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ON RANEY'S THEOREMS FOR COMPLETELY DISTRIBUTIVE COMPLETE LATTICES

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In this note we prove the theorems of G. N. Raney characterizing completely distributive complete lattices, using the notion of semifilter of sets and their associated lower and upper limits. The proofs in this paper are simpler than the original ones.

Let L be a complete lattice: its minimum and maximum elements are denoted by 0 and 1, respectively. Let $\mathscr{A}$ be a family of subsets of L. Define the *lower limit* of $\mathscr{A}$ by

$$\operatorname{Lim\,inf} \mathscr{A} = \bigvee \{\bigwedge A\colon\ A \in \mathscr{A}\}.$$

Dually, define the *upper limit* of $\mathscr{A}$ by

$$\operatorname{Lim\,sup} \mathscr{A} = \bigwedge \{\bigvee A\colon\ A \in \mathscr{A}\}.$$

Let $\mathscr{A}$, $\mathscr{B}$ be families of subsets of L. The family $\mathscr{A}$ is *coarser* than $\mathscr{B}$ ($\mathscr{B}$ is *finer* than $\mathscr{A}$; $\mathscr{A} \leqslant \mathscr{B}$) if for every $A \in \mathscr{A}$ there is $B \in \mathscr{B}$ such that $B \subset A$. Families $\mathscr{A}$, $\mathscr{B}$ are said to be *equivalent* ($\mathscr{A} \equiv \mathscr{B}$) if $\mathscr{A} \leqslant \mathscr{B}$ and $\mathscr{B} \leqslant \mathscr{A}$. Evidently, $\mathscr{A} \leqslant \mathscr{B}$ implies

$$\operatorname{Lim\,inf} \mathscr{A} \leqslant \operatorname{Lim\,inf} \mathscr{B}$$

in L and, dually,

$$\operatorname{Lim\,sup} \mathscr{A} \geqslant \operatorname{Lim\,sup} \mathscr{B}.$$

In particular, if $\mathscr{A} \equiv \mathscr{B}$, then

$$\operatorname{Lim\,inf} \mathscr{A} = \operatorname{Lim\,inf} \mathscr{B} \quad \text{and} \quad \operatorname{Lim\,sup} \mathscr{A} = \operatorname{Lim\,sup} \mathscr{B}.$$

The *grill* $\mathscr{A}^{\#}$ of a family of subsets of L is defined by

$$\mathscr{A}^{\#} = \{B \subset L\colon\ B \cap A \neq \emptyset \text{ for every } A \in \mathscr{A}\}.$$

It is clear that $\mathscr{A} \leqslant \mathscr{B}$ if and only if $\mathscr{A}^{\#} \supset \mathscr{B}^{\#}$. It follows from the definition of grill that

(1) $$\mathscr{A} \subset \mathscr{B}^{\#} \Rightarrow \operatorname{Lim\,inf} \mathscr{A} \leqslant \operatorname{Lim\,sup} \mathscr{B}.$$

In particular, $\operatorname{Lim\,inf} \mathscr{A} \leqslant \operatorname{Lim\,sup} \mathscr{A}^{\#}$ because $\mathscr{A} \subset (\mathscr{A}^{\#})^{\#}$. A complete lattice L is said to be *completely distributive* [2] if for every family $\mathscr{A}$ of subsets of L

$$\operatorname{Lim\,inf} \mathscr{A} = \operatorname{Lim\,sup} \mathscr{A}^{\#}$$

holds. A more familiar and equivalent definition of complete distributivity is given in [1].

Let $\mathscr{A}$ be a family of subsets of L. The family $\mathscr{A}$ is a *semifilter* on L if $A \in \mathscr{A}$ and $A \subset B \subset L$ imply $B \in \mathscr{A}$. In other words, $\mathscr{A}$ is a semifilter on L if and only if $\mathscr{A}^{\#\#} = \mathscr{A}$. More generally, for any $\mathscr{A}$ the family $\mathscr{A}^{\#\#}$ is the smallest semifilter containing $\mathscr{A}$; hence $\mathscr{A}^{\#\#} \equiv \mathscr{A}$. The set $\operatorname{sf} L$ of all semifilters on L is a complete ring of sets. The operator of grill is a dual automorphism of $\operatorname{sf} L$ which is an involution.

Let x be an element of a complete lattice L. The semifilters $\mathscr{N}_-(x)$ and $\mathscr{N}_+(x)$ are defined by

(2) $$\mathscr{N}_-(x) = \{A \subset L:\ \bigvee A^c \ngeqslant x\} \quad \text{and} \quad \mathscr{N}_+(x) = \{A \subset L:\ \bigwedge A^c \nleqslant x\}.$$

It can be shown that $\mathscr{N}_-(x)$ is the intersection of all semifilters $\mathscr{A}$ on L such that $\operatorname{Lim\,inf} \mathscr{A} \geqslant x$; and, dually, $\mathscr{N}_+(x)$ is the intersection of those which verify $\operatorname{Lim\,sup} \mathscr{A} \leqslant x$.

LEMMA 1. *A complete lattice L is completely distributive if and only if, for every $x \in L$, $\operatorname{Lim\,inf} \mathscr{N}_-(x) = x$.*

Proof. We observe first that for every $x \in L$

(3) $$\mathscr{N}_-(x)^{\#} = \{A \subset L:\ \bigvee A \geqslant x\} \quad \text{and} \quad \operatorname{Lim\,sup} \mathscr{N}_-(x)^{\#} = x.$$

Now, if L is completely distributive, we have

$$\operatorname{Lim\,inf} \mathscr{N}_-(x) = \operatorname{Lim\,sup} \mathscr{N}_-(x)^{\#}.$$

Hence by (3) we infer that for every $x \in L$

$$\operatorname{Lim\,inf} \mathscr{N}_-(x) = x.$$

Next, we observe that, for every family $\mathscr{A}$ of subsets of L, $\operatorname{Lim\,sup} A^{\#} = x$ implies $\mathscr{A}^{\#} \subset \mathscr{N}_-(x)^{\#}$, because

$$\bigwedge \{\bigvee A:\ A \in \mathscr{A}^{\#}\} \geqslant x.$$

Hence

(4) $$\operatorname{Lim\,sup} \mathscr{A}^{\#} = x \Rightarrow \mathscr{A} \geqslant \mathscr{N}_-(x).$$

Suppose now that for every $x \in L$

$$\operatorname{Lim\,inf} \mathscr{N}_-(x) = x.$$

We will show that L is completely distributive. Let $\mathscr{A}$ be a family of subsets of L, and $x = \operatorname{Lim\,sup} \mathscr{A}^{\#}$. By (4) we obtain $\mathscr{A} \geqslant \mathscr{N}_{-}(x)$. Hence

$$x = \operatorname{Lim\,inf} \mathscr{N}_{-}(x) \leqslant \operatorname{Lim\,inf} \mathscr{A}. \tag{5}$$

But $\operatorname{Lim\,inf} \mathscr{A} \leqslant \operatorname{Lim\,sup} \mathscr{A}^{\#} = x$, therefore by (5) we have

$$\operatorname{Lim\,inf} \mathscr{A} = \operatorname{Lim\,sup} \mathscr{A}^{\#};$$

that is, L is completely distributive.

LEMMA 2. *Let L be a complete lattice and $\{\mathscr{A}_i\}_i \subset \mathrm{sf}\, L$. Then*

$$\operatorname{Lim\,inf}(\bigcup \mathscr{A}_i) = \bigvee \operatorname{Lim\,inf} \mathscr{A}_i.$$

Moreover, if L is completely distributive, then

$$\operatorname{Lim\,inf}(\bigcap \mathscr{A}_i) = \bigwedge \operatorname{Lim\,inf} \mathscr{A}_i.$$

The proof of this lemma is immediate.

THEOREM 1 (Raney [5], Theorem 1). *Every completely distributive lattice is a complete image of a complete ring of sets.*

Proof. Let L be a completely distributive lattice. The lattice $\mathrm{sf}\, L$ is a complete ring of sets. By Lemma 2, L is a complete image of $\mathrm{sf}\, L$, because

$$\operatorname{Lim\,inf}\colon \mathrm{sf}\, L \to L$$

is a complete homomorphism.

Let L be a complete lattice. A subset M of L is called a *lower semi-ideal* of L if, for every $x \in M$, $[0, x] \subset M$.

THEOREM 2 (Raney [5], Theorem 1). *Let $K(x)$ be the intersection of the family of lower semi-ideals M of L such that $\bigvee M \geqslant x$. Then L is completely distributive if and only if, for every x, $\bigvee K(x) = x$.*

Proof. Since, for every $x \in L$,

$$K(x) = \{\bigwedge A\colon A \in \mathscr{N}_{-}(x)\},$$

Lemma 1 implies the required characterization of the complete distributivity.

THEOREM 3 (Raney [7], Theorems 4 and 5). *Let L be a complete lattice. The following properties are equivalent*:

(a) *L is completely distributive*;

(b) *for every $x \in L$,*

$$\bigvee \{\bigwedge [0, y]^c\colon y \in [x, 1]^c\} = x;$$

(c) *for every pair $x, y \in L$ with $x \nleqslant y$ there exist $x', y' \in L$ such that $x' \nleqslant y$, $x \nleqslant y'$ and*

$$[0, y'] \cup [x', 1] = L. \tag{6}$$

Proof. Since $\mathscr{N}_{-}(x) \equiv \{[0, y]^c\colon y \in [x, 1]^c\}$, Lemma 1 implies that (a) and (b) are equivalent. Now we prove that (a) $\Rightarrow$ (c). Let $x, y \in L$ with $x \nleqslant y$.

By Lemma 1 and its dual, we have

$$\operatorname{Lim\,inf}\mathcal{N}_-(x) \nleqslant \operatorname{Lim\,sup}\mathcal{N}_+(y).$$

Hence, by (1), there are $A \in \mathcal{N}_-(x)$ and $B \in \mathcal{N}_+(y)$ such that $A \cap B = \emptyset$. Setting $y' = \bigvee A^c$ and $x' = \bigwedge B^c$, by (2) we obtain $y' \ngeqslant x$ and $x' \nleqslant y$. Since

$$[0, y'] \supset A^c \quad \text{and} \quad [x', 1] \supset B^c,$$

by $A \cap B = \emptyset$ we have (6). Therefore, the implication (a) $\Rightarrow$ (c) is proved. Now, using Lemma 1 we verify that (c) $\Rightarrow$ (a). Let $x \in L$. Since

$$\operatorname{Lim\,inf}\mathcal{N}_-(x) \leqslant \operatorname{Lim\,sup}\mathcal{N}_-(x)^{\#},$$

by (3) we have $\operatorname{Lim\,inf}\mathcal{N}_-(x) \leqslant x$. Let $x' \nleqslant \operatorname{Lim\,inf}\mathcal{N}_-(x)$ and y' satisfy (6). Using the inequality

$$x' \nleqslant \operatorname{Lim\,inf}\mathcal{N}_-(x),$$

we have $[x', 1]^c \in \mathcal{N}_-(x)^{\#}$; and by (6) we obtain $[x', 1]^c \subset [0, y']$. Since $\mathcal{N}_-(x)^{\#}$ is a semifilter on L, we infer that $[0, y'] \in \mathcal{N}_-(x)^{\#}$. Hence by (3) we obtain $y' \geqslant x$. Therefore, for every x', $y' \in L$, $x' \nleqslant \operatorname{Lim\,inf}\mathcal{N}_-(x)$ and (6) imply $y' \geqslant x$. Hence by (c)' we obtain

$$x \leqslant \operatorname{Lim\,inf}\mathcal{N}_-(x).$$

Since we have verified that $x \geqslant \operatorname{Lim\,inf}\mathcal{N}_-(x)$, Lemma 1 gives (c) $\Rightarrow$ (a).

THEOREM 4 (Raney [6], Theorem A, and [7], Theorem 7, Papert Strauss [4], Theorem 6). *Every completely distributive complete lattice is isomorphic to a closed sublattice of a product of copies of the unit interval* $[0, 1] \subset \boldsymbol{R}$.

Proof (sketch). Let Ω be the set of all complete homomorphisms from a complete lattice L to $[0, 1]$. For every $x \in L$, let $\Phi(x)\colon \Omega \to [0, 1]$ be defined for every $\omega \in \Omega$ by

$$\Phi(x)(\omega) = \omega(x).$$

Φ is a complete homomorphism from L to $[0, 1]^{\Omega}$. If L is completely distributive, then Φ is also a one-to-one mapping of L into $[0, 1]^{\Omega}$. In fact, if $x, y \in L$ with $x \nleqslant y$, by induction the property (c) (see Theorem 3) implies that there exist $\{a_s\}_{s \in D}$, $\{b_s\}_{s \in D} \subset L$ (D is the set of all dyadic rational numbers belonging to $[0, 1]$) such that

$$a_0 = 0, \quad a_1 = x, \quad b_0 = y, \quad b_1 = 1,$$

$$[0, b_s] \cup [a_s, 1] = L \quad \text{for every } s \in D,$$

$$a_s \nleqslant b_t \quad \text{for every } s, t \in D \text{ with } s > t.$$

Moreover, since

$$\sup\{s \in D\colon x \geqslant a_s\} = \inf\{s \in D\colon x \leqslant b_s\}$$

for every $x \in L$, it is easy to verify that ω: $L \to [0, 1]$, defined for every $x \in L$ by

$$\omega(x) = \sup\{s \in D: \ x \geqslant a_s\},$$

is a complete homomorphism such that $\omega(x) = 1$ and $\omega(y) = 0$. Therefore, for a completely distributive lattice L the mapping Φ is a complete isomorphism between L and the complete lattice of real-valued functions $L^* = \{\Phi(x): \ x \in L\}$.

Remark. Other characterizations of the completely distributive complete lattices are also obtained by the author, using notions of lower limit and upper limit [2] and that of fuzzy integral [3].

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