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CAUCHY'S PROBLEM FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL COEFFICIENTS

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1. Introduction. Let $A = (a_{ij})$ be a matrix such that $\hat{a}_{ij}$ is a continuous function of finite variation in R^1 and $a_{ij} = (\hat{a}_{ij})'$ for $i, j = 1, 2, \ldots, n$, and let f be a vector all components of which are measures. In this note we consider the system of equations

(i) $$y' = Ay+f,$$

where y is an unknown vector. The derivative is understood in the distributional sense. We prove some theorems on the existence and uniqueness of solutions of Cauchy's problem for system (i). All distributions considered in this paper are real in R^1. The sequential theory of distributions is used (see [4]).

I am indebted to P. Antosik for his suggestions which have improved the presentation of this paper.

2. The mean value of a distribution at a point and integrals.

Definition 1. By a *δ-sequence* we mean (see [3] and [4], p. 75) a sequence of smooth non-negative scalar functions $\{\delta_k\}$ satisfying

(a) $\int_{-\infty}^{\infty} \delta_k(x)dx = 1$,

(b) $\delta_k(x) = 0$ for $|x| \geqslant a_k$, where $\{a_k\}$ is a sequence of positive numbers with $a_k \to 0$,

(c) $\delta_k(x) = \delta_k(-x)$.

Definition 2. By a *regular sequence* for a distribution u we mean any sequence of the form

$$\varphi_k(x) = (u*\delta_k)(x) = \int_{-\infty}^{\infty} u(x-t)\,\delta_k(t)\,dt,$$

where $\{\delta_k\}$ is a δ-sequence (see [4], p. 117 and 153).

Definition 3 (see [3]). A distribution u takes the *mean value a at a point x_0* if the distribution

$$v(x) = \frac{1}{2}[u(x+x_0)+u(-x+x_0)]$$

takes the value a at 0 in the sense of [4], p. 240, i.e.

$$\lim_{k\to\infty}\varphi_k(0) = a$$

for each regular sequence $\{\varphi_k\}$ for v.

We introduce the notation

$$(2.1)\qquad \int_a^b u(x+t)\,dt = \psi(x+b)-\psi(x+a), \quad \text{where } \psi' = u.$$

The mean value of the distribution $\int_a^b u(x+t)\,dt$ at $x = 0$, if exists, will be denoted (see [3]) by

$$(2.2)\qquad \int_a^b u(t)\,dt.$$

Definition 4. By a *non-negative distribution* we understand a distribution for which there exists a fundamental sequence whose terms are non-negative functions.

We write $u \geqslant v$ if the difference $u-v$ is a non-negative distribution (see [2]).

Definition 5 ([2], p. 718). If, for every regular sequence $\{\varphi_k\}$ for a distribution u, the sequence $\{|\varphi_k|\}$ is distributionally convergent, then we say that the *modulus $|u|$ of u exists* and put

$$|u| = \lim_{k\to\infty}(d)\,|\varphi_k|.$$

The consistency of Definition 5 follows from the fact that the interlaced sequence of two δ-sequences is also a δ-sequence.

In the sequel, when saying that u is a function of finite variation, we mean that u is of bounded variation on each finite interval and that

$$u(x) = \frac{1}{2}[\lim_{y\to x^+} u(y)+\lim_{y\to x^-} u(y)] \quad \text{for every } x.$$

In this case the mean value of u at each point x equals $u(x)$.

We shall need the following properties of integral (2.2) which are proved in [3]. We assume that all integrals appearing exist. Then we can write

$$\int_a^b (\alpha u(x)+\beta v(x))\,dx = \alpha\int_a^b u(x)\,dx+\beta\int_a^b v(x)\,dx \tag{2.3}$$

for all numbers α and β.

If $u \leqslant v$ and $a \leqslant b$, then

$$\int_a^b u(x)\,dx \leqslant \int_a^b v(x)\,dx. \tag{2.4}$$

Moreover, if u is a measure, i.e. u is the derivative (of the first order in the distributional sense) of a function of finite variation, then, for all a and b, the integral $\int_a^b u(x)\,dx$ exists and

$$\left|\int_a^b u(x)\,dx\right| \leqslant \int_a^b |u(x)|\,dx. \tag{2.5}$$

The following property is of great importance: if u is a function of finite variation, then

$$u(x) = \int_a^x u'(t)\,dt+u(a). \tag{2.6}$$

3. Main results. At first we shall introduce some notation. Throughout this paper the small letters y, z, f, g, o will stand for n-dimensional vectors and the capital letters A, B, Y, Z will denote matrices of type $n\times n$. If $A = (a_{ij})$ is a matrix whose all elements are measures, $y = (y_1, \ldots, y_n)$ is a vector all components of which are also measures, and $\{\delta_k\}$ is a δ-sequence, then we put

$$a_{ijk} = a_{ij}*\delta_k, \quad A_k = A*\delta_k = (a_{ijk}), \quad \hat{A}_k = \int_{x_0}^x A_k = \Big(\int_{x_0}^x a_{ijk}\Big),$$

$$y_k = y*\delta_k = (y_{1k}, \ldots, y_{nk}), \quad \hat{y}_k = \int_{x_0}^x y_k = \Big(\int_{x_0}^x y_{1k}, \ldots, \int_{x_0}^x y_{nk}\Big),$$

$$k = 1, 2, \ldots, \quad \|A\| = \sum_{i,j=1}^n |a_{ij}|, \quad |y| = \sum_{i=1}^n |y_i|.$$

Moreover, by V we denote the class of all vectors all components of which are functions of finite variations.

Definition 6 (cf. [4], p. 242). We say that the *product of distributions u and v exists* if the sequence $\{(u*\delta_k)(v*\delta_k)\}$ is distributionally convergent for every δ-sequence $\{\delta_k\}$.

THEOREM 1. *Let A be a matrix whose elements are derivatives of the first order (in the distributional sense) of continuous functions of finite* *variation. Then the problem*

$$y' = Ay, \quad y(x_0) = 0 \tag{ii}$$

has only the zero solution in the class V.

Proof. Suppose that y is a non-zero solution of problem (ii) and that $y \in V$. Since the elements of A are derivatives of the first order of continuous functions of finite variation, there exists an interval (a, b) such that

$$\int_a^b \|A(t)\| dt < 1, \tag{3.1}$$

$x_0 \in (a, b)$, and the vector y is continuous at the points a and b. Having integrated (ii), from (2.3)-(2.6) we obtain the inequality

$$\sum_{i=1}^{n} \sup_{x \in [a,b]} |y_i(x)| \leqslant \Big(\sum_{i=1}^{n} \sup_{x \in [a,b]} |y_i(x)|\Big) \int_a^b \|A(t)\| dt. \tag{3.2}$$

If y is a non-zero vector whose all components are functions of finite variation in (a, b), then, in view of (3.2), we get

$$\int_a^b \|A(t)\| dt \geqslant 1$$

which contradicts (3.1). Thus our assertion follows.

Remark 1. The assumption of the continuity of $\hat{a}_{ij}$ in Theorem 1 is essential. This is seen from the following

Example 1. Consider the problem

$$y' = 2\delta y, \quad y(-1) = 0, \tag{3.3}$$

where δ denotes Dirac's delta distribution. From the equality $HH = H$, where H denotes Heaviside's distribution, we obtain $H\delta = \frac{1}{2}\delta$ (1). Since we also have $H' = \delta$, it follows that the distribution $y = H$ is a solution of problem (3.3).

THEOREM 2. *Let A satisfy the assumptions of Theorem 1. Then the problem*

$$y' = Ay, \quad y(x_0) = y^0 \tag{iii}$$

has exactly one solution in the class V.

Proof. Let $A_k = A * \delta_k$. We consider the sequence $\{\tilde{g}_k\}$ defined by

$$\tilde{g}_k(x) = \int_{x_0}^{x} A_k(s)\tilde{g}_k(s) ds + y^0. \tag{3.4}$$

(1) This equality has been observed by P. Antosik.

Hence, by Bellman's inequality, we have

$$(3.5)\qquad |\tilde{g}_k(x)| \leqslant |y^0| \exp\left| \int_{x_0}^{x} \|A_k(s)\|\, ds \right| .$$

Similarly,

$$(3.6)\qquad |\tilde{g}_k(x) - \tilde{g}_l(x)| \leqslant u_l^k(I) \exp\left| \int_{x_0}^{x} \|A_l(s)\|\, ds \right| ,$$

where I is an arbitrary compact interval, $x \in I$ and

$$u_l^k(I) = \max_{x \in I} \left| \|\hat{A}_k(s) - \hat{A}_l(s)\| \, |\tilde{g}_k(s)| \Big|_{x_0}^{x} \right| + \\ + \max_{x \in I} \left| \int_{x_0}^{x} \|\hat{A}_k(s) - \hat{A}_l(s)\| \left| (\tilde{g}_k(s))' \right| ds \right| .$$

From (3.5) and (3.6) we infer that $\{\tilde{g}_k\}$ is almost uniformly convergent in R^1. Put

$$g = \lim_{k\to\infty} \tilde{g}_k, \qquad g_k = g * \delta_k .$$

In view of (3.4), we have $g(x_0) = y^0$. Moreover, for every number $\varepsilon > 0$ there exists an integer k_0 such that

$$(3.7)\qquad \left| \int_{x_0}^{x} A_k(s) \big(g_k(s) - \tilde{g}_k(s)\big) ds \right| \leqslant \varepsilon \left| \int_{x_0}^{x} \|A_k(s)\|\, ds \right|$$

for all $k > k_0$ and $x \in I$. Hence, by [1], p. 259, we have

$$\lim_{k\to\infty} (d) A_k \tilde{g}_k = Ag .$$

Thus g is a solution of problem (iii). An application of Theorem 1 completes the proof.

Example 2. The problem

$$(3.8)\qquad y' = 2\delta y, \quad y(-1) = 1$$

has no solution in V. In fact, if we suppose to the contrary that y is a solution of problem (3.8) in V, then $y = cH + c_1$, where c and c_1 are arbitrary constants. From the initial condition we have $y = cH + 1$, so that y does not satisfy (3.8).

Example 3. The sequence of solutions of the problems

$$(3.9)\qquad y' = \delta_k y, \quad y(-1) = 1$$

is not convergent to the solution of the problem

$$(3.10)\qquad y' = \delta y, \quad y(-1) = 1 .$$

In fact, it is seen that $y = \exp(H_k)$, where $H_k = H*\delta_k$, is a solution of problem (3.9). Since

$$\lim_{k\to\infty}(d)\exp(H_k) = \exp(H),$$

then

$$\delta\exp(H) = (e-1)H\delta+\delta = \frac{e+1}{2}\delta.$$

On the other hand,

$$\big(\exp(H)\big)' = \big((e-1)H+1\big)' = (e-1)\delta.$$

Hence $\exp(H)$ is not a solution of problem (3.10). One can easily verify that the distribution $2H+1$ is a solution of problem (3.10).

We adopt the definition of a fundamental matrix of solutions of system (ii) analogous to that of [7], p. 137. Then one can show (cf. [7], Theorem 4, p. 137) that

THEOREM 3. *If a matrix A satisfies the assumptions of Theorem* 1, *then there exists a fundamental matrix of solutions of system* (ii).

THEOREM 4. *Let A satisfy the assumptions of Theorem* 1 *and let f be a vector all components of which are measures. Then the problem*

$$y' = Ay+f, \quad y(x_0) = y^0 \tag{iv}$$

has exactly one solution in the class V.

At first we prove

LEMMA. *Suppose that Z is a matrix whose elements are continuous functions of finite variation and y is a vector whose components are functions of finite variation. If A satisfies the assumptions of Theorem* 1, *then*

$$A(Zy) = (AZ)y. \tag{3.11}$$

Proof. Let $Y = AZ$, $h = Zy$, $A_k = A*\delta_k$, $Z_k = Z*\delta_k$, $y_k = y*\delta_k$, $Y_k = Y*\delta_k$ and $h_k = h*\delta_k$. Then

$$\int_{x_0}^{x}\big(A_k(s)Z_k(s)-A_l(s)Z_l(s)\big)ds = \big(\hat{A}_k(s)-\hat{A}_l(s)\big)Z_k(s)\big|_{x_0}^{x} - \tag{3.12}$$
$$-\int_{x_0}^{x}\big(\hat{A}_k(s)-\hat{A}_l(s)\big)\big(Z_k(s)\big)'ds + \int_{x_0}^{x}\big(Z_k(s)-Z_l(s)\big)A_l(s)ds.$$

Hence $\Big\{\int_{x_0}^{x} A_k(s)Z_k(s)ds\Big\}$ is almost uniformly convergent to the matrix

$$\lim_{k\to\infty}\int_{x_0}^{x} Y_k(s)ds.$$

From the equalities

$$(3.13)\qquad Ah = (\hat{A}h)' - \hat{A}h',$$

$$(3.14)\qquad Yy = (\hat{Y}y)' - \hat{Y}y'$$

and [1], p. 259, we infer that both sides of (3.11) make sense. Moreover,

$$(3.15)\qquad \int_{x_0}^{x} y_k(s)[A_k(s)Z_k(s) - Y_k(s)]ds = y_k(s)B_k(s)|_{x_0}^{x} - \int_{x_0}^{x} B_k(s)\big(y_k(s)\big)'ds,$$

where

$$B_k(x) = \int_{x_0}^{x} \big(A_k(s)Z_k(s) - Y_k(s)\big)ds.$$

By (3.15), we get

$$(3.16)\qquad \lim_{k\to\infty}(d)A_kZ_ky_k = (AZ)y.$$

On the other hand,

$$(3.17)\qquad A_kZ_ky_k - A_kh_k = [\hat{A}_k(Z_ky_k - h_k)]' - \hat{A}_k(Z_ky_k - h_k)'.$$

Since

$$(3.18)\qquad \int_{x_0}^{x} \hat{A}_k(s)[Z_k(s)y_k(s) - h_k(s)]'ds = \int_{x_0}^{x} \hat{A}_k(x)[Z_k(s)y_k(s) - h_k(s)]'ds + \\ + \int_{x_0}^{x} \big(\hat{A}_k(s) - \hat{A}_k(x)\big)[Z_k(s)y_k(s) - h_k(s)]'ds,$$

by (3.17) we have

$$(3.19)\qquad \lim_{k\to\infty}(d)A_kZ_ky_k = A(Zy).$$

Relations (3.16) and (3.19) complete the proof of the Lemma.

Remark 2. The assumption of the continuity of the elements of Z in the Lemma is essential. In fact,

$$(3.20)\qquad H(H\delta) \neq (HH)\delta.$$

Proof of Theorem 4. Uniqueness assertion results from Theorem 1. Thus it is enough to prove the existence of a solution of problem (iv). Let $Z = (z_{ij})$ be the fundamental matrix of system (ii) such that $z_{ij}(x_0) = 0$

for $i \neq j$ and $z_{ii}(x_0) = 1$ for $i, j = 1, 2, \ldots, n$. We claim that

$$y = Zg, \tag{3.21}$$

where

$$g' = Z^{-1}f, \quad g(x_0) = y^0 \tag{3.22}$$

(Z^{-1} denotes the inverse matrix of Z) is a solution of problem (iv). In fact, using (3.21), the Lemma and [1], p. 261, we obtain

$$y' = Z'g + Zg' = (AZ)g + Z(Z^{-1}f) = A(Zg) + f. \tag{3.23}$$

Hence Zg satisfies (iv). Moreover, by (3.21) and (3.22), we get $y(x_0) = y^0$ which implies our assertion.

Example 4. The non-homogeneous equation

$$y' = [-2\delta(x) + 2\delta(x-1)]y + \delta(x) \tag{3.24}$$

has no solutions in the class V. In fact, if there exists a solution y in V, then

$$y = c_1 H(x) + c_2 H(x-1) + H(x) + c_3. \tag{3.25}$$

From (3.24) and (3.25) we get

$$c_1 + c_3 = -\tfrac{1}{2}, \quad c_1 + c_3 = -1$$

which is of course impossible.

Remark 3. The distributional solutions of systems of ordinary linear differential equations with smooth coefficients can be found in [5]. In paper [6] some homogeneous systems of equations with distributional coefficients are considered. But the existence of the product of a measure and a continuous function does not result from the definition of the product of two distributions in [6]. Moreover, all theorems of that paper are published without proofs.

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