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A REPRESENTATION OF THE CATEGORY OF ALGEBRAS OVER DIFFERENT MONADS BY THE CATEGORY OF ALGEBRAS OVER ONE SUITABLE MONAD IN A CATEGORY OF POINTED MONADS

BY

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In this paper we consider the categories $\mathrm{Alm}(A)$ and $\mathrm{Mon}_*(A)$ of all monad algebras and of all pointed monads, respectively, over an arbitrary 2-category A. For each pair $R = \langle M, B\rangle$ which has the property (S), where M is a subcategory of $\mathrm{Mon}_*(A)$ and B is a subcategory of $\mathrm{Alm}(A)$, we give the adjunction

$$\Sigma_R = \langle H_R, L_R, \eta_R^*, \varepsilon_R^*\rangle\colon M \rightharpoonup B$$

for which the comparison functor K_R from B to M^{D_R} is an isomorphism, where D_R is the monad defined by Σ_R, and M^{D_R} is the category of D_R-algebras in M. The pairs

$$\langle \mathrm{Mon}_*(A), \mathrm{Alm}(A)\rangle, \quad \langle \mathrm{Mon}_*(A)(X, -), \mathrm{Alm}(A)(X, -)\rangle,$$
$$\langle \mathrm{Mon}_*(A)(-, X'), \mathrm{Alm}(A)(-, X')\rangle,$$
$$\langle \mathrm{Mon}_*(A)(X, X'), \mathrm{Alm}(A)(X, X')\rangle,$$
$$\langle \mathrm{Mon}_*(A)(X, T), \mathrm{Alm}(A)(X, T)\rangle, \quad \langle \mathrm{Mon}_*(A)(-, T), \mathrm{Alm}(A)(-, T)\rangle,$$

where X and X' are any 0-cells in A and T is any monad in A, have the property (S).

For any 2-category A we also introduce the diagonal 2-category 2-$d_a A$ of any diagonal type a and the notion of right monad algebra in A. In Section 3 we give a category of R-automata, where R is a pair with the property (S), and we formulate a problem. In the last part of Section 2 we formulate a representation theorem for the category of all automata in the category of functors from X to X and we give other examples.

1. PRELIMINARY DEFINITIONS AND PROPOSITIONS

The reader is assumed to be familiar with the basic concepts from theory of categories [4].

1.1. Remarks on foundations. The foundations are based on standard Zermelo-Fraenkel axioms from the set theory with the notion of a universe and the following axiom (see [6]):

(i) *Every set is an element of a universe.*

Let U be any universe. A category C is a *U-category* if $\mathrm{Ob}(C)$ and $\mathrm{Mor}(C)$ are subclasses of U. A U-category C is *small* if $\mathrm{Ob}(C) \in U$ and $\mathrm{Mor}(C) \in U$; in the opposite case $-$ C is large. By (i) there exists a universe U_1 with $U \in U_1$. Hence every U-category C is a small U_1-category. The small sets over U, i.e. the elements $v \in U$, as objects with the mappings as morphisms, define a U-category $\mathrm{Set}(U)$. All sets over U, i.e. all $v \subset U$ with the mappings, define a U_1-category $\mathrm{SET}(U)$ which is a full subcategory of $\mathrm{Set}(U_1)$. The small U-categories, as objects with the functors, define a U-category $\mathrm{Cat}(U)$. All U-categories with functors define a U_1-category $\mathrm{CAT}(U)$ which is a full subcategory of $\mathrm{Cat}(U_1)$.

1.2. 2-categories. We use the notion of a 2-category in the sense of [3] and [5]. Let U be any universe. A *2-category* A over U consists of a U-category A_0 (called the *local discrete category* of A) with the factorization

$$\begin{tikzcd} & \mathrm{Cat}(U) \arrow[dr, "|(-)|"] & \\ A_0^{\mathrm{op}} \times A_0 \arrow[ur, "A(-,-)"] \arrow[rr, "A_0(-,-)"'] & & \mathrm{Set}(U) \end{tikzcd}$$

such that for all objects X, Y, Z in A_0 there are functors

$$A(X, Y) \times A(Y, Z) \xrightarrow{\circ = \circ_{X,Y,Z}} A(X, Z)$$

which are natural, associative and unitary in all variables and agree with composition in A_0 on objects.

Let A be any 2-category over U. The objects and morphisms of A_0 are called *0-cells* and *1-cells* of A, respectively. For any 0-cells X, Y of A the morphisms of the category $A(X, Y)$ the objects of which are 1-cells from X to Y are called the *2-cells* of A from X to Y. The compositions $\circ = \circ_{X,Y,Z}$ are called *strong* in A and we write $\alpha \circ \beta = \beta\alpha$. The composition in the category $A(X, Y)$ is said to be the *weak* one of A and it is denoted by $\cdot$. If φ: $m'f \to gm$ is a 2-cell in A, then it is presented by the diagram square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\varphi\downarrow"] & X' \arrow[d, "m'"] \\
Y \arrow[r, "g"'] & Y'
\end{tikzcd}$$

If φ: $gm \to m'f$ is a 2-cell in A, then the diagram square for φ is obtained from the above one by replacing $\varphi\downarrow$ by $\varphi\uparrow$. Let A be any 2-category over U and let X, Y, Z be any 0-cells in A. Moreover, let f, h, h' and g, g', g'' be any 1-cells in A from X to Y and from Y to Z, respectively. Since $\circ$ are functors, we have the following well-known facts [3]:

1.2.1. Proposition. (a$_1$) $(\psi' \cdot \psi)(\varphi' \cdot \varphi) = (\psi'\varphi') \cdot (\psi\varphi)$, *where* φ: $f \to h$, φ': $h \to h'$, ψ: $g \to g'$ *and* ψ': $g' \to g''$ *are any 2-cells in* A.

(a$_2$) $g(\varphi' \cdot \varphi) = (g\varphi') \cdot (g\varphi)$, *where* φ: $f \to h$ *and* φ': $h \to h'$ *are 2-cells in* A.

(a$_3$) $(\psi' \cdot \psi)f = (\psi'f) \cdot (\psi f)$, *where* ψ: $g \to g'$ *and* ψ': $g' \to g''$ *are 2-cells in* A.

(a$_4$) $a \cdot h = h' \cdot a = a$, *where* a: $h \to h'$ *is any 2-cell in* A.

(a$_5$) $(ga) \cdot (\beta h) = (\beta h') \cdot (fa) = \beta a$ *and* $Ya = aX = a$, *where* a: $h \to h'$ *and* β: $g \to g'$ *are any 2-cells in* A.

(a$_6$) *For any diagram in* A *of the form*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\varphi\downarrow"] & X' \arrow[r, "f'"] \arrow[d, "m'"] \arrow[dr, phantom, "\varphi'\downarrow"] & X'' \arrow[d, "m''"] \\
Y \arrow[r, "g"'] \arrow[d, "n"'] \arrow[dr, phantom, "\psi\downarrow"] & Y' \arrow[r, "g'"'] \arrow[d, "n'"] \arrow[dr, phantom, "\psi'\downarrow"] & Y'' \arrow[d, "n''"] \\
Z \arrow[r, "h"'] & Z' \arrow[r, "h'"'] & Z''
\end{tikzcd}$$

we have

$$(\psi' \top \psi) \bigtriangledown (\varphi' \top \varphi) = (\psi' \bigtriangledown \varphi') \top (\psi \bigtriangledown \varphi),$$

where $\psi \bigtriangledown \varphi = (\psi m) \cdot (n'\varphi)$ *and* $\varphi' \top \varphi = (g'\varphi) \cdot (\varphi' f)$ *are the compositions of vertical and horizontal squares, respectively, or 2-cells in* A.

(a$_6'$) *For any diagram in* A *obtained from the diagram in* (a$_6$) *by replacing the squares* $\varphi\downarrow$, $\varphi'\downarrow$, $\psi\downarrow$ *and* $\psi'\downarrow$ *by the inverse squares* $\varphi\uparrow$, $\varphi'\uparrow$, $\psi\uparrow$ *and* $\psi'\uparrow$, *respectively, we have*

$$(\psi' \perp \psi) \bigtriangleup (\varphi' \perp \varphi) = (\psi' \bigtriangleup \varphi') \perp (\psi \bigtriangleup \varphi),$$

where $\psi \bigtriangleup \varphi = (n'\varphi) \cdot (\psi m)$ *and* $\varphi' \perp \varphi = (\varphi' f) \cdot (g'\varphi)$ *are the compositions of inverse vertical and inverse horizontal squares, respectively, or 2-cells in* A.

For any 2-category A there is a total category A_t of A the objects of which are all 0-cells in A, morphisms from X to Y are all 2-cells in A from X to Y, and the composition agrees with $\circ$. The categories $\mathrm{Cat}(U)$ and $\mathrm{CAT}(U)$ with the natural transformations of functors as 2-cells define the 2-categories 2-$\mathrm{Cat}(U)$ and 2-$\mathrm{CAT}(U)$ over U and U_1, respectively.

1.3. The diagonal categories and 2-categories. Let A be any 2-category over U. Let us denote by d_1A (respectively, d_3A) the set of all 1-cells in A, as the set of objects, together with:

(b$_1$) the correspondence which with any 1-cells $f\colon X \to X'$ and $g\colon Y \to Y'$ of A associates the set $d_1A(f, g)$ (respectively, $d_3A(f, g)$) of all 3-tuples (m, m', φ), as the set of morphisms from f to g, in such a way that in A there is a square of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\varphi\downarrow"] & X' \arrow[d, "m'"] \\ Y \arrow[r, "g"'] & Y' \end{tikzcd} \qquad \text{for } d_1A(f, g)$$

and

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\varphi\uparrow"] & X' \arrow[d, "m'"] \\ Y \arrow[r, "g"'] & Y' \end{tikzcd} \qquad \text{for } d_3A(f, g),$$

and the identity morphism of f is the 3-tuple (X, X', f);

(b$_2$) the composition of vertical squares (respectively, inverse vertical squares) of A, as the composition in d_1A (respectively, d_3A), i.e.

$$(n, n', \psi)(m, m', \varphi) = (nm, n'm', \psi \nabla \varphi) \quad \text{in } d_1A,$$

$$(n, n', \psi)(m, m', \varphi) = (nm, n'm', \psi \triangle \varphi) \quad \text{in } d_3A.$$

1.3.1. THEOREM. *For any 2-category A over U, d_1A and d_3A are U-categories (called diagonal and inverse diagonal categories of A, respectively). There are the extensions of d_1A and d_3A to the 2-categories.*

1.3.2. THEOREM. *For any 2-category A over U there are 2-categories 2-d_1A and 2-d_3A over U such that:*

(c$_1$) *d_1A and d_3A are the local discrete categories of 2-d_1A and 2-d_3A, respectively.*

(c$_2$) *For any objects $f\colon X \to X'$ and $g\colon Y \to Y'$ in d_1A or d_3A the category 2-$d_1A(f, g)$ or 2-$d_3A(f, g)$ consists of*

(1) *the set of all morphisms from f to g in d_1A or d_3A as the set of objects;*

(2) *the set of pairs of 2-cells in A of the form*

$$\{\langle\alpha, \beta\rangle \mid \alpha\colon m \to m_1, \beta\colon m' \to m_1' \text{ and } (g\alpha)\cdot\varphi = \varphi_1\cdot(\beta f)\}$$

or the set of pairs of 2-cells in A of the form

$$\{\langle \alpha, \beta\rangle \mid \alpha\colon m \to m_1, \beta\colon m' \to m_1' \text{ and } (\beta f)\cdot\varphi = \varphi_1\cdot(g\alpha)\},$$

as the set of morphisms from $\boldsymbol{a} = (m, m', \varphi)\colon f \to g$ to

$$\boldsymbol{b} = (m_1, m_1', \varphi_1)\colon f \to g$$

in $2\text{-}d_1A(f, g)$ or $2\text{-}d_3A(f, g)$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are any morphisms from f to g in d_1A or d_3A and with $\langle m, m'\rangle$ as the identity morphism of $\boldsymbol{a}$;

(3) *the direct square of the weak composition in A, as the composition in $2\text{-}d_1A(f, g)$ or $2\text{-}d_3A(f, g)$.*

(c$_3$) *The composition rules in $2\text{-}d_1A$ or $2\text{-}d_3A$ are the functors*

$$2\text{-}d_iA(f, g) \times 2\text{-}d_iA(g, h) \xrightarrow{\circ} 2\text{-}d_iA(f, h), \quad i = 1, 3,$$

defined by the mappings

$$\begin{array}{ccc} \big((m, m', \varphi), (n, n', \psi)\big) & & (nm, n'm', \psi \,\square\, \varphi) \\ \Big\downarrow{\scriptstyle (\langle\alpha,\beta\rangle,\langle\alpha_1,\beta_1\rangle)} & \longmapsto & \Big\downarrow{\scriptstyle \langle\alpha_1\alpha,\beta_1\beta\rangle} \\ \big((m_1, m_1', \varphi'), (n_1, n_1', \psi')\big) & & (n_1m_1, n_1'm_1', \psi' \,\square\, \varphi') \end{array}$$

where $\square$ is ∇ for $i = 1$ and $\square$ is $\triangle$ for $i = 3$.

Proof. By (a$_4$), $\langle m, m'\rangle$ is indeed a morphism of $\boldsymbol{a}$. By (a$_2$) and (a$_3$), the composition of morphisms in $2\text{-}d_1A(f, g)$, given by $\langle\alpha', \beta'\rangle\langle\alpha, \beta\rangle = \langle\alpha'\cdot\alpha, \beta'\cdot\beta\rangle$, fulfils the equalities

$$g(\alpha'\cdot\alpha)\cdot\varphi = g\alpha'\cdot g\alpha\cdot\varphi = g\alpha'\cdot\varphi_1\cdot\beta f = \varphi_2\cdot\beta' f\cdot\beta f = \varphi_2\cdot(\beta'\cdot\beta)f,$$

and thus it is also a morphism in $2\text{-}d_1A(f, g)$. Hence $2\text{-}d_1A(f, g)$ is a category. The last part follows easily from the definitions.

1.3.3. Definition. For any 2-category A over U we denote by d_0A (respectively, d_2A) the subcategory of d_1A (respectively, d_3A) determined by objects of the form $f\colon X \to X$, where X is any 0-cell in A, and by morphisms of the form (m, m, φ), briefly written as (m, φ). The categories $2\text{-}d_0A$ and $2\text{-}d_2A$ are subcategories of $2\text{-}d_1A$ and $2\text{-}d_3A$ defined by 0-cells and 1-cells in d_1A and d_3A and by 2-cells of the form $\langle\alpha, \beta\rangle$ with $\alpha = \beta$.

1.3.4. Definition. By $A \downarrow A$ we denote the subcategory of d_1A defined by all objects and by morphisms (m, m', φ) with $\varphi = \mathrm{id}$. Moreover, $2\text{-}A \downarrow A$ is the full subcategory with respect to 2-cells of the 2-category $2\text{-}d_1A$.

1.3.5. Definition. Let $\boldsymbol{a} = (a_1, a_2)$ be any pair with $a_i \in \{0, 1, 2, 3\}$. The *diagonal category* of type $\boldsymbol{a}$ of a 2-category A is the category $d_{\boldsymbol{a}}A$ which is the subcategory of the direct product $d_{a_1}A \times d_{a_2}A$ defined by all objects (f_1, f_2) with $\mathrm{dom} f_2 = \mathrm{codom} f_1$ and by all morphisms of the form $\big((m_1, m_1', \varphi_1), (m_2, m_2', \varphi_2)\big)$ with $m_1' = m_2$.

By $2\text{-}d_{\boldsymbol{a}}A$ we denote the subcategory of $2\text{-}d_{a_1}A \times 2\text{-}d_{a_2}A$ defined by all 0-cells and 1-cells in $d_{\boldsymbol{a}}A$ and by 2-cells of the form $(\langle a_1, \beta_1\rangle, \langle a_2, \beta_2\rangle)$ with $\beta_1 = a_2$.

In an analogical way we can define the categories $d_{\boldsymbol{a}}A$ and $2\text{-}d_{\boldsymbol{a}}A$ for any diagonal type $\boldsymbol{a} = (a_1, \ldots, a_n)$ with $a_i \in \{0, 1, 2, 3\}$.

1.3.6. Definition. Let A be any 2-category over U. By $\Gamma(A)$ and $r\text{-}\Gamma(A)$ we denote the sets

$$\{(X, -), (-, X'), (X, X'), (X, T), (-, T)\}$$

and

$$\{(X, -), (-, X'), (X, X'), (T_0, X'), (T_0, -)\},$$

respectively, where X, X' are any 0-cells in A and $T\colon X' \to X'$, $T_0\colon X \to X$ are any 1-cells in A. For any diagonal type $\boldsymbol{a} = (a_1, a_2)$ with $a_i \in \{0, 1, 2, 3\}$ and for any element $\varkappa$ in $\Gamma(A)$ or in $r\text{-}\Gamma(A)$ we define a category $d_{\boldsymbol{a}}A\varkappa$ which is a subcategory of $d_{\boldsymbol{a}}A$ determined by all objects (f_1, f_2) and morphisms $\big((m_1, m_1', \varphi_1), (m_2, m_2', \varphi_2)\big)$ such that

(1) $\operatorname{dom} f_1 = X$, $m_1 = X$ if $\varkappa = (X, -)$;
(2) $\operatorname{codom} f_1 = X'$, $m_1' = X'$ if $\varkappa = (-, X')$;
(3) $\operatorname{dom} f_1 = X$, $\operatorname{codom} f_1 = X'$, $m_1 = X$, $m_1' = X'$ if $\varkappa = (X, X')$;
(4) $\operatorname{dom} f_1 = X$, $f_2 = T$, $m_1 = X$, $\varphi_2 = T$ if $\varkappa = (X, T)$;
(5) $f_2 = T$, $\varphi_2 = T$ if $\varkappa = (-, T)$;
(6) $f_1 = T_0$, $\operatorname{codom} f_2 = X'$, $\varphi_1 = T_0$, $m_2' = X'$ if $\varkappa = (T_0, X')$;
(7) $f_1 = T_0$, $\varphi_1 = T_0$ if $\varkappa = (T_0, -)$.

For any $\varkappa$ in $\Gamma(A)$ or in $r\text{-}\Gamma(A)$ there is a 2-category $2\text{-}d_{\boldsymbol{a}}A\varkappa$ which is a subcategory of $2\text{-}d_{\boldsymbol{a}}A$ defined by all 0-cells and 1-cells in $d_{\boldsymbol{a}}A\varkappa$ and by 2-cells of the form

$$(\langle a_i, \beta_i\rangle)\colon \big((m_i, m_i', \varphi_i)\big) \to \big((n_i, n_i', \varphi_i')\big), \quad i = 1, 2,$$

such that

(1′) $a_1 = m_1 = X$ if $\varkappa \in \{(X, -), (X, X'), (X, T)\}$;
(2′) $\beta_1 = m_1' = X'$ if $\varkappa \in \{(-, X'), (X, X')\}$;
(3′) $\beta_1 = n_1' = \operatorname{dom} T$, $\beta_2 = n_2' = \operatorname{codom} T$ if $\varkappa \in \{(X, T), (-, T)\}$;
(4′) $\beta_2 = m_2' = X'$ if $\varkappa = (T_0, X')$;
(5′) $a_1 = m_1 = \operatorname{dom} T_0$, $a_2 = m_2 = \operatorname{codom} T$ if $\varkappa \in \{(T_0, X), (T_0, -)\}$.

1.3.7. Definition. Let A be any 2-category over U and let X be any 0-cell in A. By $d_1A(X, -)$ we denote the subcategory of d_1A defined by all 1-cells f with $\operatorname{dom} f = X$ and by morphisms of the form (n, n', φ) with $n = X$. By $2\text{-}d_1A(X, -)$ we denote the subcategory of $2\text{-}d_1A$ defined by 0-cells and 1-cells in $d_1A(X, -)$ and by 2-cells of the form $\langle a, \beta\rangle$ with $a = X$. Let f, g be any 1-cells in A with $\operatorname{dom} f = \operatorname{dom} g = X$. The terminal object, if there is any, of the category $2\text{-}d_1A(X, -)(f, g)$ is called the *right Kan extension* of g along f and is denoted by $\big(X, r_f(g), \varepsilon\big)$.

The right Kan extension $(X, r_f(g), \varepsilon)$ is said to be *preserved* by 1-cells if for any 1-cell h: $Y' \to Y''$ in A, where f: $X \to X'$ and g: $X \to Y'$, there exists a right Kan extension $(X, r_f(hg), a)$ which is isomorphic in the category $2\text{-}d_1A(X, -)(f, hg)$ to the object $(X, hr_f(g), h\varepsilon)$. By $\mathrm{Kan}(A \downarrow A)$ we denote the full subcategory of $A \downarrow A$ defined by all 1-cells g for which there is a right Kan extension along g and this extension is preserved by 1-cells in A.

1.4. The monad categories and 2-categories. Let A be any 2-category over U. A *monad* (*1-cell*) in A is any 4-tuple $\boldsymbol{T} = \langle X, T, \eta, \mu\rangle$ consisting of any endocell T: $X \to X$ in A and 2-cells η: $X \to T$, μ: $T^2 \to T$ in A such that

$$\mu\cdot T\mu = \mu\cdot\mu T \quad \text{and} \quad \mu\cdot T\eta = \mu\cdot\eta T = T.$$

1.4.1. PROPOSITION. *The above-defined 4-tuple $\boldsymbol{T} = \langle X, T, \eta, \mu\rangle$ is a monad in A if and only if*

$$\mu \triangledown \mu = \mu \top \mu \quad \textit{and} \quad \eta \top \mu = \mu \triangledown \eta = T.$$

If $\boldsymbol{T} = \langle X, T, \eta, \mu\rangle$ is a monad in A, then $T = F\boldsymbol{T}$, η, μ are the endocell, the unity and the operation of $\boldsymbol{T}$, respectively, and $\boldsymbol{T}$ is said to be a *monad* of the 0-cell X in A. Let $\boldsymbol{T} = \langle X, T, \eta, \mu\rangle$ and $\boldsymbol{T}' = \langle X', T', \eta', \mu'\rangle$ be two monads in A. Then a monad morphism (respectively, inverse monad morphism) from $\boldsymbol{T}$ to $\boldsymbol{T}'$ is any morphism (m, φ) in d_0A (respectively, d_2A) from $F\boldsymbol{T}$ to $F\boldsymbol{T}'$ such that

$$\eta' m = \varphi\cdot m\eta \quad \text{and} \quad \varphi\cdot m\mu = \mu' m\cdot T'\varphi\cdot\varphi T$$

(respectively, $m\eta = \varphi\cdot\eta' m$ and $\varphi\cdot\mu' m = m\mu\cdot\varphi T\cdot T'\varphi$). Hence we have

1.4.2. PROPOSITION. *A morphism (m, φ): $F\boldsymbol{T} \to F\boldsymbol{T}'$ in d_0A (respectively, d_2A) is a monad morphism (respectively, inverse monad morphism) in A from $\boldsymbol{T}$ to $\boldsymbol{T}'$ if and only if*

$$\eta' \top m = \varphi \triangledown \eta \quad \textit{and} \quad \varphi \triangledown \mu = \varphi \top (\mu' \triangledown \varphi)$$

(*respectively,* $m \triangledown \eta = \varphi \top \eta'$ *and* $\mu' \top \varphi = (\varphi \top \mu) \triangledown \varphi$).

Consider the composition $h = (m_1, \varphi_1)(m, \varphi) = (m_1 m, \varphi_1 \triangledown \varphi)$ of monad morphisms in d_0A. Then we have

$$\begin{aligned}(\eta'' m_1)m &= (\varphi_1\cdot m_1\eta')m = \varphi_1 m\cdot m_1\eta' m = \varphi_1 m\cdot m_1(\eta' m)\\ &= \varphi_1 m\cdot m_1(\varphi\cdot m\eta) = \varphi_1 m\cdot m_1\varphi\cdot m_1 m\eta = (\varphi_1 \triangledown \varphi)\cdot m_1 m\eta\end{aligned}$$

and

$$\begin{aligned}(\varphi_1 \triangledown \varphi) \triangledown \mu &= \varphi_1 \triangledown (\varphi \top (\mu' \triangledown \varphi)) = (\varphi_1 \triangledown \mu') \triangledown (\varphi \top \varphi)\\ &= [\varphi_1 \top (\mu'' \triangledown \varphi_1)] \triangledown (\varphi \top \varphi)\\ &= (T'' \top \mu'') \triangledown [(\varphi_1 \triangledown \varphi) \top (\varphi_1 \triangledown \varphi)]\\ &= [(T'' \triangledown \varphi_1) \triangledown \varphi] \top [\mu'' \triangledown (\varphi_1 \triangledown \varphi_1)]\\ &= (\varphi_1 \triangledown \varphi) \top [\mu'' \triangledown (\varphi_1 \triangledown \varphi)],\end{aligned}$$

and thus h is also a monad morphism in A. Moreover, the identity of FT is a monad morphism. We can prove analogically that the composition in d_2A of inverse monad morphisms is also an inverse monad morphism and the identity of FT is an inverse monad morphism.

1.4.3. THEOREM. *For any 2-category A over U the monads in A with the monad morphisms (respectively, inverse monad morphisms) define a U-category* $\mathrm{Mon}(A)$ *(respectively,* $\mathrm{Mon}^{\leftarrow}(A)$*), and the functor* $F\colon \mathrm{Mon}(A)\to d_0A$ *(respectively,* $F\colon \mathrm{Mon}^{\leftarrow}(A)\to d_2A$*), defined above, is forgetful.*

$\mathrm{Mon}(A)$ *and* $\mathrm{Mon}^{\leftarrow}(A)$ *are the local discrete categories of 2-categories* $2\text{-}\mathrm{Mon}(A)$ *and* $2\text{-}\mathrm{Mon}^{\leftarrow}(A)$ *such that* $2\text{-}\mathrm{Mon}(A)(T, T')$ *and* $2\text{-}\mathrm{Mon}^{\leftarrow}(A)(T, T')$ *are the full subcategories of* $2\text{-}d_0A(FT, FT')$ *and* $2\text{-}d_2A(FT, FT')$ *determined by all monad morphisms and inverse monad morphisms.*

For any 0-cell X in A we have the category $\mathrm{Mon}(A)(X)$ (respectively, $\mathrm{Mon}^{\leftarrow}(A)(X)$) of all monads of X which is a subcategory of $\mathrm{Mon}(A)$ (respectively, $\mathrm{Mon}^{\leftarrow}(A)$) determined by all monads of X and all morphisms of the form (X, φ).

Any object $g\colon X\to X'$ of $\mathrm{Kan}(A\downarrow A)$ defines a monad $T = \langle X', T, \eta, \mu\rangle$ in A such that

$$\langle X, \eta\rangle\colon (X, X', g)\to (X, T, \varepsilon) \quad\text{and}\quad \langle X, \mu\rangle\colon (X, T^2, \varepsilon\cdot T\varepsilon)\to (X, T, \varepsilon)$$

are the unique morphisms in $2\text{-}d_1A(X, -)(g, g)$ defined by the terminal object (X, T, ε) which is the right Kan extension of g along g. Indeed, $g = \varepsilon\cdot\eta g$, $\varepsilon\cdot T\varepsilon = \varepsilon\cdot\mu g$ and, by (a_5),

$$\eta g\cdot\varepsilon = T\varepsilon\cdot\eta Tg \quad\text{and}\quad T\varepsilon\cdot\mu Tg = \mu g\cdot T^2\varepsilon.$$

Hence

$$\begin{aligned}\varepsilon\cdot(\mu\cdot\mu T)g &= \varepsilon\cdot\mu g\cdot\mu Tg = \varepsilon\cdot T\varepsilon\cdot\mu Tg = \varepsilon\cdot\mu g\cdot T^2\varepsilon = \varepsilon\cdot T\varepsilon\cdot T^2\varepsilon\\ &= \varepsilon\cdot T(\varepsilon\cdot T\varepsilon) = \varepsilon\cdot T(\varepsilon\cdot\mu g) = \varepsilon\cdot T\varepsilon\cdot T\mu g = \varepsilon\cdot\mu g\cdot T\mu g = \varepsilon\cdot(\mu\cdot T\mu)g,\end{aligned}$$

and thus, by the property of the terminal object, we have

$$\mu\cdot\mu T = \mu\cdot T\mu.$$

Analogically we prove that $\mu\cdot T\eta = \mu\cdot\eta T = T$.

1.4.4. THEOREM. *For any 2-category A over U the mappings*

$$\begin{array}{ccc} X\xrightarrow{g}X' & & T = \langle X', T, \eta, \mu\rangle \\ \Big\downarrow{\scriptstyle (m,m')} & \longmapsto & \Big\downarrow{\scriptstyle (m',\gamma)} \\ Y\xrightarrow[g']{}Y' & & T' = \langle Y', T', \eta', \mu'\rangle \end{array}$$

define a functor $\mathrm{Str}\colon \mathrm{Kan}(A\downarrow A)\to\mathrm{Mon}^{\leftarrow}(A)$, *where* (X, T, ε) *and* (Y, T', ε') *are the right Kan extensions of g and of g' along g and g', T and*

T' are monads of those extensions, $\gamma = z^{-1}\cdot h$ with $\langle X, z\rangle$: $(X, m'T, m\varepsilon) \to (X, T'', a)$ is the isomorphism in $2\text{-}d_1A(X, -)(g, m'g)$ to the right Kan extension (X, T'', a) of $m'g$ along g, and $\langle X, h\rangle$: $(X, T'm', \varepsilon'm) \to (X, T'', a)$ is the unique morphism in this category to the terminal object.

Proof. Since

$$a\cdot(z\cdot m'\eta)g = a\cdot zg\cdot m'\eta g = m'\varepsilon\cdot m'\eta g = m'(\varepsilon\cdot\eta g) = m'g = g'm$$

and

$$a\cdot(z\cdot\gamma\cdot\eta'm')g = a\cdot hg\cdot\eta'm'g = \varepsilon'm\cdot\eta'm'g = \varepsilon'm\cdot\eta'g'm$$
$$= (\varepsilon'\cdot\eta'g')m = g'm,$$

we have $z\cdot m'\eta = z\cdot\gamma\cdot\eta'm'$, and thus $m'\eta = \gamma\cdot\eta'm'$. Since

$$a\cdot(z\cdot\gamma\cdot\mu'm')g = a\cdot(z\cdot m'\mu\cdot\gamma T\cdot T'\gamma)g,$$

we obtain $\gamma\cdot\mu m' = m'\mu\cdot\gamma T\cdot T'\gamma$. Hence (m', γ): $\boldsymbol{T} \to \boldsymbol{T}'$ is an inverse monad morphism. In an analogical way we prove that Str is a functor.

1.5. Pointed monad categories and 2-categories. Let A be any 2-category over U. A *pointed monad* (respectively, *right pointed monad*) in A is any pair $(f, \boldsymbol{T})$ (respectively, $(\boldsymbol{T}, f)$), where f is any 1-cell in A and $\boldsymbol{T}$ is any monad of the 0-cell with $\operatorname{codom} f$ (respectively, $\operatorname{dom} f$) in A. A *pointed monad morphism* (respectively, *right pointed monad morphism*) from $(f, \boldsymbol{T})$ (respectively, $(\boldsymbol{T}, f)$) to $(f', \boldsymbol{T}')$ (respectively, $(\boldsymbol{T}', f')$) is any morphism $\boldsymbol{t} = (t_1, t_2)$ in $d_{(1,0)}A$ (respectively, $d_{(0,1)}A$) from $(f, \boldsymbol{FT})$ (respectively, $(\boldsymbol{FT}, f)$) to $(f', \boldsymbol{FT}')$ (respectively, $(\boldsymbol{FT}', f')$) such that t_2 (respectively, t_1) is a monad morphism in $\mathrm{Mon}(A)$ from $\boldsymbol{T}$ to $\boldsymbol{T}'$. The composition of pointed monad morphisms or right pointed monad morphisms is the same as the composition in $d_{(1,0)}A$ or $d_{(0,1)}A$. In this way we obtain the categories $\mathrm{Mon}_*(A)$ and $r\text{-}\mathrm{Mon}_*(A)$ and the forgetful functors

$$G\colon \mathrm{Mon}_*(A) \to d_{(1,0)}A \quad \text{and} \quad G_r\colon r\text{-}\mathrm{Mon}_*(A) \to d_{(0,1)}A.$$

Those categories are local discrete categories of the 2-categories $2\text{-}\mathrm{Mon}_*(A)$ and $2\text{-}r\text{-}\mathrm{Mon}_*(A)$ such that the categories $2\text{-}\mathrm{Mon}_*(A)(h, h')$ and $2\text{-}r\text{-}\mathrm{Mon}_*(A)(h, h')$ are the full subcategories of $2\text{-}d_{(1,0)}A(Gh, Gh')$ and $2\text{-}d_{(0,1)}A(G_rh, G_rh')$ defined by the pointed monad and right pointed monad morphisms.

1.5.1. Definition. Let A be any 2-category over U. By $\Gamma_m(A)$ and $r\text{-}\Gamma_m(A)$ we denote the sets

$$\{(X, -), (-, X'), (X, X'), (X, \boldsymbol{T}), (-, \boldsymbol{T})\}$$

and

$$\{(X, -), (-, X'), (X, X'), (\boldsymbol{T}_0, X), (\boldsymbol{T}_0, -)\},$$

respectively, where X, X' are any 0-cells in A and $T\colon X' \to X'$, $T_0\colon X \to X$ are any monads in A. These sets may be considered as subsets of the sets $\Gamma(A)$ and r-$\Gamma(A)$ defined in 1.3.6. For any element $\varkappa$ in $\Gamma_m(A)$ or r-$\Gamma_m(A)$ we denote by $\mathrm{Mon}_*(A)(\varkappa)$ or r-$\mathrm{Mon}_*(A)(\varkappa)$ the subcategory of $\mathrm{Mon}_*(A)$ or r-$\mathrm{Mon}_*(A)$ determined by the objects h and the morphisms p such that Gh and Gp or $G_r h$ and $G_r p$ are in $d_{(1,0)}A(\varkappa)$ or $d_{(0,1)}A(\varkappa)$.

In a similar way we define the 2-categories 2-$\mathrm{Mon}_*(A)(\varkappa)$ and 2-r-$\mathrm{Mon}_*(A)(\varkappa)$ for $\varkappa$ in $\Gamma_m(A)$ or r-$\Gamma_m(A)$, using the 2-categories 2-$d_{(1,0)}A(\varkappa)$ and 2-$d_{(0,1)}A(\varkappa)$.

If $A = $ 2-$\mathrm{Cat}(U)$ or $A = $ 2-$\mathrm{CAT}(U)$, then the category $\mathrm{Mon}_*(A)(\mathbf{1}, -)$ is denoted by $\mathrm{Mon}_*(U)$ or $\mathrm{MON}_*(U)$. Since any functor $f\colon \mathbf{1} \to X'$ is uniquely determined by the object $f1$ of category X', and any 2-cell

$$
\begin{tikzcd}
 & X' \arrow[dd, "m'"] \\
\mathbf{1} \arrow[ur, "f"] \arrow[dr, "f'"'] \arrow[r, phantom, "\varphi\Downarrow"] & {} \\
 & Y'
\end{tikzcd}
$$

is uniquely determined by the morphism $m'f1 \to f'1$ of the category Y', then the objects of $\mathrm{Mon}_*(U)$ or $\mathrm{MON}_*(U)$ may be considered as the pairs (f, T) such that f is any object of X' and $T = \langle X', T, \eta, \mu\rangle$ is any monad of the category X', and the morphisms $\big((1, m', \varphi), (m', \psi)\big)$ may be considered as the 3-tuple (φ, m', ψ) such that $\varphi\colon m'f \to f'$ is a morphism in Y' and $(m', \psi)\colon T \to T'$ is a monad morphism.

1.6. The algebra categories and 2-categories. Let A be any 2-category over U and let $s, t \in \{0, 1, 2, 3\}$. Let us consider the category $C(s, t)$ such that

$$C(s, t) = d_s 2\text{-}d_t A.$$

The objects of this category are morphisms $(c_1, c_2, a)\colon g_1 \to g_2$ in $d_t A$. Using Theorem 1.3.2 we can prove

1.6.1. THEOREM. *Let*

$$c = (c_1, c_2, a)\colon g_1 \to g_2 \quad \text{and} \quad c' = (c_1', c_2', a)\colon g_1' \to g_2'$$

be any objects in $C(s, t)$. Then the morphisms from $\mathbf{c}$ to $\mathbf{c}'$ in $C(s, t)$ are all 3-tuples (m^*, n^*, ψ^*), where $m^* = (m_1, m_2, \varphi)\colon g_1 \to g_1'$, $n^* = (n_1, n_2, \varphi')\colon g_2 \to g_2'$ are 1-cells in 2-$d_t A$, $\psi^* = (\psi, \psi')$ is any 2-cell in 2-$d_t A$ from $n^*\mathbf{c}$ to $\mathbf{c}'m^*$ if $t \in \{0, 1\}$ and from $\mathbf{c}'m^*$ to $n^*\mathbf{c}$ if $t \in \{2, 3\}$ such that

(1) $m^* = n^*$ *if* $s \in \{0, 2\}$,
(2) $g_2'\psi \cdot (\varphi' \nabla a) = (a' \nabla \varphi) \cdot \psi' g_1$ *if* $t \in \{0, 1\}$,
(3) $\psi' g_1 \cdot (a' \top \varphi) = (\varphi' \top a) \cdot g_2' \psi$ *if* $t \in \{2, 3\}$.

1.6.2. Definitions. (I) The objects of the category $C(0, 1)$ of the form

$$(\mathrm{dom} f, T, a)\colon f \to f,$$

i.e. the diagrams in A of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "X"'] \arrow[dr, phantom, "a\downarrow"] & X' \arrow[d, "T"] \\ X \arrow[r, "f"'] & X' \end{tikzcd},$$

are called *endoalgebras* in A. If $(\mathrm{dom} f, T, a)\colon f \to f$ is an endoalgebra in A, then a is a *T-structure* on f and (f, T) is the *strong type* of this algebra.

(II) The objects of the category $C(1, 0)$ of the form

$$(f, f, a)\colon T \to \mathrm{codom} f,$$

i.e. the diagrams in A of the form

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "a\downarrow"] & X \arrow[d, "f"] \\ X' \arrow[r, "X'"'] & X' \end{tikzcd},$$

are called *right endoalgebras* in A. If $(f, f, a)\colon T \to X'$ is a right endoalgebra in A, then a is a *right T-structure* on f and (T, f) is the *strong type* of this algebra.

The full subcategory of $C(0, 1)$ (respectively, $C(1, 0)$) defined by all endoalgebras (respectively, right algebras) in A is denoted by $\mathrm{wAl}(A)$ (respectively, r-$\mathrm{wAl}(A)$). The morphisms of these categories are called *weak algebra* (respectively, *right algebra*) *homomorphisms* in A. Those categories have extensions to 2-categories 2-$\mathrm{wAl}(A)$ and 2-r-$\mathrm{wAl}(A)$ which are subcategories of $2\text{-}d_s 2\text{-}d_t A$. By 1.6.1 and 1.6.2 (I), the weak algebra homomorphism Q from $(\mathrm{dom} f, T, a)\colon f \to f$ to $(\mathrm{dom} f', T', a')\colon f' \to f'$ may be considered as the pair $\big((m, m', \varphi), (\psi, \psi')\big)$ such that the 1-cells in A, i.e.

$$m\colon \mathrm{dom} f \to \mathrm{dom} f', \qquad m'\colon \mathrm{dom} T \to \mathrm{dom} T',$$

and the 2-cells in A, i.e.

$$\varphi\colon m'f \to f'm, \qquad \psi\colon m \to m, \qquad \psi'\colon m'T \to T'm,$$

fulfil one of the following equivalent equalities:

$$(\mathrm{q}_1) \qquad f'\psi \cdot (\varphi \nabla a) = (a' \nabla \varphi) \cdot \psi' f,$$

$$(\mathrm{q}_2) \qquad f'\psi \cdot \varphi \cdot m'a = a'm \cdot T'\varphi \cdot \psi' f,$$

$$(\mathrm{q}_3) \qquad (\varphi \nabla a) \top \psi = \psi' \top (a' \nabla \varphi).$$

By 1.6.1 and 1.6.2 (II), the weak right algebra homomorphism Q from $(f, f, a)\colon T \to X'$ to $(f', f', a')\colon T' \to Y'$ is determined by a 3-tuple $\big((m, \varphi, (n, \varphi'), (\psi, \psi')\big)$ such that the 1-cells in A, i.e.

$$m\colon \operatorname{dom} T \to \operatorname{dom} T', \qquad n\colon X' \to Y',$$

and the 2-cells in A, i.e.

$$\varphi\colon mT \to T'm, \qquad \varphi'\colon n \to n, \qquad \psi, \psi'\colon nf \to f'n,$$

fulfil one of the following equivalent equalities:

$$(\mathrm{rq}_1) \qquad \psi\cdot(\varphi' \nabla a) = (a' \nabla \varphi)\cdot\psi' T,$$

$$(\mathrm{rq}_2) \qquad \psi\cdot\varphi' f\cdot na = a' m\cdot f'\varphi\cdot\psi' T,$$

$$(\mathrm{rq}_3) \qquad (\varphi' \nabla a) \top \psi = \psi' \top (a' \nabla \varphi).$$

A weak algebra homomorphism $\big((m, m', \varphi), (\psi, \psi')\big)$ in A is said to be an *algebra homomorphism* in A provided $\psi = m$. Conditions (q_i) with $\psi = m$ are denoted by $(\mathrm{q}_{i\mathrm{h}})$ and they have the following forms:

$$(\mathrm{q}_{1\mathrm{h}}) \qquad \varphi \nabla a = (a' \nabla \varphi)\cdot\psi' f,$$

$$(\mathrm{q}_{2\mathrm{h}}) \qquad \varphi\cdot m' a = a' m\cdot T'\varphi\cdot\psi' f,$$

$$(\mathrm{q}_{3\mathrm{h}}) \qquad \varphi \nabla a = \psi' \top (a' \nabla \varphi).$$

A weak right algebra homomorphism $\big((m, \varphi), (n, \varphi'), (\psi, \psi')\big)$ in A is said to be a *right algebra homomorphism* in A provided $\varphi' = n$ and $\psi = \psi'$. Conditions (rq_i) with $\varphi' = n$ and $\psi = \psi'$ are denoted by $(\mathrm{rq}_{i\mathrm{h}})$ and they have the forms

$$(\mathrm{rq}_{1\mathrm{h}}) \qquad \psi\cdot(n \nabla a) = (a' \nabla \varphi)\cdot\psi T,$$

$$(\mathrm{rq}_{2\mathrm{h}}) \qquad \psi\cdot na = a' m\cdot f'\varphi\cdot\psi T,$$

$$(\mathrm{rq}_{3\mathrm{h}}) \qquad \psi \nabla a = a' \nabla (\psi \top \varphi).$$

Obviously, the composition of algebra or right algebra homomorphisms in $\mathrm{wAl}(A)$ or $r\text{-}\mathrm{wAl}(A)$ is an algebra or right algebra homomorphism. Hence the endoalgebras or right algebras in A with algebra or right algebra homomorphisms determine the categories $\mathrm{Al}(A)$ and $r\text{-}\mathrm{Al}(A)$. Moreover, there are functors

$$L'\colon \mathrm{Al}(A) \to d_{(1,0)}A \quad \text{and} \quad L'_r\colon r\text{-}\mathrm{Al}(A) \to d_{(0,1)}A$$

such that $L'c$ and $L'_r c$ are the strong types of c, and

$$L'\Big(\big((m, m', \varphi), (m, \psi')\big)\Big) = \big((m, m', \varphi), (m', \psi')\big),$$

$$L'_r\Big(\big((m, \varphi), (n, n), (\psi, \psi)\big)\Big) = \big((m, \varphi), (m, n, \psi)\big)$$

for morphisms.

Obviously, the categories of algebras have the extensions to the 2-categories 2-Al(A) and 2-r-Al(A).

1.6.3. Definition. A *monad algebra* (respectively, *right algebra*) in a 2-category A over U is any 3-tuple $\boldsymbol{a} = \langle \boldsymbol{T}, f, \alpha\rangle$ consisting of a monad $\boldsymbol{T} = \langle X', T, \eta, \mu\rangle$ in A, a 1-cell $f\colon X \to X'$ (respectively, $f\colon X' \to X$) in A, a 2-cell $\alpha\colon Tf \to f$ (respectively, $\alpha\colon fT \to f$) in A such that

$$(\mathrm{m}_1) \qquad \alpha\cdot T\alpha = \alpha\cdot\mu f \quad \text{and} \quad \alpha\cdot\eta f = f$$

for the monad algebra, and

$$(\mathrm{rm}_1) \qquad \alpha\cdot\alpha T = \alpha\cdot f\mu \quad \text{and} \quad \alpha\cdot f\eta = f$$

for the right algebra.

Conditions (m_1) and (rm_1) are equivalent to

$$(\mathrm{m}_2) \qquad \alpha \bigtriangledown \alpha = \mu \top \alpha \quad \text{and} \quad \eta \top \alpha = f$$

and

$$(\mathrm{rm}_2) \qquad \alpha \top \alpha = \alpha \bigtriangledown \mu \quad \text{and} \quad \alpha \bigtriangledown \eta = f,$$

respectively.

If $\boldsymbol{a} = \langle \boldsymbol{T}, f, \alpha\rangle$ is a monad algebra (respectively, right algebra), then the pointed monad $(f, \boldsymbol{T})$ (respectively, $(\boldsymbol{T}, f)$) is said to be the *strong type* of the algebra $\boldsymbol{a}$.

Any monad algebra or right algebra $\boldsymbol{a} = \langle \boldsymbol{T}, f, \alpha\rangle$ determines the endoalgebra $(\mathrm{dom} f, T, \alpha)\colon f \to f$ or right endoalgebra $(f, f, \alpha)\colon T \to \mathrm{codom} f$, denoted by $\boldsymbol{a}_e$ or $\boldsymbol{a}_{re}$, respectively.

A *monad algebra* (respectively, *right algebra*) *homomorphism* in A from a monad algebra $\boldsymbol{a}$ to a monad algebra $\boldsymbol{b}$ in A is any algebra (respectively, right algebra) homomorphism Q from $\boldsymbol{a}_e$ (respectively, $\boldsymbol{a}_{re}$) to $\boldsymbol{b}_e$ (respectively, $\boldsymbol{b}_{re}$) such that $L'Q$ (respectively, L'_rQ) is a pointed (respectively, right pointed) monad morphism from the strong type of $\boldsymbol{a}$ to the strong type of $\boldsymbol{b}$. The composition in Al(A) or r-Al(A) of monad algebra or right algebra homomorphisms is a monad algebra or right algebra homomorphism.

1.6.4. THEOREM. *The monad algebras or right algebras in A with the monad algebra or right algebra homomorphisms define the category* Alm(A) *or* r-Alm(A). *Moreover, there are functors*

$$L\colon \mathrm{Alm}(A) \to \mathrm{Mon}_*(A), \qquad -_e\colon \mathrm{Alm}(A) \to \mathrm{Al}(A)$$

and

$$L_r\colon r\text{-}\mathrm{Alm}(A) \to r\text{-}\mathrm{Mon}_*(A), \qquad -_{re}\colon r\text{-}\mathrm{Alm}(A) \to r\text{-}\mathrm{Al}(A),$$

where L and L_r are the strong types of algebras on objects, $L = L'$, $L_r = L'_r$, and $-_e$ and $-_{re}$ are the identity mappings on morphisms.

Obviously, the monad algebra categories in A have the extensions to the 2-categories 2-Alm(A) and 2-r-Alm(A).

For any element $\varkappa$ in $\Gamma_m(A)$ or r-$\Gamma_m(A)$ we denote by $\mathrm{Alm}(A)(\varkappa)$ or r-$\mathrm{Alm}(A)(\varkappa)$ the subcategory of Alm(A) or r-Alm(A), defined by all objects $\boldsymbol{a}$ with $\boldsymbol{a}_e \in \mathrm{Al}(A)(\varkappa)$ or $\boldsymbol{a}_{re} \in r\text{-}\mathrm{Al}(A)(\varkappa)$, and by monad algebra homomorphisms in $\mathrm{Al}(A)(\varkappa)$ or r-$\mathrm{Al}(A)(\varkappa)$. The subcategory $\mathrm{Al}(A)(\varkappa)$ or r-$\mathrm{Al}(A)(\varkappa)$ of the category Al(A) or r-Al(A) is defined by objects $\boldsymbol{c}$ with $L'\boldsymbol{c}$ or $L'_r\boldsymbol{c}$ in $d_{(1,0)}A(\varkappa)$ or $d_{(0,1)}A(\varkappa)$, and by algebra homomorphisms $\boldsymbol{h}$ with $L'\boldsymbol{h}$ or $L'_r\boldsymbol{h}$ in $d_{(1,0)}A(\varkappa)$ or $d_{(0,1)}A(\varkappa)$ (¹).

1.6.5. Proposition. *For each $\varkappa$ in $\Gamma_m(A)$ or r-$\Gamma_m(A)$ the formulas*

$$L_\varkappa = L\,|\,\mathrm{Alm}(A)(\varkappa), \qquad L'_\varkappa = L\,|\,\mathrm{Al}(A)(\varkappa),$$

$$L_{r\varkappa} = L_r\,|\,r\text{-}\mathrm{Alm}(A)(\varkappa) \quad \text{and} \quad L'_{r\varkappa} = L'_r\,|\,r\text{-}\mathrm{Al}(A)(\varkappa)$$

define the functors of the form

$$L_\varkappa\colon \mathrm{Alm}(A)(\varkappa) \to \mathrm{Mon}_*(A)(\varkappa), \qquad L'_\varkappa\colon \mathrm{Al}(A)(\varkappa) \to d_{(1,0)}A(\varkappa), \qquad \varkappa \in \Gamma_m(A),$$

$$L_{r\varkappa}\colon r\text{-}\mathrm{Alm}(A)(\varkappa) \to r\text{-}\mathrm{Mon}_*(A)(\varkappa), \qquad L'_{r\varkappa}\colon r\text{-}\mathrm{Al}(A)(\varkappa) \to d_{(0,1)}A(\varkappa), \qquad \varkappa \in r\text{-}\Gamma_m(A).$$

If $A = 2\text{-Cat}(U)$ or $A = 2\text{-CAT}(U)$ and $\varkappa = (\mathbf{1}, \boldsymbol{T})$, where $\mathbf{1}$ is a one-object category and $\boldsymbol{T}$ is a monad of the category X, then $\mathrm{Alm}(A)(\varkappa)$, denoted by $X^{\boldsymbol{T}}$, is the category of all monad $\boldsymbol{T}$-algebras of the category X.

2. MAIN THEOREMS

2.1. Theorem. *Let A be any 2-category over U. Then*

(i) *there are functors*

$$H\colon \mathrm{Mon}_*(A) \to \mathrm{Alm}(A) \quad \text{and} \quad H_r\colon r\text{-}\mathrm{Mon}_*(A) \to r\text{-}\mathrm{Alm}(A)$$

such that

$$H\big((f, \boldsymbol{T})\big) = \langle \boldsymbol{T}, \boldsymbol{T}f, \mu f\rangle \quad \text{and} \quad H_r\big((\boldsymbol{T}, f)\big) = \langle \boldsymbol{T}, f\boldsymbol{T}, f\mu\rangle$$

on objects, and

$$H\Big(\big((m, m', \varphi), (m', \psi')\big)\Big) = \big((m, m', \psi' \top \varphi), (m, \psi')\big),$$

$$H_r\Big(\big((m', \psi'), (m', m, \varphi)\big)\Big) = \big((m', \psi'), (m, m), (\varphi \top \psi', \varphi \top \psi')\big)$$

on morphisms;

(¹) The categories $\mathrm{Al}(A)(\varkappa)$ with $A = 2\text{-CAT}(U)$ and $\varkappa = (\mathbf{1}, -)$ or $(\mathbf{1}, X)$ are considered in [1] and [2] as the categories of process system dynamics.

(ii) *the formulas*

$$H_\varkappa = H|\mathrm{Mon}_*(A)(\varkappa) \quad \textit{and} \quad H_{r\varkappa} = H_r|r\text{-}\mathrm{Mon}_*(A)(\varkappa)$$

define the functors

$$H_\varkappa\colon \mathrm{Mon}_*(A)(\varkappa) \to \mathrm{Alm}(A)(\varkappa), \quad \textit{where } \varkappa \in \Gamma_m(A),$$

$$H_{r\varkappa}\colon r\text{-}\mathrm{Mon}_*(A)(\varkappa) \to r\text{-}\mathrm{Alm}(A)(\varkappa), \quad \textit{where } \varkappa \in r\text{-}\Gamma_m(A).$$

Proof. Obviously, $\langle T, Tf, \mu f\rangle$ and $\langle T, fT, f\mu\rangle$ are the monad algebra and the right algebra, respectively, in A. Since

$$\begin{aligned}(\psi' \top \varphi) \nabla \mu f &= (\psi' \top \varphi) \nabla (\mu \top f) = (\psi' \nabla \mu) \top (\varphi \nabla f)\\ &= [\psi' \top (\mu' \nabla \psi')] \top \varphi = [\psi' \top (\mu' \nabla \psi')] \top (f' \nabla \varphi)\\ &= \psi' \top [(\mu' \nabla \psi') \top (f' \nabla \varphi)]\\ &= \psi' \top [(\mu' \top f') \nabla (\psi' \top \varphi)] = \psi' \top [\mu' f' \nabla (\psi' \top \varphi)],\end{aligned}$$

therefore, by (q_{3h}), H is well defined on morphisms. H also preserves the composition, and thus H is a functor. Analogically we prove that H_r is a functor. H-images and H_r-images of objects and morphisms in $\mathrm{Mon}_*(A)(\varkappa)$ and $r\text{-}\mathrm{Mon}_*(A)(\varkappa)$ belong to $\mathrm{Alm}(A)(\varkappa)$ and $r\text{-}\mathrm{Alm}(A)(\varkappa)$.

Let us consider the sets of morphisms

(p_1) $\quad \eta^* = \{\eta^*_{(f,T)} = ((X, X', \eta f), (X', T)) : (f, T) \in \mathrm{Mon}_*(A)\}$,

(p_2) $\quad \varepsilon^* = \{\varepsilon^*_{\langle T,f,a\rangle} = ((X, X', a), (X, T)) : \langle T, f, a\rangle \in \mathrm{Alm}(A)\}$,

(rp_1) $\quad r\text{-}\eta^* = \{\eta^*_{(T,f)} = ((X, T), (X, X', f\eta)) : (T, f) \in r\text{-}\mathrm{Mon}_*(A)\}$,

(rp_2) $\quad r\text{-}\varepsilon^* = \{\varepsilon^*_{\langle T,f,a\rangle} = ((X, T), (X', X'), (a, a)) : \langle T, f, a\rangle \in r\text{-}\mathrm{Alm}(A)\}$,

where $X = \mathrm{dom} f$ and $X' = \mathrm{codom} f$.

For any $\varkappa \in \Gamma_m(A)$ the sets $\eta^*|\mathrm{Mon}_*(A)(\varkappa)$ and $\varepsilon^*|\mathrm{Alm}(A)(\varkappa)$ are denoted by $\eta^*_\varkappa$ and $\varepsilon^*_\varkappa$, respectively. For any $\varkappa \in r\text{-}\Gamma_m(A)$ the sets $r\text{-}\eta^*|r\text{-}\mathrm{Mon}_*(A)$ and $r\text{-}\varepsilon^*|r\text{-}\mathrm{Alm}(A)(\varkappa)$ are denoted by $r\text{-}\eta^*_\varkappa$ and $r\text{-}\varepsilon^*_\varkappa$, respectively. •

2.2. Definitions. Let A be any 2-category over U.

(I) We say that a pair $R = \langle M, B\rangle$ of categories *fulfils property* (S) under (S_1)-(S_4) if

(S_1) M is a subcategory of $\mathrm{Mon}_*(A)$ such that, for each object (f, T) in M, $\eta^*_{(f,T)}$ is a morphism in M;

(S_2) B is a subcategory of $\mathrm{Alm}(A)$ such that, for each object $\langle T, f, a\rangle$ in B, $\varepsilon^*_{\langle T,f,a\rangle}$ is a morphism in B;

(S_3) $H(Q)$ is a morphism in B for any morphism Q in M;

(S_4) $L(Q)$ is a morphism in M for any morphism Q in B.

(II) We say that a pair $R = \langle M, B\rangle$ of categories *fulfils property* (rS) under (rS_1)-(rS_4) if

(rS_1) M is a subcategory of r-$\mathrm{Mon}_*(A)$ such that, for each object $\langle T, f\rangle$ in M, $\eta^*_{\langle T,f\rangle}$ is a morphism in M;

(rS_2) B is a subcategory of r-$\mathrm{Alm}(A)$ such that, for each object $\langle T, f, a\rangle$ in B, $\varepsilon^*_{\langle T,f,a\rangle}$ is a morphism in B;

(rS_3) $H_r(Q)$ is a morphism in B for any morphism Q in M;

(rS_4) $L_r(Q)$ is a morphism in M for any morphism Q in B.

(III) We say that a pair $R = \langle M, B\rangle$ *has property* (S) *or* (rS) if it has property (S) under (S_1)-(S_4) or property (rS) under (rS_1)-(rS_4) and the following condition (S_5) or (rS_5) holds:

(S_5) (a) For each morphism of the form $\big((X, X', a), (X', T)\big)$ from $\langle Tf, T\rangle$ to $\langle f, T\rangle$ in M, where $X = \mathrm{dom} f$ and $X' = \mathrm{codom} f$, if $\langle T, f, a\rangle$ is a monad algebra in A, then it is an object in B.

(S_5) (b) If

$$\big((n, n', \gamma), (n', \psi')\big)\colon \langle f, T\rangle \to \langle f', T'\rangle,$$

where

$$f\colon X \to X', \quad T\colon X' \to X' \quad \text{and} \quad f'\colon Y \to Y', \quad T'\colon Y' \to Y',$$

is a morphism in M,

$$Q' = \big((n, n', \gamma), (n, \psi')\big)\colon \langle T, f, a\rangle \to \langle T', f', a'\rangle$$

is a morphism in $\mathrm{Alm}(A)$,

$$\big((X, X', a), (X', T)\big)\colon \langle Tf, T\rangle \to \langle f, T\rangle$$

and

$$\big((Y, Y', a'), (Y', T')\big)\colon \langle T'f', T'\rangle \to \langle f', T'\rangle$$

are morphisms in M, then Q' is a morphism in B.

(rS_5) (a) For each morphism of the form $\big((X, T), (X, X', a)\big)$ from $\langle T, fT\rangle$ to $\langle T, f\rangle$ in M, if $\langle T, f, a\rangle$ is a monad right algebra in A, then it is an object in B.

(rS_5) (b) If

$$\big((n', \psi'), (n', n, \gamma)\big)\colon \langle T, f\rangle \to \langle T', f'\rangle,$$

where

$$T\colon X \to X, \quad f\colon X \to X' \quad \text{and} \quad T'\colon Y \to Y, \quad f'\colon Y \to Y',$$

is a morphism in M,

$$Q' = \big((n', \psi'), (n, n), (\gamma, \gamma)\big)$$

is a morphism in r-$\mathrm{Alm}(A)$ from $\langle T, f, a\rangle$ to $\langle T', f', a'\rangle$ and, moreover,

$$\big((X, T), (X, X', a)\big)\colon \langle T, fT\rangle \to \langle T, f\rangle$$

and

$$\big((Y, T), (Y, Y', a')\big)\colon \langle T', f'T'\rangle \to \langle T', f'\rangle$$

are morphisms in M, then Q' is a morphism in B.

Let $R = \langle M, B\rangle$ be any pair which fulfils property (S) under (S_1)-(S_4) or property (rS) under (rS_1)-(rS_4). By L_R and ε_R^* we denote the cuttings of L or L_r and of ε^* or r-ε^* to B, respectively. By H_R and η_R^* we denote the cuttings of H or H_r and of η^* or r-η^* to M, respectively. Obviously, L_R: $B \to M$ and H_R: $M \to B$ are functors. Let us observe that the pairs

$$\langle \mathrm{Mon}_*(A), \mathrm{Alm}(A)\rangle, \quad \langle \mathrm{Mon}_*(A)(\varkappa), \mathrm{Alm}(A)(\varkappa)\rangle \quad \text{with } \varkappa \in \Gamma_m(A)$$

have property (S), and the pairs

$$\langle r\text{-}\mathrm{Mon}_*(A), r\text{-}\mathrm{Alm}(A)\rangle, \quad \langle r\text{-}\mathrm{Mon}_*(A)(\varkappa), r\text{-}\mathrm{Alm}(A)(\varkappa)\rangle$$

have property (rS). The functors L_R, H_R of those pairs are the same as L, H, $L_\varkappa$, $H_\varkappa$, L_r, H_r, $L_{r\varkappa}$ and $H_{r\varkappa}$, respectively.

2.3. THEOREM. *Let A be any 2-category over U.*

(I) *For each pair $R = \langle M, B\rangle$ of categories which fulfils property* (S) *under* (S_1)-(S_4) *or property* (rS) *under* (rS_1)-(rS_4) *we have the adjunction*

$$\Sigma_R = \langle H_R, L_R, \eta_R^*, \varepsilon_R^*\rangle\colon M \rightharpoonup B.$$

(II) *We have the following adjunctions*:

(a) $$\Sigma = \langle H, L, \eta^*, \varepsilon^*\rangle\colon \mathrm{Mon}_*(A) \rightharpoonup \mathrm{Alm}(A),$$

(b) $$\Sigma_\varkappa = \langle H_\varkappa, L_\varkappa, \eta_\varkappa^*, \varepsilon_\varkappa^*\rangle\colon \mathrm{Mon}_*(A)(\varkappa) \rightharpoonup \mathrm{Alm}(A)(\varkappa)$$

for each element $\varkappa$ in $\Gamma_m(A)$,

(ra) $$\Sigma_r = \langle H_r, L_r, r\text{-}\eta^*, r\text{-}\varepsilon^*\rangle\colon r\text{-}\mathrm{Mon}_*(A) \rightharpoonup r\text{-}\mathrm{Alm}(A),$$

(rb) $$\Sigma_{r\varkappa} = \langle H_{r\varkappa}, L_{r\varkappa}, r\text{-}\eta_\varkappa^*, r\text{-}\varepsilon_\varkappa^*\rangle\colon r\text{-}\mathrm{Mon}_*(A)(\varkappa) \rightharpoonup r\text{-}\mathrm{Alm}(A)(\varkappa)$$

for each element $\varkappa$ in r-$\Gamma_m(A)$.

Proof. Using definitions (p_i), (rp_i), equalities (q_{ih}), (rq_{ih}) and Proposition 1.2.1, we see that

$$\eta_R^*\colon 1_M \to L_R H_R \quad \text{and} \quad \varepsilon_R^*\colon H_R L_R \to 1_B$$

are the natural transformations of functors. Now we verify that

$$\varepsilon_R^* H_R \cdot H_R \eta_R^* = H_R \quad \text{and} \quad L_R \varepsilon_R^* \cdot \eta_R^* L_R = L_R,$$

and thus Σ_R is indeed an adjunction. Part (II) follows from (I).

Let us denote by D_R, D and $D_\varkappa$ the monads defined by the adjunctions Σ_R, (a) and (b) from Theorem 2.3, and by D_r and $D_{r\varkappa}$ the monads defined by the adjunctions (ra) and (rb) from this theorem, respectively.

2.4. THEOREM. *Let A be any 2-category over U.*

(I) *Let $R = \langle M, B\rangle$ be any pair of categories which has property* (S) *or* (rS). *Then the category B is isomorphic to the category M^{D_R} of all monad*

D_R-algebras in the category M. This isomorphism is given by the comparison functor defined by the adjunction Σ_R from Theorem 2.3 (I).

(II) *We have the isomorphisms*

$$\text{(a)} \qquad \mathrm{Alm}(A) \simeq \mathrm{Mon}_*(A)^D,$$

$$\text{(b)} \qquad \mathrm{Alm}(A)(\varkappa) \simeq \mathrm{Mon}_*(A)(\varkappa)^{D\varkappa} \quad \textit{for } \varkappa \in \Gamma_m(A),$$

$$\text{(ra)} \qquad r\text{-}\mathrm{Alm}(A) \simeq r\text{-}\mathrm{Mon}_*(A)^{Dr},$$

$$\text{(rb)} \qquad r\text{-}\mathrm{Alm}(A)(\varkappa) \simeq r\text{-}\mathrm{Mon}_*(A)(\varkappa)^{Dr\varkappa} \quad \textit{for } \varkappa \in r\text{-}\Gamma_m(A),$$

which are given by the comparison functors defined by the adjunctions (a), (b), (ra) *and* (rb) *from Theorem* 2.3 (II).

Proof. Consider the comparison functor $K\colon B \to M^{D_R}$ defined by the adjunction Σ_R and assume that R has property (S). Then

$$K(\boldsymbol{a}) = \langle D_R, L_R\boldsymbol{a}, L_R\varepsilon_R^*\boldsymbol{a}\rangle = \langle D_R, (f, T), ((X, X', a), (X', T))\rangle,$$

where $\boldsymbol{a} = \langle T, f, a\rangle$, and

$$L_R\varepsilon_R^*\boldsymbol{a} = ((X, X', a), (X', T))\colon D_R(f, T) = (Tf, T) \to (f, T)$$

is a morphism in M. Moreover, $KQ = L_RQ$ on morphisms.

Now we will define a functor $G\colon M^{D_R} \to B$. For this let $\boldsymbol{b}^* = \langle D_R, b, Q\rangle$ be any monad D_R-algebra in M. Then $b = (f, T)$, and

$$Q = ((m, m', \varphi), (m', \psi'))\colon (Tf, T) \to (f, T)$$

is a morphism in M such that

$$Q\cdot\eta_R^* b = b, \quad Q\cdot D_R Q = Q\cdot L_R\varepsilon_R^* H_R b \quad \text{in } M.$$

From the first equality we obtain

$$((m, m', \varphi \nabla \eta f), (m', \psi')) = ((X, X', f), (X', T)),$$

and thus

$$m = X, \quad m' = X', \quad \psi' = T \quad \text{and} \quad \varphi \nabla \eta f = \varphi\cdot\eta f = f,$$

where $X = \mathrm{dom} f$ and $X' = \mathrm{codom} f$. By the second equality we have

$$((X, X', \varphi), (X', T))((X, X', T\varphi), (X', T))$$
$$= ((X, X', \varphi), (X', T))((X, X', \mu f), (X', T)),$$

and thus $\varphi \nabla T\varphi = \varphi \nabla \mu f$, i.e. $\varphi\cdot T\varphi = \varphi\cdot\mu f$. Hence $G\boldsymbol{b}^* = \langle T, f, \varphi\rangle$ is, by (m_1), a monad algebra in A and, by (S_5) (a), it is an object in B.

Consider any D_R-algebra morphism $q = ((n, n', \gamma), (n', \psi'))$ in M from any D_R-algebra $\boldsymbol{b}^* = \langle D_R, b, Q\rangle$ to D_R-algebra $\boldsymbol{b}_1^* = \langle D_R, b_1, Q_1\rangle$ in M. Then $q\cdot Q = Q_1\cdot D_R q$. But, by the above,

$$b = (f, T), \quad Q = ((X, X', \varphi), (X', T)),$$

where $X = \mathrm{dom} f$ and $X' = \mathrm{codom} f = \mathrm{dom} T$,

$$b_1 = (f', T'), \quad Q_1 = \big((Y, Y', \varphi'), (Y', T')\big),$$

where $Y = \mathrm{dom} f'$ and $Y' = \mathrm{codom} f' = \mathrm{dom}\, T'$, and thus the last equality gives

$$\big((n, n', \gamma \nabla \varphi), (n', \psi')\big) = \big((n, n', \varphi' \nabla (\psi' \top \gamma)), (n', \psi')\big),$$

i.e. $\gamma \nabla \varphi = \varphi' \nabla (\psi' \top \gamma)$. Hence

$$\gamma \nabla \varphi = \varphi' \nabla (T'\gamma \cdot \psi' f) = \varphi' n \cdot (T'\gamma \cdot \psi' f) = \varphi' n \cdot T'\gamma \cdot \psi' f = (\varphi' \nabla \gamma) \cdot \psi' f,$$

and thus $Gq = \big((n, n', \gamma), (n, \psi')\big)$ is, by $(\mathrm{q_{1h}})$, a monad algebra homomorphism in A from Gb^* to Gb_1^* which, by (S_5) (b), belongs to the category B. Moreover, G preserves the composition and $GK = 1_B$, $KG = 1_M$. Hence the functor K is an isomorphism. The proof for the pair with property (rS) is obtained in an analogical way. Part (II) follows from (I).

2.5. Examples. Let $A = 2\text{-CAT}(U)$. The categories $\mathrm{Alm}(A)(\varkappa)$ and $\mathrm{Mon}_*(A)(\varkappa)$ with $\varkappa = (1, X')$ will be denoted by $\mathrm{Alm}(X')$ and $\mathrm{Mon}_*(X')$, respectively.

Example 1 (the category of all group actions). Let $X' = \mathrm{Set}(U)$. Each group G in X' defines a monad

$$T(G) = \langle X', T(G), \eta_G, \mu_G \rangle$$

such that $T(G)(f) = G \times f$, $\eta_G(f)(e) = \langle u, e \rangle$, u being the unit of G, and $\mu_G(f)\big((g, (g', e))\big) = (gg', e)$. The G-group action $\alpha\colon G \times f \to f$ on the set f is exactly the monad $T(G)$-structure on f and it defines the monad $T(G)$-algebra $\langle T(G), f, \alpha \rangle$. The monad algebra homomorphism in $\mathrm{Alm}(X')$ of the form

$$\big((1, X', \varphi), (1, \psi)\big)\colon \langle T(G), f, \alpha \rangle \to \langle T(G'), f', \alpha' \rangle,$$

where G and G' are groups, can be regarded as the group action morphism, since $\psi\colon G \to G'$ is a group morphism and $\varphi\colon f \to f'$ is a set morphism with $\varphi(ge) = \psi(g)\varphi(e)$ for $g \in G$ and $e \in f$. Let us denote by Alag and Mag the full subcategories of $\mathrm{Alm}(X')$ and $\mathrm{Mon}_*(X')$ defined by all objects $\langle T(G), f, \alpha \rangle$ and $\langle f, T(G) \rangle$, respectively, where G and f are any groups and sets. The pair $R = \langle \mathrm{Mag}, \mathrm{Alag} \rangle$ has property (S), and thus, by Theorem 2.4, we obtain

2.5.1. THEOREM. *The comparison functor* $K\colon \mathrm{Alag} \to \mathrm{Mag}^{D_R}$ *defined by the adjunction* Σ_R *with* $R = \langle \mathrm{Mag}, \mathrm{Alag} \rangle$ *is an isomorphism.*

Remark. The consideration of Example 1 also holds if instead of groups and of $X' = \mathrm{Set}$ we take the monoids and monoidal category.

Example 2 (the category of all modules). Let $X' = \mathrm{Ab}$ be the category of all Abelian groups. Every ring P defines a monad of X' of the form

$$\boldsymbol{T}(P) = \langle X', T(P), \eta_P, \mu_P \rangle,$$

where $T(P)(G) = P \otimes G$ is the tensor product, $\eta_P(G)(g) = 1 \otimes g$, and $\mu_P(G)\big((p_1, (p_2, g))\big) = (p_1 p_2, g)$. A monad $\boldsymbol{T}(P)$-algebra $\langle \boldsymbol{T}(P), G, a\rangle$ in the category Ab is precisely a left P-module. The monad algebra homomorphism in $\mathrm{Alm}(X')$, which is of the form $\big((1, X', \varphi), (1, \psi)\big)$ from $\langle \boldsymbol{T}(P), G, a\rangle$ to $\langle \boldsymbol{T}(P'), G', a'\rangle$, can be regarded as a ring module morphism, since $\psi\colon P \to P'$ and $\varphi\colon G \to G'$ are ring and set morphisms, respectively, with $\varphi(pg) = \psi(p)\varphi(g)$ for all $p \in P$ and $g \in G$. Denote by Mod and Mr the full subcategories of $\mathrm{Alm}(X')$ and $\mathrm{Mon}_*(X')$ defined by all objects $\langle \boldsymbol{T}(P), G, a\rangle$ and $\langle G, \boldsymbol{T}(P)\rangle$, respectively, where P and G are any ring and Abelian group. The pair $R = \langle \mathrm{Mr}, \mathrm{Mod}\rangle$ has property (S).

2.5.2. THEOREM. *The comparison functor* $K\colon \mathrm{Mod} \to \mathrm{Mr}^{D_R}$ *defined by the adjunction* Σ_R *with* $R = \langle \mathrm{Mr}, \mathrm{Mod}\rangle$ *is an isomorphism.*

Example 3 (functor automata). Let X' be any U-category. The notion of a monad in the category $e(X')$ of all functors from X' to X' is equivalent to the notion of a monoid in $e(X')$. Thus the monad algebras in $e(X')$ can be regarded as the automata in this category and $\mathrm{Alm}\big(e(X')\big)$ is the category of functor automata in X'. Moreover, the pair $R = \langle \mathrm{Mon}_*\big(e(X')\big), \mathrm{Alm}\big(e(X')\big)\rangle$ has property (S).

2.5.3. THEOREM. *The comparison functor*

$$K\colon \mathrm{Alm}\big(e(X')\big) \to \mathrm{Mon}_*\big(e(X')\big)^{D_R}$$

defined by the adjunction Σ_R *with* $R = \langle \mathrm{Mon}_*\big(e(X')\big), \mathrm{Alm}\big(e(X')\big)\rangle$ *is an isomorphism.*

3. *R-AUTOMATA*

Let A be any 2-category over U and let $R = \langle M, B\rangle$ be any pair of categories which has property (S) under (S_1)-(S_4). For fixed objects I^* and Y in M we define the category $R(I^*, Y)$ of all *R-automata* with input state I^* and output state Y. The objects of this category are all R-automata $\mathfrak{M} = \langle \boldsymbol{a}, \boldsymbol{h}, \tau, \beta\rangle$, where $\boldsymbol{a} = (f, \boldsymbol{T})$ is any object in M, $\langle \boldsymbol{T}, f, h\rangle = \langle \boldsymbol{a}, \boldsymbol{h}\rangle$ is an object in B, $\tau\colon I^* \to \boldsymbol{a}$, $\beta\colon \boldsymbol{a} \to Y$ are morphisms in M, and $\boldsymbol{T} = \langle X', T, \eta, \mu\rangle$ is a monad with $X' = \mathrm{codom} f = \mathrm{dom}\, T$ in A. Each R-automaton $\mathfrak{M}$ defines in M the diagram

$$\begin{array}{ccccc} (Tf, \boldsymbol{T}) & \xrightarrow{h^1} & (f, \boldsymbol{T}) & \xrightarrow{\beta} & Y, \\ & & \uparrow{\scriptstyle\tau} & & \\ & & I^* & & \end{array}$$

where $h^1 = L_R\varepsilon_R^*\langle a, h\rangle = \big((X, X', h), (X', T)\big)$ is the next state morphism, τ and β are the initial and output morphisms of $\mathfrak{M}$, respectively.

Given R-automata $\mathfrak{M}$ and $\mathfrak{M}'$, an ***R-simulation*** $\sigma\colon \mathfrak{M} \to \mathfrak{M}'$ is a morphism $\sigma\colon \langle T, f, h\rangle \to \langle T', f', h'\rangle$ in B such that the diagram

$$\begin{tikzcd}
 & (f, T) \arrow[dd, "L_R\sigma"] \arrow[dr, "\beta"] & \\
I^* \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & & Y \\
 & (f', T') \arrow[ur, "\beta'"'] &
\end{tikzcd}$$

commutes in M. The morphisms of the category $R(I^*, Y)$ are the R-simulations.

Now we define the external behaviour of R-automaton $\mathfrak{M} = \langle (f, T), h, \tau, \beta\rangle$, where $T = \langle X', T, \eta, \mu\rangle$ and $f\colon X \to X'$ are a monad and a 1-cell in A, respectively. By Theorem 2.3, there is an R-free algebra in B over every object in M. Let $\langle I^*, I, Ii, \mu^0 i\rangle$, where $I^* = (i, I)$ and $I = \langle J, I, \eta^0, \mu^0\rangle$, be an R-free monad algebra in B over I^*. Hence the initial morphism $\tau\colon I^* \to (f, T)$ has the unique extension morphism τ^* from $\langle I, Ii, \mu^0 i\rangle$ to $\langle T, f, h\rangle$ in B such that the diagram

$$\begin{tikzcd}
 & (I^2 i, I) \arrow[r, "\mu^0 i"] \arrow[dl, "D_R^2\tau"'] \arrow[d, "H_R L_R\tau^*"] & (Ii, I) \arrow[r, "\eta^0 i"] \arrow[dl, "D_R\tau"'] \arrow[d, "L_R\tau^*"] & (i, I) = I^* \arrow[dl, "\tau"] \\
(T^2 f, T) \arrow[r, "\mu f"'] & (Tf, T) \arrow[r, "h^1"'] & (f, T) \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

commutes in M. If $L_R\tau^*$ is an epimorphism in M, then $\mathfrak{M}$ is said to be *reachable*. The morphism $\beta L_R\tau^*\colon (Ii, I) \to Y$ in M is denoted by $g_{\mathfrak{M}}$ and it is called the *external behaviour* of $\mathfrak{M}$. Conversely, an R-realization of a morphism $g\colon (Ii, I) \to Y$ in M is any R-automaton $\mathfrak{M}$ whose behaviour is g, i.e. $g_{\mathfrak{M}} = g$. A realization $\mathfrak{M}$ is *minimal* if $\mathfrak{M}$ is reachable and for any other reachable R-realization $\mathfrak{M}'$ there is an R-simulation $\sigma'\colon \mathfrak{M}' \to \mathfrak{M}$. We ask under what condition minimal R-realizations exist, i.e. under what condition the category R_g, which is a full subcategory of $R(I^*, Y)$ determined by all reachable R-realizations of g, has the terminal object? (**P 1164**). The category R_g always has an initial object. The general open problem is determining all pairs R for which this problem is solved.

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