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LOCAL HOMEOMORPHISMS ONTO TREE-LIKE CONTINUA

BY

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All spaces considered in this paper are compact metric, and all mappings are continuous surjective. A continuum is a compact connected space.

A *simple chain* is a finite collection of open sets $G_1, G_2, \ldots, G_k$ such that G_i intersects G_j if and only if $|i-j| \leqslant 1$. If the links of a chain are of diameter less than ε, the chain is called an *ε-chain*. A continuum is called *arc-like* if for each positive number ε it can be covered by an ε-chain. Sometimes the arc-like continua are also called *chainable* or *snake-like*.

A collection $G_1, G_2, \ldots, G_k$ is a *circular chain* if it has more than two links and G_i intersects G_j whenever either $|i-j| \leqslant 1$ or $|i-j| = k-1$. A collection $\mathscr{G}$ is *coherent* if it has more than two links and if, for each proper subcollection $\mathscr{G}'$ of it, an element of $\mathscr{G}'$ intersects an element of $\mathscr{G} \setminus \mathscr{G}'$. A finite coherent collection $\mathscr{G}$ of open sets is called a *tree-chain* if no three of the open sets have a point in common and no subcollection of $\mathscr{G}$ is a circular chain. A continuum is called *tree-like* if for each positive number ε there is a tree-chain covering it so that each element of the tree chain is of diameter less than ε (for the above-mentioned definitions see [1], p. 653).

Recall that every *λ-dendroid*, i.e., a hereditarily decomposable and hereditarily unicoherent continuum (see [3], Theorem 1, p. 16), is tree-like (see [4], Corollary, p. 20), whence it follows, in particular, that every *dendrite* (i.e., a locally connected continuum without a simple closed curve) is also tree-like.

A continuous mapping f from a space X onto a space Y is said to be

(i) a *local homeomorphism* if for every point $x \in X$ there exists an open neighbourhood U of x such that $f(U)$ is an open neighbourhood of $f(x)$, and f restricted to U is a homeomorphism between U and $f(U)$ (see [14], p. 199);

(ii) *open* if f maps every set open in X onto a set open in Y.

We have the following characterization of local homeomorphisms (see [10], Theorem 4, p. 856, and [14], (6.21), p. 200):

PROPOSITION. *A mapping $f: X \to Y$ of X onto a continuum Y is a local homeomorphism if and only if f is an open mapping and there is a natural number n such that* $\operatorname{card} f^{-1}(y) = n$ *for each $y \in Y$.*

It is known that a local homeomorphism of a continuum onto a dendrite (a λ-dendroid) is a homeomorphism (see [14], Corollary, p. 199, and [10], Theorem 9; cf. also [9], Théorème 3, p. 56). Similarly, a local homeomorphism of an arc-like continuum is a homeomorphism (see [13], Theorem 2.0, p. 261). Some other results of this kind are contained in [5] and in [11] (see also [2]).

We face the problem: does it follow that if a local homeomorphism f maps a continuum onto a tree-like continuum, then f is a homeomorphism? (See [10], Problem 12, p. 858; [11], Problem 14; and cf. [13], Question 4, p. 262.)

The following theorem answers this question. This theorem is a consequence of Theorem (6.1) of [6], but the methods used in the proof are different. The author is very much indebted to Professor A. Lelek who paid the author's attention to Fox's theorem.

THEOREM. *Each local homeomorphism of a continuum onto a tree-like continuum is a homeomorphism.*

Proof. Let a local homeomorphism f map a continuum X onto a tree-like continuum Y. Since f is a local homeomorphism and since X is compact, we conclude that

(1) There exists a positive δ such that, for every two different points $x, x' \in X$, $f(x) = f(x')$ implies $\varrho(x, x') \geqslant 4\delta$.

Further, by the Proposition we have

(2) There is a natural n such that $\operatorname{card} f^{-1}(y) = n$ for each $y \in Y$.

Moreover,

(3) There exists a positive ε such that, for any $y, y' \in Y$, it follows from $\varrho(y, y') < \varepsilon$ that for each $x \in f^{-1}(y)$ there is $x' \in f^{-1}(y')$ satisfying $\varrho(x, x') < \delta$, where δ is given as in (1).

In fact, let y be an arbitrary point of Y and assume that $f^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$ (cf. (2)). Since f is a local homeomorphism, there are open sets $U_1, U_2, \ldots, U_n$ such that $x_i \in U_i$, U_i is of diameter less than $\delta/2$ (cf. (1)), $f(U_i)$ is an open set and $f|U_i$ is a homeomorphism for any $i = 1, 2, \ldots, n$. Put

$$V = \bigcap_{i=1}^{n} f(U_i).$$

Since $y \in V$ and V is open, we infer that there is a positive ε_y such that $B(y, \varepsilon_y) \subset V$, where $B(y, \varepsilon_y)$ denotes an open ball in Y with the centre in y

and with the radius ε_y. Let $\varrho(y, y') < \varepsilon_y$. Then $y' \in B(y, \varepsilon_y) \subset V$. Since

$$f\big(U_i \cap f^{-1}(V)\big) = V,$$

we conclude that, for each $i = 1, 2, \ldots, n$,

$$f^{-1}(y') \cap \big(U_i \cap f^{-1}(V)\big) \neq \emptyset.$$

Therefore, for each $x \in f^{-1}(y)$ there is $x' \in f^{-1}(y')$ satisfying $\varrho(x, x') < \delta/2$, since U_i is of diameter less than $\delta/2$.

For each $y \in Y$ take an open ball $B(y, \varepsilon_y)$, where ε_y is as before. Since Y is compact, there is a finite covering

$$B(y_1, \varepsilon_{y_1}), B(y_2, \varepsilon_{y_2}), \ldots, B(y_m, \varepsilon_{y_m}).$$

It follows from Corollary 4d in [8], § 41, VI, p. 24, that there is a positive ε such that if $\varrho(y, y') < \varepsilon$, then $y, y' \in B(y_j, \varepsilon_{y_j})$ for some $j = 1, 2, \ldots$ $\ldots, m$. This ε satisfies the required condition (3). Indeed, if $\varrho(y, y') < \varepsilon$, then $y, y' \in B(y_j, \varepsilon_{y_j})$ for some $j = 1, 2, \ldots, m$. Therefore,

$$\varrho(y, y_j) < \varepsilon_{y_j} \quad \text{and} \quad \varrho(y', y_j) < \varepsilon_{y_j}.$$

Thus for each $x'' \in f^{-1}(y_j)$ there are $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$ such that $\varrho(x'', x) < \delta/2$ and $\varrho(x'', x') < \delta/2$. Since

$$\operatorname{card} f^{-1}(y_j) = \operatorname{card} f^{-1}(y) = \operatorname{card} f^{-1}(y') = n$$

(cf. (2)), we infer that for each $x \in f^{-1}(y)$ there is (even exactly one) $x' \in f^{-1}(y')$ such that $\varrho(x, x') < \delta$. Thus (3) holds.

Let $\mathscr{G} = \{G_1, G_2, \ldots, G_k\}$ be a tree-chain covering Y such that

(4) G_i is of diameter less than ε for $i = 1, 2, \ldots, k$

and let $b_1, b_2, \ldots, b_k$ be a sequence of points of Y such that

(5) $b_i \in G_i \setminus (G_1 \cup G_2 \cup \ldots \cup G_{i-1} \cup G_{i+1} \cup \ldots \cup G_k)$ for $i = 1, 2, \ldots, k$.

Put, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$ (cf. (2)),

$$f^{-1}(b_i) = \{a_1^i, a_2^i, \ldots, a_n^i\} \quad \text{and} \quad H_j^i = f^{-1}(G_i) \cap B(a_j^i, \delta),$$

where $B(z, \delta)$ denotes an open ball in X with the centre in z and with the radius δ.

We have

(6) $f(H_j^i) = G_i$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$.

In fact, the definition of H_j^i implies that $f(H_j^i) \subset G_i$. Let $y \in G_i$. Then $\varrho(y, b_i) < \varepsilon$ by (4) and (5). Since $a_j^i \in H_j^i$ and $f(a_j^i) = b_i$, we conclude that there is $x \in f^{-1}(y)$ satisfying $\varrho(x, a_j^i) < \delta$ by (3). Therefore, $x \in f^{-1}(y)$ and $x \in B(a_j^i, \delta)$. Thus

$$x \in f^{-1}(y) \cap f^{-1}(G_i) \cap B(a_j^i, \delta), \quad \text{i.e.,} \quad x \in f^{-1}(y) \cap H_j^i.$$

This implies that $y \in f(H_j^i)$. Consequently, $G_i \subset f(H_j^i)$. Hence $f(H_j^i) = G_i$, i.e., (6) holds.

Further,

(7) $H_j^i \cap H_t^i = \emptyset$ for $j \neq t$, $j, t = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, k$.

Suppose, on the contrary, that $z \in H_j^i \cap H_t^i$ for i, j, and t as in (7). Then

$$\varrho(a_j^i, z) < \delta \quad \text{and} \quad \varrho(a_t^i, z) < \delta.$$

Therefore $\varrho(a_j^i, a_t^i) < 2\delta$. Since $f(a_j^i) = f(a_t^i)$, we obtain a contradiction by (1), since $j \neq t$.

We have also

(8) If $H_j^i \cap H_t^s \neq \emptyset$, then $H_j^i \cap H_w^s = \emptyset$ for $t \neq w, i \neq s$, $i, s = 1, 2, \ldots, k$ and $j, t, w = 1, 2, \ldots, n$.

Suppose, on the contrary, that $z \in H_j^i \cap H_t^s$ and $z' \in H_j^i \cap H_w^s$ for i, j, s, t, and w as in (8). Then

$$\varrho(a_t^s, z) < \delta, \quad \varrho(a_j^i, z) < \delta, \quad \varrho(a_j^i, z') < \delta \quad \text{and} \quad \varrho(a_w^s, z') < \delta.$$

Therefore $\varrho(a_t^s, a_w^s) < 4\delta$. Since $f(a_t^s) = f(a_w^s)$, and $t \neq w$, we obtain a contradiction by (1).

(9) If $G_i \cap G_s \neq \emptyset$, then for any $j = 1, 2, \ldots, n$ there are $t = 1, 2, \ldots$ $\ldots, n$ such that $H_j^i \cap H_t^s \neq \emptyset$, where $i, s = 1, 2, \ldots, k$.

In fact, if $z \in G_i \cap G_s$, then $\varrho(z, b_s) < \varepsilon$. It follows from (6) that there is $z' \in f^{-1}(z) \cap H_j^i$. From (3) we infer that $\varrho(z', a_t^s) < \delta$ for some $t = 1, 2, \ldots$ $\ldots, n$. Therefore, $z' \in H_j^i \cap H_t^s$ by the definition of H_t^s, since $z' \in f^{-1}(z)$ $\subset f^{-1}(G_s)$.

(10) If $H_j^i \cap H_t^s \cap H_w^r \neq \emptyset$, then either $i = s$ and $j = t$, or $s = r$ and $t = w$, or $i = r$ and $j = w$, where $i, s, r = 1, 2, \ldots, k$ and $j, t, w = 1, 2, \ldots, n$.

Indeed, if $H_j^i \cap H_t^s \cap H_w^r \neq \emptyset$, then $G_i \cap G_s \cap G_r \neq \emptyset$ by (6). Thus either $i = s$ or $s = r$ or $i = r$ (cf. the definition of a tree-chain). Assume that $i = r$. Then $j = w$ by (7). Therefore (10) holds.

Moreover, we have

(11) The collection $\mathscr{H} = \{H_j^i : i = 1, 2, \ldots, k \text{ and } j = 1, 2, \ldots, n\}$ is an open cover of X.

It is clear that sets H_j^i are open. It suffices to show that $\mathscr{H}$ is a cover of X. Let $x \in X$. Then $y = f(x) \in G_i$ for some $i = 1, 2, \ldots, k$. Therefore

$\varrho(y, b_i) < \varepsilon$; we conclude from (3) that for each $j = 1, 2, \ldots, n$ there is $x_j \in f^{-1}(y)$ such that $\varrho(x_j, a_j^i) < \delta$. Condition (1) implies that points $x_1, x_2, \ldots, x_n$ are different. Thus $x = x_j$ for some $j = 1, 2, \ldots, n$, since $\operatorname{card} f^{-1}(y) = n$ (cf. (2)) and $y = f(x)$. Hence we have $x \in H_j^i$. This means that $\mathscr{H}$ covers X.

Let P be a nerve of $\mathscr{G}$ and let $p_1, p_2, \ldots, p_k$ be vertices of P which correspond to $G_1, G_2, \ldots, G_k$, respectively (for the definition of a nerve see [7], § 28, V, p. 318). Since $\mathscr{G}$ is a tree-chain, we infer that

(12) P is a one-dimensional polyhedron containing no simple closed curve.

Similarly, let Q be a nerve of $\mathscr{H}$ and let q_j^i be a vertex of Q which corresponds to H_j^i, where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$. By (10) and (11) we conclude that

(13) Q is a one-dimensional polyhedron.

We define a mapping h of Q onto P as follows: $h(q_j^i) = p_i$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$, and h is a homeomorphism of the simplex $q_j^i q_t^s$ onto the simplex $p_i p_s$ provided $G_i \cap G_s \neq \emptyset$ and t is as in (9) (such t is exactly one by (8)).

It follows from the definition of h that h is continuous and $\operatorname{card} h^{-1}(p) = n$ for each $p \in P$ (moreover, one can prove that h is a local homeomorphism). Since h maps a continuum onto a dendrite (cf. (12) and (13)), we infer that $n = 1$ by Theorem 8 in [11] (cf. [10], Theorem 9, p. 857, and [14], Corollary, p. 199). Therefore, f is a homeomorphism (cf. (2)). The proof of the Theorem is complete.

Since a locally homeomorphic image of a tree-like continuum is tree-like (see [12], p. 472; cf. our Proposition), by the Theorem we have the following corollary:

COROLLARY. *Each local homeomorphism of a tree-like continuum is a homeomorphism.*

REFERENCES

[1] R. H. Bing, *Snake-like continua*, Duke Mathematical Journal 18 (1951), p. 653-663.

[2] K. Borsuk and R. Molski, *On a class of continuous mappings*, Fundamenta Mathematicae 45 (1957), p. 84-98.

[3] J. J. Charatonik, *On decompositions of λ-dendroids*, ibidem 67 (1970), p. 15-30.

[4] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, ibidem 68 (1970), p. 19-22.

[5] S. Eilenberg, *Sur quelques propriétés des transformations localement homéomorphes*, ibidem 24 (1935), p. 35-42.

[6] R. H. Fox, *On shape*, ibidem 74 (1972), p. 47-71.
[7] K. Kuratowski, *Topology*, Vol. I, New York - London - Warszawa 1966.
[8] — *Topology*, Vol. II, New York - London - Warszawa 1968.
[9] A. Lelek, *Sur l'unicohérence, les homéomorphies locales et les continus irréductibles*, Fundamenta Mathematicae 45 (1957), p. 51-63.
[10] T. Maćkowiak, *A note on local homeomorphisms*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 21 (1973), p. 855-858.
[11] — *Mappings of constant degree*, ibidem 23 (1975), p. 885-891.
[12] T. B. McLean, *Confluent images of tree-like curves are tree-like*, Duke Mathematical Journal 39 (1972), p. 465-473.
[13] I. Rosenholtz, *Open maps of chainable continua*, Proceedings of the American Mathematical Society 42 (1974), p. 258-264.
[14] G. T. Whyburn, *Analytic topology*, New York 1942.

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