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NON-ARCHIMEDEAN ADJOINTS

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A natural non-Archimedean analogue of the notion of adjoint transformation on a complex Hilbert space is the following:

Definition. Let $A: X \to Y$ be a linear transformation between non-Archimedean normed linear spaces. A linear transformation $B: Y_0 \to X$ defined on a subspace Y_0 of Y containing the range of A is called an *adjoint* of A if, for $x \in X$ and $y \in Y_0$, $Ax \perp y$ implies $x \perp By$. The set of all adjoints of A defined on Y_0 will be denoted by $(A: X \to Y_0)'$, and A' will denote $(A: X \to Y)'$.

Very loosely speaking, our main result is that A has an adjoint iff there is a restriction $A|_M$ of A such that $A|_M$ has the same norm as A, $A(M)$ is "almost" equal to Y_0, and $A|_M$ is "almost" an isometry. If these conditions are fulfilled, $(A: X \to Y_0)'$ consists of those operators that are "almost" inverses of $A|_M$.

Some notation from [1] and [2], to which the reader is referred for background material, is recalled below.

X and Y will denote non-Archimedean normed linear spaces over the complete non-trivially valued field F. In case F has discrete valuation, the symbol π will be reserved for a prime element of the ring of integers of F. An asterisk will denote "non-zero elements of"; e.g., $X^* = X - \{0\}$.

The space of bounded linear transformations from X to Y will be denoted by $L(X, Y)$. If A is a linear transformation on X, then $R(A) = \{Ax \mid x \in X\}$. The cardinality of the collection of cosets $\{\|x\| \, |F^*| \mid x \in X^*\}$ will be denoted by $e(X)$.

The relation $u \operatorname{non}\perp v$ (u is not orthogonal to v) is an equivalence relation on X^*, and $\hat{x}$ denotes the equivalence class of x. The conventions $0 \operatorname{non}\perp x$ and $x \operatorname{non}\perp 0$ for all $x \in X$ are adopted. X is *singular* if $u \operatorname{non}\perp v$ for all $u, v \in X^*$, and X is an *immediate extension* of its subspace M if, for all $x \in X^*$, $x \operatorname{non}\perp M$ (i.e., $x \operatorname{non}\perp m$ for some $m \in M^*$). A linear transformation $A: X \to Y$ is *orthogonal* if $u \perp v$ implies $Au \perp Av$. (According to [1], an orthogonal transformation A on a non-singular space X is "almost an

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isometry", in the sense that $\|Ax\|/\|x\|$, $x \in X^*$, is almost constant. The assumption in [1] that X is complete is not essential.) Define $\mathcal{N}(X)$ to be the set

$$\{A \in L(X, X)^* \mid Ax \text{ non} \perp x \text{ for all } x \in X^*\}.$$

We assume from this point on that A is a non-zero linear transformation from X to Y (possibly $Y = X$), Y_0 is a subspace of Y containing $R(A)$, and X and Y_0 are non-singular.

One observes immediately that the notion of adjoint is not vacuous: $A^{-1} \in A'$ if A is *biorthogonal*, i.e., if A is bijective and A and A^{-1} are orthogonal; and $(i\colon M \to X) \in (E\colon X \to M)'$ if E is an orthogonal projection on a non-singular subspace M, and i is the injection mapping. Adjoints, when they exist, are never unique, since a non-zero scalar multiple of an adjoint is also an adjoint; more generally,

$$B \in (A\colon X \to Y_0)' \text{ implies } \mathcal{N}(X)B\mathcal{N}(Y_0) \subset (A\colon X \to Y_0)'$$

(see Lemma 1 below). Relations such as

$$(aA)' = A' \quad \text{for } a \in F^*$$

and

$$(A\colon X \to Y)'(B\colon Y \to Z)' \subset (BA\colon X \to Z)'$$

are obvious.

LEMMA 1. *Let $D\colon X \to Y$ be an orthogonal transformation. If F has non-discrete valuation or $e(X) = e(Y) = 1$, then $\|Dx\| = \|D\|\,\|x\|$ for all $x \in X$.*

For the proof, apply Theorem 1 of [1].

LEMMA 2. *Each operator in $\mathcal{N}(X)$ is a scalar multiple of an isometry.*

Proof. First it is established that an operator $D \in \mathcal{N}(X)$ is 1-1. Suppose that $Du = 0$ for $u \neq 0$. If $x \perp u$, then, choosing $a \in F^*$ such that $\|x\| < \|au\|$, we obtain

$$Dx = D(x + au) \in \hat{u} \cup \{0\}.$$

Since also $Dx \in \hat{x} \cup \{0\}$, we have $Dx = 0$. If $v \in \hat{u}$, the above argument with x replaced by v and u replaced by x implies $Dv = 0$. Thus $D = 0$, contradicting the choice of D. Since D is 1-1 and u non$\perp$ v is an equivalence relation on X^*, D is orthogonal. The conclusion of the lemma is now a simple consequence of Theorem 1 from [1] (cf. the proof of Lemma 8.2 in [2]).

LEMMA 3. *If $(A\colon X \to Y_0)'$ is not empty, then Y_0 is an immediate extension of $R(A)$.*

Proof. Suppose that Y_0 is not an immediate extension of $R(A)$ and that $B\colon Y_0 \to X$ is a linear transformation. Choose a vector $y \in Y_0$ which is orthogonal to $R(A)$. Let $x = By$ if $ABy \neq 0$; let $x = By + p$, where $Ap \neq 0$ and $\|p\| < \|By\|$, if $ABy = 0$ and $By \neq 0$; and let x be any vector such that $Ax \neq 0$ if $By = 0$. Then $Ax \perp y$ and x non$\perp$ By, so that B is not an adjoint of A.

LEMMA 4. *If $B \in (A\colon X \to Y_0)'$, then B is 1-1, $A[(By)\hat{}\,] \subset \hat{y}$ for all $y \in Y_0^*$ and $AB \in \mathcal{N}(Y_0)$.*

Proof. Suppose that $B \in (A\colon X \to Y_0)'$. Since the non-singular space Y_0 is an immediate extension of $R(A)$, $R(A)$ is non-singular. If $By = 0$, then Ax non$\perp$ y for all $Ax \in R(A)$. Since $R(A)$ is non-singular, this implies that $y = 0$ and B is 1-1.

For all $y \in Y_0^*$ and $x \in (By)\hat{}$, Ax non$\perp$ y. Equivalently, $Ax \in \hat{y} \cup \{0\}$. The assumption $Ax = 0$ leads to a contradiction: choose $y_1 \in Y_0$ such that $y_1 \perp y$, choose x_1 such that $Ax_1 \in \hat{y}_1$ and $\|x_1\| < \|x\|$, and let $u = x + x_1$. Then $Ax = 0$ implies $Au \perp y$ but u non$\perp$ By, contradicting the hypothesis that B is adjoint to A. It now follows that $[A(By)\hat{}\,] \subset \hat{y}$, and this readily implies that $AB \in \mathcal{N}(Y_0)$.

THEOREM 1. *Suppose that F has non-discrete valuation or $e(X) = e(Y_0) = 1$. Then $B \in (A\colon X \to Y_0)'$ iff $B = (A|_M)^{-1}T$, where*

(1) *M is a subspace of X such that $A|_M$ (is 1-1 and) has an orthogonal inverse (defined on $A(M)$);*

(2) *$T \in \mathcal{N}(Y_0)$ and $R(T) = A(M)$; and*

(3) $\|A|_M\| = \|A\|$.

Proof. Assume that $B \in (A\colon X \to Y_0)'$. Let $M = R(B)$ and let $T = AB$. Then $T \in \mathcal{N}(Y_0)$ by Lemma 4, and $R(T) = A(M)$. Also, $A|_M$ is 1-1, since T is 1-1. The orthogonality of $(A|_M)^{-1}$ is established as follows. If $u \perp v$, where $u, v \in A(M)$, then

$$A(A|_M)^{-1}(u) \perp T^{-1}(v) \quad \text{and} \quad (A|_M)^{-1}(u) \perp BT^{-1}(v) = (A|_M)^{-1}(v).$$

Next we prove that condition (3) is satisfied if F has discrete valuation and $e(X) = e(Y_0) = 1$; the proof for the non-discrete case is similar. Let x be any element of X such that $Ax \neq 0$. Choose an element $y \in Y_0$ such that $Ax \perp y$ and, applying Lemma 1 to B, choose $a \in F$ such that

$$|a| = \frac{\|x\|}{|\pi|\,\|B\|\,\|y\|}.$$

Then (by Lemmas 2 and 4, and Lemma 1.3 from [1])

$$\|aBy\| = |a|\,\|B\|\,\|y\| = \frac{\|x\|}{|\pi|},$$

$$aBy + x \in (By)\hat{}\,, \qquad aABy + Ax \in \hat{y},$$

$$\|Ax\| < \|aABy\| = |a|\,\|T\|\,\|y\| = \frac{\|A|_M\|\,\|x\|}{|\pi|}, \qquad \|Ax\| \leqslant \|A|_M\|\,\|x\|.$$

Therefore $\|A|_M\| = \|A\|$.

Now assume that $B = (A|_M)^{-1}T$, where M and T satisfy conditions (1) through (3), and let $Ax \perp y$. If $x \perp M$, then $x \perp By$. If x non$\perp$ M, $x = m+p$, where $m \in M$ and $\|p\| < \|m\|$. Hence

$$\|Am\| = \|A\|\|m\| > \|A\|\|p\| \geqslant \|Ap\|,$$

which implies

$$Am \text{ non}\perp (Am+Ap) = Ax.$$

Therefore

$$Ax \perp y \Rightarrow Am \perp y \Rightarrow Am \perp Ty \Rightarrow m \perp (A|_M)^{-1}Ty \Rightarrow x \perp (A|_M)^{-1}Ty.$$

Examination of the proof of Theorem 1 shows that, without any restrictive hypothesis concerning F, X or Y, each $B \in (A: X \to Y_0)'$ is of the form $(A|_M)^{-1}T$, where M and T satisfy conditions (1) and (2) and, if F is discretely valued, then

$$\|A|_M\| \geqslant |\pi|\|A\|.$$

COROLLARY 1. *Adjoints are orthogonal transformations.*

COROLLARY 2. *Unbounded transformations have no adjoints.*

COROLLARY 3. *Assume that $A \in L(X, X)^*$, $e(X) = 1$ and X is finite-dimensional and orthogonalizable. Then A' is non-empty iff A is a non-zero scalar multiple of an isometry, and in this case*

$$A' = \{a(A^{-1}+D) \mid a \in F^*, \|D\| < \|A^{-1}\|\}.$$

Proof. If A' is non-empty, $A = A|_M$ is a non-zero scalar multiple of an isometry by Corollary 1.3 from [1]. The description of A' follows from Theorem 1 and Lemma 8.5 in [2].

THEOREM 2. *If X is complete and $B \in (A: X \to R(A))'$, then the extension by continuity of B to $\overline{R(A)}$ is the unique extension of B belonging to $(A: X \to \overline{R(A)})'$.*

THEOREM 3. *If X is spherically complete, Y is an immediate extension of Y_0, and $B_0 \in (A: X \to Y_0)'$ satisfies $\|B_0 y\| = \|B_0\|\|y\|$ for all $y \in Y_0$, then there exists an extension $B \in A'$ of B_0 which satisfies $\|By\| = \|B\|\|y\|$ for all $y \in Y$.*

For the proof assume that B is any extension of B_0 to Y satisfying $\|B\| = \|B_0\|$.

COROLLARY 4. *Suppose that X is spherically complete, Y is an immediate extension of Y_0 and either F has non-discrete valuation or $e(X) = e(Y) = 1$. Then each $B_0 \in (A: X \to Y_0)'$ has an extension $B \in A'$.*

Example. Let F be discretely valued, let X have an orthonormal base $\{x_1, x_2\}$, and let Y have an orthogonal base $\{y_1, y_2\}$ with

$$1 = \|y_1\| > \|y_2\| > |\pi|.$$

Let A be the linear transformation carrying x_i into y_i $(i = 1, 2)$, and let $B = A^{-1}$. Then $B \in A'$; thus adjoints do not necessarily have orthogonal inverses. Also, $B' = \emptyset$, showing that $B \in A'$ does not imply $A \in B'$ and that $A'B' \neq (BA)'$.

REFERENCES

[1] N. Shilkret, *Orthogonal transformations in non-Archimedean spaces*, Archiv der Mathematik (Basel) 23 (1972), p. 285-291.
[2] — *Non-Archimedean orthogonality*, ibidem 27 (1976), p. 67-78.

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