COLLOQUIUM MATHEMATICUM

VOL. XLIX 1984 FASC. 1

BIREGULAR EDGE-SYMMETRIC GRAPHS

BY

DAVID BURNS (BIG RAPIDS, MICHIGAN)

A graph G is *edge-symmetric* if for any two edges e and f of G there exists an induced edge-automorphism α of G such that $\alpha(e) = f$. Edge-symmetric graphs have been studied by Bouwer [2], [3], Dauber and Harary [5], Folkman [6], and Foster [7], [8]. We observe that, for all positive integers m and n, the graphs nK_m, $K(m, n)$, $K_{m(n)}$, and nC_m are edge-symmetric as are a number of other well-known graphs such as the Heawood and Petersen graphs and the graphs of the five regular polyhedra. Our terminology follows that of Behzad et al. [1].

If G and H are graphs such that $G = H \cup K_1$, then G is edge-symmetric if and only if H is edge-symmetric. With no loss, then, we focus our attention on edge-symmetric graphs containing no isolated vertices.

In [4] it was shown that if G is an edge-symmetric graph which has no isolated vertices and if $\mathscr{D}_G$ is the degree set of G, then $|\mathscr{D}_G| = 1$ or $|\mathscr{D}_G| = 2$. We call a graph *biregular* if $|\mathscr{D}_G| = 2$. Our goal is to establish non-structural characterizations of those biregular edge-symmetric graphs containing no isolated vertices.

We begin work towards that goal with some new terminology and notation. For unequal positive integers a and b and for a graph G with $\{a, b\} \subset \mathscr{D}_G$, we denote by V_a (respectively, V_b) the set of vertices of G with degree a (respectively, b). For such a graph we define an equivalence relation on $V_a \cup V_b$ as follows: two vertices v_1 and v_2 are equivalent if and only if $N(v_1) = N(v_2)$, where $N(v)$ denotes the neighborhood of v. Note that if two vertices are equivalent, then they have the same degree. Hence each equivalence class so formed is a subset of either V_a or V_b. We denote these classes by

$$V_a(1),\ V_a(2), \ldots, V_a(N_a),\ V_b(1),\ V_b(2), \ldots, V_b(N_b).$$

If G is a bipartite graph, then a *transitive bipartition* of G is a bipartition of G such that for any two vertices u and v of G belonging to the same partite set there exists an automorphism θ of G such that $\theta(u) = v$.

Our first result for these graphs appears in [4].

THEOREM 1. *If G is an edge-symmetric graph containing no isolated vertices where $\mathscr{D}_G = \{a, b\}$, then G is bipartite and has a unique transitive bipartition $V_a \cup V_b$.*

Our characterization theorem involves the identification of a class of graphs containing edge-symmetric graphs as a proper subset. A graph G is called *neighborhood symmetric* if for any two pairs $\{u_1, v_1\}$, $\{u_2, v_2\}$ of adjacent vertices of G with $\deg u_1 = \deg u_2$ and $\deg v_1 = \deg v_2$ there exists an automorphism α of G such that $\alpha(N(u_1)) = N(u_2)$ and $\alpha(N(v_1)) = N(v_2)$. The tree shown in Fig. 1 is neighborhood symmetric but not edge-symmetric.

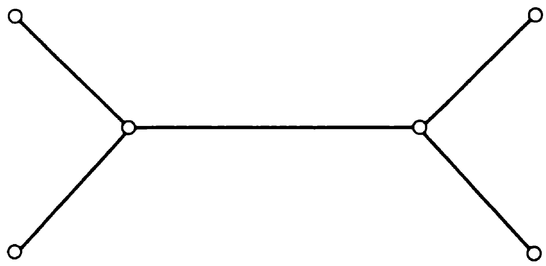

Fig. 1

THEOREM 2. *Let G be a graph containing no isolated vertices where $\{a, b\} \subset \mathscr{D}_G$. Then G is edge-symmetric and biregular if and only if G has a bipartition $V_a \cup V_b$ and G is neighborhood symmetric.*

Proof. Assume first that G is edge-symmetric and biregular. It is immediate from Theorem 1 that G has a bipartition $V_a \cup V_b$. To show that G is neighborhood symmetric, let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be two pairs of adjacent vertices in G such that $\deg u_1 = \deg u_2 = a$ and $\deg v_1 = \deg v_2 = b$. Since G is edge-symmetric, there exists an automorphism α of G whose induced edge-automorphism maps $u_1 v_1$ to $u_2 v_2$. Thus $\alpha\{u_1, v_1\} = \{u_2, v_2\}$, and since a and b are unequal, $\alpha(u_1) = u_2$ and $\alpha(v_1) = v_2$. Hence $\alpha(N(u_1)) = N(\alpha(u_1)) = N(u_2)$ and $\alpha(N(v_1)) = N(\alpha(v_1)) = N(v_2)$.

Suppose, conversely, that G has a bipartition $V_a \cup V_b$ and is neighborhood symmetric. It is obvious that G is biregular. To show that G is edge-symmetric let $e_1 = u_1 v_1$ and $e_2 = u_2 v_2$ be edges of G where, say, $u_i \in V_a$ and $v_i \in V_b$ for $i = 1, 2$. We consider two cases depending on whether or not $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are equivalent pairs of vertices.

Case 1. Assume that $N(u_1) = N(u_2)$ and $N(v_1) = N(v_2)$. Hence there exist integers i and j such that $1 \leqslant i \leqslant n_a$ and $1 \leqslant j \leqslant n_b$, where $\{u_1, u_2\} \subset V_a(i)$ and $\{v_1, v_2\} \subset V_b(j)$. Let π: $V_a(i) \to V_a(i)$ be a bijection such that $\pi(u_1) = u_2$. Define φ: $V(G) \to V(G)$ as follows:

$$\varphi(v) = \begin{cases} \pi(v) & \text{if } v \in V_a(i), \\ v & \text{otherwise.} \end{cases}$$

Similarly, let λ: $V_b(j) \to V_b(j)$ be a bijection with $\lambda(v_1) = v_2$ and define θ: $V(G) \to V(G)$ as follows:

$$\theta(v) = \begin{cases} \lambda(v) & \text{if } v \in V_b(j), \\ v & \text{otherwise}. \end{cases}$$

The maps φ and θ are automorphisms of G. Thus $\theta\varphi$ is an automorphism of G mapping u_1 to u_2 and v_1 to v_2. Therefore, $\theta\varphi$ induces an edge-automorphism of G which maps e_1 to e_2.

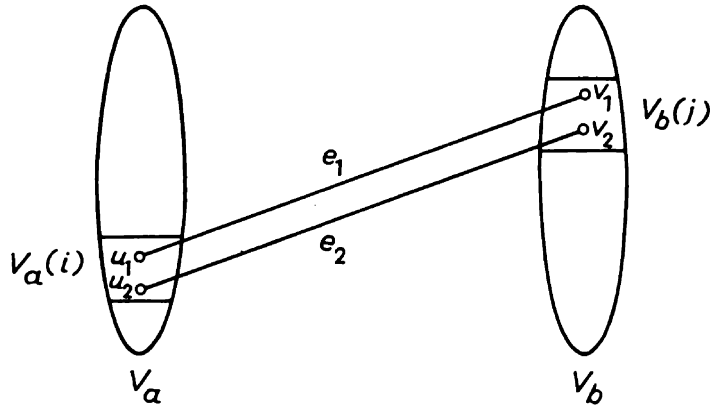

Fig. 2

Case 2. Assume that either $N(u_1) \neq N(u_2)$ or $N(v_1) \neq N(v_2)$. Since G is neighborhood symmetric and since $\deg u_1 = \deg u_2 = a$ and $\deg v_1 = \deg v_2 = b$, there exists an automorphism α of G such that $\alpha(N(u_1)) = N(u_2)$ and $\alpha(N(v_1)) = N(v_2)$. It follows that $N(\alpha(u_1)) = N(u_2)$ and $N(\alpha(v_1)) = N(v_2)$. Applying the argument of Case 1 to the equivalent pairs of vertices $\{\alpha(u_1), u_2\}$ and $\{\alpha(v_1), v_2\}$ we conclude that there exists an automorphism β of G such that $\beta(\alpha(u_1)) = u_2$ and $\beta(\alpha(v_1)) = v_2$. Thus $\beta\alpha$ is an automorphism of G whose induced edge automorphism maps e_1 to e_2. This completes the proof.

There exist bipartite biregular neighborhood symmetric graphs which are not edge-symmetric; see, for example, the graph of Fig. 1. By Theorem 2 this graph fails to be edge-symmetric because it has no bipartition $V_1 \cup V_3$. The requirement that G be neighborhood symmetric is also necessary in Theorem 2. The graph in Fig. 3, for example, is biregular and is bipartite with bipartition $V_3 \cup V_2$ yet neither edge-symmetric nor neighborhood symmetric.

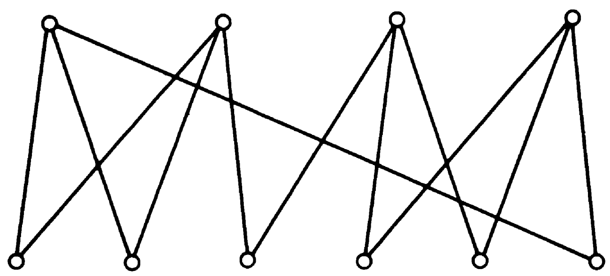

Fig. 3

A second characterization is now easy to establish.

COROLLARY. *Let G be a graph containing no isolated vertices where $\{a, b\} \subset \mathscr{D}_G$. Then G is edge-symmetric and biregular if and only if G has a transitive bipartition and is neighborhood symmetric.*

Proof. Assume first that G is edge-symmetric and biregular. By Theorem 1, G has a transitive bipartition and, by Theorem 2, G is neighborhood symmetric. Suppose, conversely, that G has a transitive bipartition and is neighborhood symmetric. Vertices belonging to the same partite set of the transitive bipartition must be of the same degree so that the transitive bipartition is $V_a \cup V_b$ and the result follows by Theorem 2.

REFERENCES

[1] M. Behzad, G. Chartrand and L. Lesniak-Foster, *Graphs and digraphs*, Boston 1979.

[2] I. Z. Bouwer, *An edge but not vertex transitive cubic graph*, Canadian Mathematical Bulletin 11 (1968), p. 533-535.

[3] — *On edge but not vertex transitive regular graphs*, Journal of Combinatorial Theory, Series B, 12 (1972), p. 32-40.

[4] D. Burns, S. F. Kapoor and P. A. Ostrand, *On line-symmetric graphs*, Fundamenta Mathematicae (to appear).

[5] E. Dauber and F. Harary, *On point-symmetric and line-symmetric graphs* (unpublished).

[6] J. Folkman, *Regular line-symmetric graphs*, Journal of Combinatorial Theory 3 (1967), p. 215-232.

[7] R. M. Foster, *Geometrical circuits of electrical networks*, Transactions of the American Institute of the Electrical Engineerings 51 (1932), p. 309-317.

[8] — *A census of trivalent symmetrical graphs. I* (unpublished).

MATHEMATICS DEPARTMENT
FERRIS STATE COLLEGE
BIG RAPIDS, MICHIGAN

Reçu par la Rédaction le 13. 10. 1980