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THE MINIMUM DIMENSION OF A RESIDUAL ANR

BY

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1. Introduction and preliminaries. Let M be a compact connected topological n-manifold. If M has empty boundary, let M_B denote M with a small open n-cell deleted; otherwise let $M_B = M$. A subset R of M is said to be *residual* in M if $M_B - R$ is a topological copy of $\mathrm{bdry}\, M_B \times (0, 1]$ which is dense in M_B. If, in addition, M_B is homeomorphic with the mapping cylinder of some surjection $\mathrm{bdry}\, M_B \to R$, then R is said to be *strongly residual* in M. In [3] it was asked whether the minimum dimension of a set residual in M is the same as the minimum dimension of a set strongly residual in M (P 820). In the present report, it is shown how recent results of several researchers can be used to shed new light on this question.

If K is a finite complex locally tamely embedded in M_B as a strong deformation retract with the inclusion map (a simple homotopy equivalence), then K is said to be a *spine* of M.

Caution. In [3] the word "spine" was used to mean what is termed here "strongly residual set". Nomenclature is changed in this report to conform to the terminology which has become generally accepted in the past few years.

A spine which is also a residual set bears some resemblance to a strongly residual set: in the latter case, the requirement concerning simple homotopy equivalence (defined in terms of elementary collapses) is replaced by the requirement that there be a global "elementary topological collapse", i.e. a pseudoisotopic deformation of M_B onto R (along the fibres of the mapping cylinder structure). It is clear that either condition imposes stringent additional requirements on a set which is merely assumed to be residual.

2. Dimension and connectivity. If the manifold M admits only $(n-1)$-dimensional residual sets, there is of course no problem concerning the minimum dimensions of the various types of its residual sets. This trivial observation gains significance from the following, which appears as Theorem 2 of [1]:

THEOREM (Doyle). *Let M be a closed manifold which has a residual set of codimension 2 or greater. Then M is simply connected.*

Before this theorem had appeared, Summerhill [6] had proved a kind of generalization which required the residual set to be rather well-behaved:

THEOREM (Summerhill). *Let $0 \leqslant k \leqslant n-3$. A closed n-dimensional manifold is k-connected if and only if it has a residual Z_{k-1}-set (necessarily of codimension $k+1$ or greater).*

Using Doyle's result, one of the inferences in Summerhill's Theorem can be deduced under considerably weakened hypotheses:

THEOREM 1. *Let M be a closed n-manifold. Suppose M has a residual ANR of codimension k or greater. Then M is $(k-1)$-connected.*

Proof. If $k = 1$, there is nothing to prove (M is assumed to be connected). If $k > 1$, we note that (by Doyle's Theorem) M is simply connected, and hence orientable.

Let $j < k$. Since R is an ANR of dimension $n-j$ or smaller, $\bar{H}^{n-j}(R) = H^{n-j}(R) = 0$. ($\bar{H}^*$ denotes Alexander-Čech cohomology, H^* singular cohomology over the integers.) The orientability of M enables us to use Alexander duality to infer that $H_j(M, M-R) = 0$. But $M-R$ is topologically an n-cell, so $H_j(M, M-R) = H_j(M, \text{point}) = H_j(M)$. An application of the Hurewicz Isomorphism Theorem proves that $\pi_j(M) = 0$.

COROLLARY 1. *Let $0 \leqslant k \leqslant n-3$. If the closed n-manifold M has a residual ANR of codimension $k+1$ or greater, then M has a residual Z_{k-1}-set.*

Proof. Direct from Theorem 1 and Summerhill's Theorem.

THEOREM 2. *Let $0 \leqslant k \leqslant n-3$ and $n < 2k+1$. If the closed n-manifold M has a residual ANR of codimension $k+1$ or greater, then M is an n-sphere.*

Proof. Using Corollary 1, we find a residual Z_{k-1}-set X for M whose dimension [5] does not exceed $n-k-1$. The General Position Theory of Summerhill [5] can now be further applied to push X off itself *via* a small homeomorphism h so that $X \cap h(X)$ is empty. M is thus represented as the union of two open n-cells, the complements of X and of $h(X)$.

Note. The inequalities in the hypotheses of Corollary 2 force $n \geqslant 6$.

3. Residual ANR's and residual spines. If, instead of Summerhill's Theorem, we apply Pedersen's Existence Theorem (Corollary 3 of [4]) to the result of Theorem 1, we obtain

THEOREM 3. *Let M be a closed n-manifold having a residual ANR of codimension $k+1$. If $n \geqslant 6$ and $k \geqslant 3$, then M has a spine of codimension k.*

Since M is simply connected, so also is its spine X whose existence is guaranteed by this theorem. Thus $Wh(\pi_1 X) = 0$ and the uniqueness portion of the Topological Tubular Neighbourhood Theorem [2] shows that X is residual in M.

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[5] R. R. Summerhill, *General position for closed subsets of euclidean space*, General Topology and its Applications 3 (1973), p. 339-345.

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