let $\alpha(M, \varphi)$ denote the α-genus of M with the spin structure determined by φ.

THEOREM 1. *Let $n = 8k+1,\ 8k+2$ and $\tilde{N}^n \to N^n$ be a d-fold covering of n-dimensional, stably parallelizable manifolds and $d = \operatorname{ord}\pi_n^S$. Assume that $\tilde{N}^n$ bounds a parallelizable manifold and N^n admits a Riemannian metric of positive scalar curvature. Let Σ^n be an exotic sphere with non-zero α-genus. Then in the covering*

$$\tilde{N}^n \# \underbrace{\Sigma^n \# \ldots \# \Sigma^n}_{d\ times} \to N^n \# \Sigma^n$$

the manifold $\tilde{N}^n \# d\cdot\Sigma^n$ bounds a parallelizable manifold but $N^n \# \Sigma^n$ does not.

Proof. Let φ be a framing $M^n = N^n \# \Sigma^n$. Then there exist a framing φ_1 on Σ^n and $\varphi+(-\varphi_1)$ on N^n such that

$$(M^n, \varphi) = \big(N^n, \varphi+(-\varphi_1)\big)+(\Sigma^n, \varphi_1) \quad \text{in } \Omega_n^{\mathrm{fr}}.$$

The above fact is obtained by a standard construction ([6], Section 4) of the framed connected sum of two framed manifolds that corresponds to the sum in π_n^S. We get the equality

$$N^n = N^n \# S^n = N^n \# \Sigma^n \# (-\Sigma^n) = M^n \# (-\Sigma^n),$$

and thus we can write

$$\varphi = [\varphi+(-\varphi_1)]+\varphi_1.$$

Since N^n admits a Riemannian metric of positive scalar curvature, we have $\alpha(N^n, \varphi-\varphi_1) = 0$ by the Hitchin–Lichnerowicz theorem ([5], [7]). Hence $\alpha(M^n, \varphi) \neq 0$, so (M^n, φ) is not a framed boundary.

On the other hand, by assumption there exists a framing $\tilde{\psi}$ on $\tilde{N}^n$ such that $(\tilde{N}^n, \tilde{\psi}) = 0$ in Ω_n^{fr}. Now, for any framing ψ on Σ^n we have

$$(\tilde{N}^n \# \underbrace{\Sigma^n \# \ldots \# \Sigma^n}_{d\ \text{times}}, \tilde{\psi} \# \underbrace{\psi \# \ldots \# \psi}_{d\ \text{times}}) = (\tilde{N}^n, \tilde{\psi})+d\cdot(\Sigma^n, \psi) = 0 \quad \text{in } \pi_n^S.$$

EXAMPLE. It is shown in [8] and [1] that for $n = 8k+1,\ 8k+2$ there exists an exotic sphere Σ^n such that $\alpha(\Sigma^n) \neq 0$. Now the following gives examples of manifolds satisfying the assumptions of Theorem 1:

$(n = 9)$: $N^9 = L(\operatorname{ord}\pi_9^S;\ q_1, q_2, q_3, q_4)\times S^2$, where $L(p;\ q_1, q_2, q_3, q_4)$ denotes the $(2n-1)$-dimensional lens space with $\pi_1 = Z_p$ and $\tilde{N}^9 = S^7\times S^2$ is the standard $(\operatorname{ord}\pi_9^S)$-fold covering of N^9;

$(n = 10)$: $N^{10} = L(\operatorname{ord}\pi_{10}^S;\ q_1, q_2, q_3, q_4)\times S^3$, and so on.

Now we will consider coverings of odd degree.

THEOREM 2. *Let* $n \not\equiv 0 \pmod 4$, $\tilde{N}^n \to N^n$ *be a p-fold covering, where* $\tilde{N}^n$, N^n *are stably parallelizable manifolds and* N^n *bounds a parallelizable manifold. Assume that a prime* $p > 2$ *divides* $\operatorname{ord} \pi_n^S$, $|\operatorname{im} J| \not\equiv 0 \pmod p$ *and that*

$$\prod_{i=1}^{n-1} \operatorname{ord} H^i(N^n, \pi_i \mathrm{SO}) < p-1.$$

Then there exists an exotic sphere Σ^n *such that in the covering*

$$\tilde{N}^n \# p \cdot \Sigma^n \to N^n \# \Sigma^n$$

the covering manifold $\tilde{N}^n \# p \cdot \Sigma^n$ *bounds a parallelizable manifold but* $N^n \# \Sigma^n$ *does not.*

Proof. Choose a cyclic subgroup $G < \pi_n^S$ of order p. Since p does not divide $\operatorname{ord}(\operatorname{im} J)$, G projects isomorphically onto a cyclic subgroup of $\pi_n^S/\operatorname{im} J$ (which will be denoted also by G).

The orbit of the cobordism class $[\varphi_0]$ under the action of $[N^n, \mathrm{SO}]$ consists of all framed cobordism classes on N^n. The obstructions to extend a homotopy between two maps $N^n \to \mathrm{SO}$ belong to $H^i(N^n, \pi_i \mathrm{SO})$. If we choose a cell decomposition of N^n with only one n-cell, then for any two maps $N^n \to \mathrm{SO}$ which are equal on the $(n-1)$-skeleton we can find a map $f\colon S^n \to \mathrm{SO}$ such that one of these maps is given by twisting the other on the n-cell by f. Hence there are framings $\varphi_1, \ldots, \varphi_l$ of N, where

$$l \leqslant \prod_{i=1}^{n-1} \operatorname{ord} H^i(N^n, \pi_i \mathrm{SO}),$$

such that any framing on N is of the form

$$(N^n, \varphi_i) \# (S^n, f),$$

where $i = 1, 2, \ldots$ or l and (S^n, f) is the image of $[f] \in \pi_n \mathrm{SO}$ by the J-homomorphism.

Now we see that there exists $g \in G \subset \pi_n^S/\operatorname{im} J$ such that its sum with any element $(N^n, \varphi) \in \pi_n^S/\operatorname{im} J$ is non-zero.

Since the Wall group of surgery obstruction in dimensions considered is 0 or Z_2, g is represented by some exotic sphere Σ^n. The above calculations show that $N^n \# \Sigma^n$ with arbitrary framing is non-zero in Ω_n^{fr}. But $\tilde{N}^n \# p \cdot \Sigma^n$ bounds a parallelizable manifold, because $\tilde{N}^n$ does and $\operatorname{ord}(g) = p$. Thus

$$\tilde{N}^n \# p \cdot \Sigma^n \to N^n \# \Sigma^n$$

satisfies the assertion of the theorem.

EXAMPLE. The smallest dimension possible is $n = 38$. Then there exists an element β_1 of order $p = 5$ in π_{38}^S (the notation β_s for $s \not\equiv 0 \pmod p$) is from

[9] and [10]). Thus $\tilde{N}^{38} = S^1 \times S^{37}$ is a 5-fold covering of itself,

$$\prod_{i=1}^{37} \operatorname{ord} H^i(N^{38}, \pi_i SO) = 2 < p-1$$

and there exists Σ^{38} such that for

$$\tilde{N}^{38} \# 5 \cdot \Sigma^{38} \to N^{38} \# \Sigma^{38}$$

the covering manifold bounds a parallelizable manifold but the base manifold does not. Another series can be obtained by using elements $\lambda' \in \pi_n^S$ (see [9] and [10]), where

$$n = 2(2p^2+1)(p-1)-5 \equiv 1 \pmod 4.$$

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