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CHORD-DECREASING HOMEOMORPHISMS IN EUCLIDEAN SPACES

BY

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1. Introduction. Let $D^2 \subset E^2$ be a topological disc bounded by a simple closed curve $C = \partial D^2$ in the plane E^2 and let

$$\varphi: C \to E^2$$

be a homeomorphism (into) with

$$\|p-q\| \leqslant \|\varphi(p)-\varphi(q)\| \quad \text{for each pair } p, q \in C,$$

where $\|\cdot\|$ denotes the Euclidean norm.

Steinhaus [2] raised the following question: Is the area $|D^2|$ of D^2 less than or equal to the area of the domain bounded by the simple closed curve $\varphi(C)$?

In this note we give a positive answer to this question.

2. Preliminaries. Let $D^n \subset E^n$ be an n-dimensional topological ball lying in the Euclidean space E^n. Let $oe_1 \ldots e_n$ denote a fixed orthonormal frame in E^n. If $l \subset E^n$ is an arbitrary straight-line, then every component of $l \cap D^n$ is called a *chord* of the boundary ∂D^n of D^n. A chord is determined by its end points $p, q \in \partial D^n$. A chord parallel to e_k is called a *k-chord*, $1 \leqslant k \leqslant n$.

Definition. A homeomorphism

$$\varphi: D^n \to E^n \tag{2.1}$$

is called *chord-decreasing* if for every chord pq with $p, q \in \partial D^n$ we have

$$\|p-q\| \geqslant \|\varphi(p)-\varphi(q)\|. \tag{2.2}$$

The transformation (2.1) is called *k-chord-decreasing* if (2.2) is satisfied for every k-chord, $1 \leqslant k \leqslant n$.

The main result of this paper is the following

CHORD THEOREM. *If the homeomorphism* (2.1) *is k-chord-decreasing, then the* (*Lebesgue*) *measure of the domain bounded by* $\varphi(\partial D^n)$ *is less than or equal to that of* D^n.

The ball $D^n \subset E^n$ can be arbitrarily closely approximated by a ball whose boundary is an analytic surface (see [3]). Since the measure of the approximation can be made arbitrarily close to that of D^n we can suppose that the ball $D^n \subset E^n$ itself has an analytic boundary and that (2.1) is an analytic diffeomorphism. Since ∂D^n is an analytic manifold, the set

$$\partial D_k^n = \{p \in \partial D^n \mid e_k \text{ is tangent to } \partial D^n \text{ at } p \in \partial D^n\} \tag{2.3}$$

for every k with $1 \leqslant k \leqslant n$ is an $(n-2)$-dimensional compact submanifold of ∂D^n.

By B_η^n we denote a tubular neighbourhood of ∂D^n, where η is the radius of the one-dimensional ball $B_\eta(p)$ orthogonal to ∂D^n at $p \in \partial D^n$. For the definition and existence of tubular neighbourhoods see [1].

Since ∂D^n is a smooth submanifold of E^n, on ∂D^n there exists a measure density induced from E^n, and since ∂D^n is compact, we have

$$|D^n|_{n-1} < \infty, \tag{2.4}$$

where $|\cdot|_{n-1}$ denotes the measure on ∂D^n defined by the introduced measure density. Since ∂D_k^n defined by (2.3) are compact $(n-2)$-dimensional submanifolds of ∂D^n, we obtain

$$\Big|\bigcup_{k=1}^{n} D_k^n\Big|_{n-1} = 0.$$

Similarly, $\varphi(\partial D_k^n)$ are compact $(n-2)$-dimensional submanifolds on $\varphi(\partial D^n)$ and, therefore, with respect to the measure defined on $\varphi(\partial D^n)$ by the measure density induced from E^n, we have

$$\Big|\bigcup_{k=1}^{n} \varphi(\partial D_k^n)\Big|_{n-1} = 0. \tag{2.5}$$

3. Finite nets. By s^k, $1 \leqslant k \leqslant n$, we denote a *$k$-dimensional segment*, i.e., a set of E^n isometric with

$$\{x \in E^n \mid 0 \leqslant x_i \leqslant a_i,\ 1 \leqslant i \leqslant k,\ x = (x_1, \ldots, x_k, 0, \ldots, 0),\ a_i \in R_+\}.$$

We write

$$d_1 = \max(a_1, \ldots, a_k) \quad \text{and} \quad d_2 = \min(a_1, \ldots, a_k). \tag{3.1}$$

Let us take n families N_k, $1 \leqslant k \leqslant n$, of straight lines in E^n with the following properties:

(i) all straight lines of N_k are parallel to e_k;

(ii) for a positive number δ, the straight lines of all the families N_k, $1 \leqslant k \leqslant n$, define a partition of E^n into n-dimensional segments such that the numbers (3.1) for $k = n$ satisfy $d_1 < \delta$ and $d_2 > \delta/2$ for every segment of that partition.

Let us take any finite number of n-dimensional segments, defined by (i) and (ii), whose union is a connected set denoted by $F^n(\delta)$.

A finite net is said to be *inscribed in* D^n if $F^n(\delta)$ is the union of all n-dimensional segments contained in D^n and the vertices of $F^n(\delta)$ belong to ∂D^n.

If δ is sufficiently small, then such finite nets exist.

Indeed, if B^n_η denotes a tubular neighbourhood of ∂D^n in E^n, then it suffices to set $\delta = \eta$. This assures the connectedness of $F^n(\delta)$. Since d_1 and d_2, $\delta/2 < d_2 \leqslant d_1 < \delta$, are arbitrary, we can achieve that the vertices of $F^n(\delta)$ belong to ∂D^n.

Into the finite net $F^n(\delta)$ inscribed in D^n we introduce, for $n \geqslant 3$, the structure of a polytope, denoted by $PF^n(\delta)$, as follows. The vertices of $PF^n(\delta)$ are exactly those of all $(n-1)$-dimensional segments contained in the boundary $\partial F^n(\delta)$ of $F^n(\delta)$. Every $(n-1)$-dimensional segment of $\partial F^n(\delta)$ is decomposed into $(n-1)$-dimensional simplexes whose vertices are those of the segment. These $(n-1)$-dimensional simplexes are called *faces* of $PF^n(\delta)$. For $n = 2$ the above construction is not necessary.

We have

$$\operatorname*{Lim}_{\delta\to 0} F^n(\delta) = \operatorname*{Lim}_{\delta\to 0} PF^n(\delta) = D^n$$

and

$$\lim_{\delta\to 0} |PF^n(\delta)| = |D^n|, \tag{3.2}$$

where $|\cdot|$ denotes the measure in E^n.

Now we define a piecewise linear mapping $\tilde{\varphi}$ of $PF^n(\delta)$ onto a polytope denoted by $\tilde{\varphi}PF^n(\delta)$. Let us take the restriction of (2.1) to the vertices of $PF^n(\delta)$. Since the faces of $PF^n(\delta)$ are simplexes, they are uniquely determined by their vertices and, therefore, every face of $PF^n(\delta)$ corresponds to the simplex spanned by the images of its vertices. This $(n-1)$-dimensional simplex is a face of $\tilde{\varphi}PF^n(\delta)$. That assignment of simplexes defines, by the use of barycentric coordinates, the piecewise linear mapping

$$\tilde{\varphi}: PF^n(\delta) \xrightarrow{\text{onto}} \tilde{\varphi}PF^n(\delta) \tag{3.3}$$

and we have

$$\operatorname*{Lim}_{\delta\to 0} \tilde{\varphi}PF^n(\delta) = \varphi(D^n), \quad \lim_{\delta\to 0} |\tilde{\varphi}PF^n(\delta)| = |\varphi(D^n)|. \tag{3.4}$$

Examples. Consider finite nets inscribed in a manifold D^2 under a chord-decreasing homeomorphism $\varphi: D^2 \to E^2$. The homeomorphism φ restricted to the vertices of $F^2(\delta)$ inscribed in D^2 induces a transformation $\tilde{\varphi}$ of the finite net $F^2(\delta)$.

1. Let us take the finite net $ABCDEF$ as shown in Fig. 1a. Its chords are AB, AC, CD, DH, FG. The area of the domain bounded by the polygon $A'B'C'D'E'F'$ in Fig. 1b is larger than that of the domain bounded

by $ABCDEF$ in Fig. 1a, though lengths of the chords satisfy the inequalities

$$|AB| \geqslant |A'B'|, \quad |AC| \geqslant |A'C'|, \quad |CD| \geqslant |C'D'|, \quad |DH| \geqslant |D'H'|,$$
$$|FG| \geqslant |F'G'|.$$

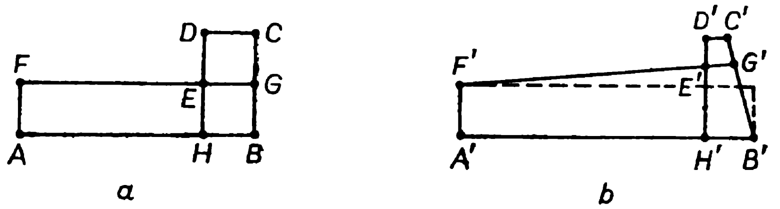

Fig. 1

2. Let us take the ellipse in two different positions with respect to the standard frame e_1, e_2, namely in the canonical position (Fig. 2a) defined by the equation

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1,$$

and in the position which arises from the canonical one by a rotation about the origin of the frame for the angle $\pi/4$ (Fig. 2b).

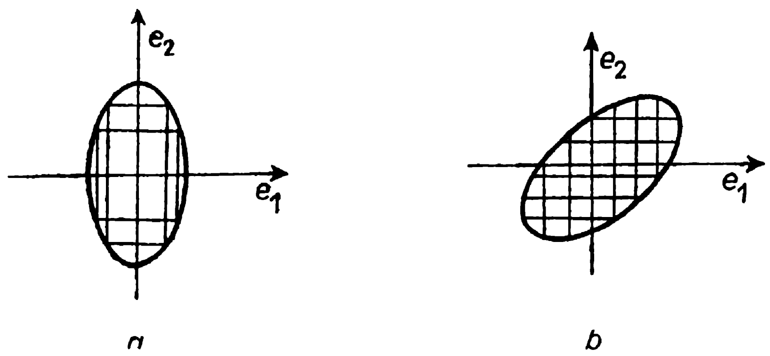

Fig. 2

In the case $n = 2$ we denote by $\tilde{\varphi}F^2(\delta)$ the polygon which is the union of all chords $\varphi(p)\varphi(q)$, where pq is a k-chord of $F^2(\delta)$, $k = 1, 2$. Let $A(F^2(\delta))$ denote the area of $F^2(\delta)$, and $A(\tilde{\varphi}F^2(\delta))$ the area of the domain bounded by $\tilde{\varphi}F^2(\delta)$.

For every chord-decreasing homeomorphism φ of the ellipse in the position as in Fig. 2a and for every finite net $F^2(\delta)$ inscribed in the ellipse we have

$$A(F^2(\delta)) \geqslant A(\tilde{\varphi}F^2(\delta)).$$

This is true for any convex domain symmetric with respect to a coordinate line defined by e_1 or e_2.

For δ sufficiently small there exist finite nets $F^2(\delta)$ inscribed in the ellipse in the position as in Fig. 2b, for which

$$A\left(F^2(\delta)\right) < A\left(\tilde{\varphi}F^2(\delta)\right)$$

for some chord-decreasing homeomorphism φ. This can be checked by applying the effect shown in Example 1.

We call the number

$$\Delta F^2(\delta) = \sup_{\varphi}\left[A\left(\tilde{\varphi}F^2(\delta)\right) - A\left(F^2(\delta)\right)\right]$$

the *area defect* of a finite net $F^2(\delta)$, where φ denotes a chord-decreasing homeomorphism. The area defect of any finite net inscribed in the ellipse shown in Fig. 2a is equal to zero, and that inscribed in the ellipse in Fig. 2b can be different from zero. From the proof of the chord theorem given in the next section it follows that for a topological disc D^2 and $F^2(\delta)$ inscribed in D^2 we have

$$\lim_{\delta\to 0} \Delta F^2(\delta) = 0.$$

The exact value of $\Delta F^2(\delta)$ for a fixed finite net $F^2(\delta)$ is not known. The concept of the area defect can be generalized to the concept of the volume defect in the case $n \geqslant 3$.

4. Proof of the chord theorem. Let $\varphi: B^n \to E^n$, $D^n \subset B^n$, denote an analytic diffeomorphism of a ball such that φ restricted to D^n is chord-decreasing. If $\varphi: D^n \to E^n$ is defined, then B^n can be defined with the use of the tubular neighbourhood B^n_η of ∂D^n by adding to D^n the exterior normal vectors multiplied by η, and $\varphi: B^n \to E^n$ denotes then an extension of $\varphi: D^n \to E^n$ to segments of length η lying on the vectors exterior to D^n with values in segments lying on vectors normal to $\varphi(D^n)$. Moreover, we suppose that the map (3.3) does not decrease the volume of the polytope $PF^n(\delta)$.

For every k, $1 \leqslant k \leqslant n$, we take in ∂D^n a tubular neighbourhood $B^{n-1}_{\eta_1,k}$ of the set ∂D^n_k defined by (2.3). We consider the set

$$D^n \setminus \bigcup_{k=1}^{n} B^{n-1}_{\eta_1,k}. \tag{4.1}$$

Take any $(n-1)$-dimensional segment $s^{n-1} \subset \partial F^n(\delta)$ of the partition defined by (i) and (ii) and orthogonal to e_k. By c_a, $1 \leqslant a \leqslant 2^{n-1}$, we denote a k-chord of the family of those k-chords of $F^n(\delta)$ which pass through the vertices of s^{n-1}. We have $c_a = p_a q_a$, $p_a, q_a \in \partial D^n$, $1 \leqslant a \leqslant 2^{n-1}$. If p_a and q_a belong to the set (4.1) for $1 \leqslant a \leqslant 2^{n-1}$, then, by the definition of the set (4.1) and by (ii), there exists such a constant $K_1(\eta_1)$ that, uniformly on (4.1), for every two points $p_a, p_\beta \in \partial D^n$ defined above, i.e. for every segment $s^{n-1} \subset \partial F^m(\delta)$ defined by (i) and (ii) and the end points p_a constructed for this segment as above, and for sufficiently small $\delta > 0$ we have

$$\|p_a - p_\beta\| < K_1(\eta_1)\delta, \quad 1 \leqslant a, \beta \leqslant 2^{n-1}. \tag{4.2}$$

In the tubular neighbourhood B^n_η of ∂D^n let us choose a compact neighbourhood $\overline{B^n_{\bar\eta}}$ of ∂D^n, $\bar\eta < \eta$. We write

$$K_2 = \sup_{p \in \overline{B^n_{\bar\eta}}} \left|\frac{\partial\varphi_i}{\partial x_j}\right|, \qquad 1 \leqslant i, j \leqslant n,$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $p = (x_1, \ldots, x_n)$. There exists a $\delta_0 > 0$ such that $\delta \leqslant \delta_0$ and $\|p-q\| < \delta$, $p, q \in \partial D^n$, imply that the segment pq is contained in $\overline{B^n_{\bar\eta}}$. Then we have the estimation

$$\begin{aligned}\|\varphi(p) - \varphi(q)\| &\leqslant \sum_{i=1}^{n} |\varphi_i(p) - \varphi_i(q)| \\ &= \sum_{i,j=1}^{n} \left|\frac{\partial\varphi_i}{\partial x_j}\right| |x_j(p) - x_j(q)| \leqslant nK_2 \sum_{j=1}^{n} |x_j(p) - x_j(q)| \\ &\leqslant \sqrt{2}\, nK_2 \|p-q\| < \sqrt{2}\, nK_2\delta,\end{aligned} \tag{4.3}$$

where the derivatives $\partial\varphi_i/\partial x_j$ are evaluated at a point of the segment $pq \subset \overline{B^n_{\bar\eta}}$.

For a fixed integer a, $1 \leqslant a$, $\beta \leqslant 2^{n-1}$, and $K_1\delta \leqslant \delta_0$ the points $\varphi(p_\beta)$ are all contained in a ball $B_\varepsilon(\varphi(p_a))$ with the center at $\varphi(p_a)$ and the radius (see (4.2) and (4.3))

$$\varepsilon = \sqrt{2}\, K_1 K_2 n\delta. \tag{4.4}$$

Up to an isometry of E^n we can suppose that

$$\varphi(p_a) = p_a \quad \text{and} \quad \varphi(p_a)\varphi(q_a) \subset p_a q_a = c_a.$$

Since φ is chord-decreasing, the length of chords c_β cannot increase and they can only change their mutual position. Therefore, the increase of the volume of $PF^m(\delta)$, which arises from the action of φ at the end points p_β of the chords c_β, is less than the volume of the cube described over the ball $B_\varepsilon(p_a)$, hence less than $(2\varepsilon)^n$. We repeat the above considerations for all $(n-1)$-dimensional segments of $\partial F^m(\delta)$ such that the end points of the orthogonal chords to these segments which pass through their vertices belong to the set (4.1). Since all edges of segments defined by (i) and (ii) are greater than $\delta/2$, we have

$$|PF^m(\delta)| + |\partial F^m(\delta)|_{n-1}\left(\frac{2\varepsilon}{\delta}\right)^{n-1} \cdot 2\varepsilon \geqslant |\tilde\varphi PF^m(\delta)| + C(\eta_1, \delta), \tag{4.5}$$

where $C(\eta_1, \delta)$ is continuous with respect to δ at $\delta = 0$ and

$$\lim_{\delta\to 0} C(\eta_1, \delta) = -\left|\varphi\left(\bigcup_{k=1}^{n} B^{n-1}_{\eta_1,k}\right)\right|_{n-1}. \tag{4.6}$$

Put $C(\eta_1) = C(\eta_1, 0)$. We denote by $|\partial F^n(\delta)|_{n-1}$ the $(n-1)$-dimensional volume of $\partial F^n(\delta)$, i.e., the sum of volumes of $(n-1)$-dimensional segments contained in $\partial F^n(\delta)$. Let $\partial F^n_k(\delta)$ denote the union of all $(n-1)$-dimensional segments of $\partial F^n(\delta)$ which are orthogonal to e_k. Using (2.4) we have

$$|\partial D^n|_{n-1} > |\partial F^n_k(\delta)|_{n-1}. \tag{4.7}$$

Since

$$|\partial F^n(\delta)|_{n-1} = \sum_{k=1}^{n} |\partial F^n_k(\delta)|_{n-1},$$

by (4.4), (4.5), and (4.7) we have

$$|PF^n(\delta)| + n|\partial D^n|_{n-1}\left(2\sqrt{2}K_1K_2n\right)^n\delta \geqslant |\tilde{\varphi}PF^n(\delta)| + C(\eta_1, \delta). \tag{4.8}$$

If δ tends to zero, then from (4.8) with the use of (3.2), (3.3), (3.4), and (4.6) we infer that

$$|D^n| \geqslant |\varphi(D^n)| + C(\eta_1).$$

By (2.5) we have

$$\lim_{\eta_1\to 0} C(\eta_1) = 0.$$

Thus

$$|D^n| \geqslant |\varphi(D^n)|,$$

and the first part of the chord theorem is proved.

For the proof of the remainder let us suppose that for a fixed k, $1 \leqslant k \leqslant n$, there exists a k-chord $pq \subset D^n$ such that

$$\|p-q\| > \|\varphi(p)-\varphi(q)\|. \tag{4.9}$$

Since φ is continuous, we suppose that (4.9) is satisfied if p (respectively, q) varies in a neighbourhood $U \subset D^n$ (respectively, $V \subset D^n$), and that U and V are disjoint with the set $\bigcup_{k=1}^{n} B^{n-1}_{\eta_1,k}$ for sufficiently small η_1.

Let us take $\delta_1 > 0$ such that for some segment $s^{n-1} \subset \partial F^n(\delta)$ orthogonal to e_k the k-chords c_a $(1 \leqslant a \leqslant 2^{n-1})$ of $F^n(\delta_1)$ which pass through the vertices of s^{n-1} satisfy $c_a = p_a q_a$ and $p_a \in U$, $q_a \in V$. There exists a number $a > 0$ such that for every a, $1 \leqslant a \leqslant 2^{n-1}$,

$$\|p_a - q_a\| \geqslant \|\varphi(p_a)-\varphi(q_a)\| + a.$$

Inequality (4.8) remains valid if we add to its right-hand side the number $\delta_1^{n-1}\cdot a/2^{n-1}$. Hence, if δ and η_1 tend to zero, we get

$$|D^n| \geqslant |\varphi(D^n)| + \frac{\delta_1^{n-1}}{2^{n-1}}\cdot a$$

and, therefore,

$$|D^n| > |\varphi(D^n)|.$$

This completes the proof of the chord theorem.

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