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ON SOME PROPERTIES OF THE FAMILY OF INDEPENDENT SETS IN ABSTRACT ALGEBRAS

BY

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1. Introduction. For every family $\boldsymbol{R}$ of subsets of a fixed set A we put

$$\nu(\boldsymbol{R}) = \sup\{|E| : E \in \boldsymbol{R}\}$$

(where $|E|$ denotes the cardinality of E), and we say that $\boldsymbol{R}$ *has property* (ω) if every set belonging to $\boldsymbol{R}$ is finite and, simultaneously, $\nu(\boldsymbol{R}) = \aleph_0$.

Moreover, we say that $\boldsymbol{R}$ *has a rank*, if 1° every set belonging to $\boldsymbol{R}$ is contained in a maximal one and 2° all maximal sets in $\boldsymbol{R}$ have the same cardinality. Of course, this cardinality is equal to $\nu(\boldsymbol{R})$; it is then called the *rank* of $\boldsymbol{R}$. Clearly

(i) *No family with a rank has property* (ω).

We adopt here the terminology and notation of [1] and [7] concerning closure operators and those of [6] and [8] concerning abstract algebras. E.g. we put $\iota(\mathfrak{A}) = \nu(\boldsymbol{Ind}(\mathfrak{A}))$ (where $\boldsymbol{Ind}(\mathfrak{A})$ denotes the family of all sets independent in $\mathfrak{A}$). We say that an algebra $\mathfrak{A}$ *has a rank* if the family $\boldsymbol{Ind}(\mathfrak{A})$ has a rank. Analogously $\mathfrak{A}$ is said to *have property* (ω) if the family $\boldsymbol{Ind}(\mathfrak{A})$ has this property.

The first example of an algebra with property (ω) was given by C. Ryll-Nardzewski. Later J. Anusiak remarked that an application of a Świerczkowski theorem [9] easily gives another algebra with this property.

The purpose of this paper is to give some examples, to prove some theorems and to raise some questions concerning the notions of rank and property (ω). And so, in Sections 2 and 3, we connect these notions with some other properties of closure spaces and general agebras, in Sections 4 and 5 we give some examples of Boolean algebras and semigroups with property (ω) and in Section 6 we consider rank and property (ω) in Abelian groups (1).

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(1) We thank Professor A. Hulanicki for his valuable advices, concerning this section.

It seems (but we do not know examples) that there exist groups, rings and separable variables algebras (see [3], p. 207) with property (ω) (**P 665**). We do not know equational classes of groups (except Abelian) in which every free group has a rank (**P 666**).

Each of the Sections 4, 5 and 6 can be read directly after Introduction.

2. Closure spaces. Let (X, C) be a closure space, or, in other terms, a set X with a closure operator C, i.e. an extensive, monotone and idempotent function $C: 2^X \to 2^X$. Let us recall that C is said to have the *exchange property* if, whenever $b \notin C(E)$ and $b \in C(E \cup \{a\})$, then $a \in C(E \cup \{b\})$ (see [1], p. 209, and [7], p. 23). If $a \notin C(I \setminus \{a\})$ for every $a \in I$, we say that I is *C-independent* and we write $I \in \boldsymbol{Ind}(C)$.

If C has the exchange property, then every minimal set of C-generators is a maximal C-independent set and conversely. Hence (see e.g. [1], p. 210, proposition (o)):

(i) *If a closure operator C has a finite character and the exchange property, then the family $\boldsymbol{Ind}(C)$ has a rank, and, consequently (in view of* 1(i)), *has not property* (ω).

Nevertheless,

(ii) *There exists a closure operator C_ω with a finite character such that $\boldsymbol{Ind}(C_\omega)$ has property* (ω).

Namely, let us consider a sequence X_n of disjoint sets with $|X_n| = n$, and put

$$X = \bigcup_n X_n, \quad \boldsymbol{J} = \bigcup_n 2^{X_n},$$

$$C_\omega(E) = \begin{cases} E & \text{if } E \in \boldsymbol{J}, \\ X & \text{if } E = 2^X \setminus \boldsymbol{J}. \end{cases}$$

It is easy to check that C_ω is a closure operator in X with a finite character and that

$$\boldsymbol{Ind}(C_\omega) = \boldsymbol{J} \cup \{E : E \subset X \text{ and } |E| = 2\}.$$

Of course this family has property (ω) and proposition (ii) is thus proved.

3. General algebras. The purpose of this section is to connect the rank and the property (ω) of abstract algebras on one hand and with different known properties of such algebras — on the other.

Let us first recall that every algebra $\mathfrak{A} = (A; \boldsymbol{F})$ induces a closure operator C in A, namely for every $E \subset A$ we take for $C(E)$ the subalgebra of $\mathfrak{A}$ generated by E. We say that $\mathfrak{A}$ is a *v_*^*-algebra*, if $\boldsymbol{Ind}(\mathfrak{A}) = \boldsymbol{Ind}(C)$, and that $\mathfrak{A}$ is a *v^*-algebra*, if, moreover, the operator C has the exchange property (cf. [11]). It follows from 2 (i) that

(i) *No v^*-algebra has property* (ω)
(which can be deduced also from (iii) or (iv), because every v^*-algebra satisfies hypotheses of these propositions). We do not know whether there exist v_*^*-algebras with this property (**P 667**).

Świerczkowski proved ([10], p. 751, Theorem 3) that

(s) *If in an algebra $\mathfrak{A}$ there is a finite set G of generators and an independent set I with $|G| < |I|$, then there is in $\mathfrak{A}$ an infinite independent set.*

It follows that

(ii) *No finitely generated algebra has property* (ω),
whence

(iii) *No algebra with a base has property* (ω).

However, there are algebras with finite bases without rank, e.g. every algebra having bases with different numbers of elements (see [6], p. 59).

We shall now prove that

(iv) *If every independent set in an algebra $\mathfrak{A}$ can be extended to a base, then* $\mathbf{Ind}(\mathfrak{A})$ *has a rank.*

Since every base is a maximal independent set, and since every independent set is contained in a maximal one, the hypothesis (iv) is equivalent to the following condition: every maximal independent set is a base.

Thus we have only to prove that all bases have the same cardinality. If not, then all bases are finite (see [6], p. 59) and we can apply theorem (s). Consequently, there is in $\mathfrak{A}$ an infinite independent set, and thus, by hypothesis, an infinite base. So we obtain a contradiction and proposition (iv) is proved.

(v) *A finite algebra has a rank and a base if and only if every independent set can be extended to a base.*

Necessity. Let B denote a base of a given algebra $\mathfrak{A} = (A; \boldsymbol{F})$. Since $\mathfrak{A}$ has a rank and B is a maximal independent set in $\mathfrak{A}$, then, by hypothesis, we have $|B| = |M|$ for every maximal independent set M in $\mathfrak{A}$. Hence we also have $|C(B)| = |C(M)|$ (see [6], p. 58, (iii)). Since $C(B) = A$ and A is finite, we obtain $C(M) = A$ and, consequently, M is a base of $\mathfrak{A}$. Thus every independent set is contained in a base.

Sufficiency follows from (iv) and from the fact that the empty set is independent.

The hypothesis on the finiteness in proposition (v) is essential. Counterexample: $(N; {}^*)$, where $N = \{0, 1, 2, \ldots\}$and $x^* = x+1$.

We shall now prove that there is a connection between the notion of rank and the condition of exchange of independent sets (EIS-condition, see [3] and [8], p. 174).

(vi) *If every subalgebra of a finite algebra $\mathfrak{A}$ has a rank, then $\mathfrak{A}$ satisfies the condition EIS.*

Let $P \cup Q, R \in \boldsymbol{Ind}(\mathfrak{A})$ and $R \subset C(P)$. Since the subalgebra $C(P)$ has a rank, there exists, in view of (v), a base R_0 of the subalgebra $C(P)$ such that $R \subset R_0$. The set R_0 is independent in $\mathfrak{A}$ (see [3], p. 202, A2). Thus (vi) follows from the "exchange theorem" ([6], p. 58, (ii)).

4. Lattices and Boolean algebras. In this section we shall prove that

(i) *There exist Boolean algebras and distributive lattices with property* (ω).

Let A be an infinite denumerable set and $\boldsymbol{K}$ the family of all subsets of A being finite or having finite complements. Let $\mathfrak{B}$ be the lattice $(\boldsymbol{K}; \cup, \cap)$ and $\mathfrak{B}'$ the Boolean algebra $(\boldsymbol{K}; \cup, \cap, ')$.

Since $\mathfrak{B}$ is a reduct of $\mathfrak{B}'$, all sets independent in $\mathfrak{B}'$ are independent in $\mathfrak{B}$, and the sets dependent in $\mathfrak{B}$ are dependent in $\mathfrak{B}'$. Thus in order to show that both $\mathfrak{B}$ and $\mathfrak{B}'$ have property (ω) it suffices to prove:

(α) for every n there exists in $\mathfrak{B}'$ an n-element independent set

and

(β) every infinite set in $\mathfrak{B}$ is dependent.

Now (α) follows immediately from the following statements: 1° for every n the algebra $\mathfrak{B}'$ contains all subsets of a certain 2^n-element set, 2° the number ι in Boolean algebra 2^{2^n} is equal to n, and 3° the following easily proved proposition:

(α^*) if $\mathfrak{B}$ is a Boolean algebra of sets and every subset Y of X belongs to $\mathfrak{B}$, and if the set $\boldsymbol{F}$ is independent in Boolean algebra 2^X, then $\boldsymbol{F}$ is independent in $\mathfrak{B}$.

In order to prove (β) let us suppose that $\boldsymbol{X} = \{X_1, X_2, \ldots\}$ is an infinite independent set of elements of $\mathfrak{B}$. If the set X_1 is finite and its power is k, then we have $X_1 \cap X_2 \cap \ldots \cap X_n = \emptyset$ for $n > k$, since in view of the independence of $\boldsymbol{X}$ we have $X_1 \cap \ldots \cap X_r \neq X_1 \cap \ldots \cap X_{r+1}$. This gives a contradiction. In the case, where the complement of X_1 is finite we apply an analogous argument replacing intersections by unions. Proposition (β), and consequently, proposition (i) is thus proved.

Since Boolean algebras satisfy the condition EIS (see [3]), we see that EIS and (ω) are not contradictory.

Let us notice incidentally that in general (on account of proposition 4 (vi) of [5], p. 142) a Boolean algebra $(X; \cup, \cap, ')$ has the property (ω) if and only if the lattice $(X; \cap, \cup)$ has this property.

5. Abelian semigroups. We shall prove that

(i) *There is an Abelian semigroup which has property* (ω).

Let $\boldsymbol{S}$ be an additive family of non-empty subsets of a fixed set X such that $X \in \boldsymbol{S}$, and let us suppose that there are in $\boldsymbol{S}$ some disjoint sets.

For every $E, F \in S$ we put

$$E \circ F = \begin{cases} E \cup F & \text{if } E \cap F = \emptyset, \\ X & \text{if } E \cap F \neq \emptyset. \end{cases}$$

It is easy to see that $\boldsymbol{S} = (S; \circ)$ is an Abelian semigroup, and that if f is an n-ary algebraic operation in $\boldsymbol{S}$ depending on every variable, then

$$f(E_1, \ldots, E_n) = E_1 \circ \ldots \circ E_n.$$

The set X is the unique algebraic constant in $\boldsymbol{S}$. Hence

(*) A family $E_1, \ldots, E_n \in S$ is an independent set in $\boldsymbol{S}$ if and only if the sets E_j are disjoint and $\bigcup_j E_j \neq X$.

For any family $\boldsymbol{P}$ of sets let us denote by $\tilde{\boldsymbol{P}}$ the family of all subfamilies of $\boldsymbol{P}$ consisting of disjoint sets.

We shall prove that

($\substack{*\\*}$) There exists an additive family $\boldsymbol{P}$ of non-void subsets of a set X such that $X \in \boldsymbol{P}$ and that $\tilde{\boldsymbol{P}}$ has property (ω).

Let X denote the set of all infinite sequences $\langle a_n \rangle$ of natural numbers such that $a_n \leqslant n$, having only finitely many elements different from zero.

Next, let $X_{ij} = \{\langle a_n \rangle \in X : a_i = j\}$ and $\boldsymbol{P}_0 = \{X_{ij} : j \leqslant i\}$. The sets $X_{n1}, X_{n2}, \ldots, X_{nn}$ are mutually disjoint and, since $n \neq m$ implies $X_{nk} \cap X_{ml} \neq 0$ for every $X_{nk}, X_{ml} \in \boldsymbol{P}_0$, there exists no infinite family of disjoint sets $X_{ij} \in \boldsymbol{P}_0$, i.e. the family $\tilde{\boldsymbol{P}}_0$ satisfies (ω). Finally, let us consider the family $\boldsymbol{P}$ consisting of all finite unions of sets $X_{ij} \in \boldsymbol{P}_0$. Of course $\tilde{\boldsymbol{P}}$ also satisfies (ω) and $X \in \boldsymbol{P}$. Proposition ($\substack{*\\*}$) is thus proved.

Propositions (*) and ($\substack{*\\*}$) give directly (i).

6. Abelian groups. We consider here three notions of independence in Abelian groups. A subset I of an Abelian group G is called *linearly independent* (in symbols $I \in \boldsymbol{Ind}_l G$) if 1° $0 \notin I$ and 2° for every $a_1, \ldots, a_n \in I$ and every integers $k_1, \ldots, k_n$ the relation

$$(*) \qquad k_1 a_1 + \ldots + k_n a_n = 0$$

implies $k_j a_j = 0$ for $j = 1, 2, \ldots, n$ (see e.g. Fuchs [2], p. 29). If (*) implies $k_j = 0$ for $j = 1, \ldots, n$, we say that I is *linearly strongly independent* (in symbols $I \in \boldsymbol{Ind}_L(G)$). Moreover, G can be regarded as an abstract algebra $\mathfrak{G} = (G, x+y, -x)$, and $I \in \boldsymbol{Ind}(G)$ if it is *independent in the general algebraic sense* (i.e. in the sense of [6] and [8]) in $\mathfrak{G}$. It is easy to see that for every Abelian group G

$$\boldsymbol{Ind}_L(G) \subset \boldsymbol{Ind}(G) \subset \boldsymbol{Ind}_l(G).$$

More precisely: $I \in \boldsymbol{Ind}_L(G)$ if and only if $I \in \boldsymbol{Ind}_l(G)$ and every element of I is of infinite order, while $I \in \boldsymbol{Ind}(G)$ if and only if $I \in \boldsymbol{Ind}_l(G)$ and every element of I is of the maximal order (i.e. $0(a) \geqslant 0(b)$ for every $a \in I$, $b \in G$).

It is well known (see e.g. Fuchs [2], p. 31, Th. 8.2) that

(i) *For every Abelian group G the family $\boldsymbol{Ind}_L(G)$ has a rank (denoted usually by $r_0(G)$) and consequently fails to have property* (ω).

On the other hand,

(ii) *There are Abelian groups (e.g. $G = Z_2 \times Z_3$) without rank of* $\boldsymbol{Ind}_l(G)$.

(iii) *In an Abelian group G the family $\boldsymbol{Ind}_l(G)$ does not have property* (ω).

Proof. If for every n there exists in G an n-element linearly independent set consisting of elements of infinite order, then the theorem follows from (i).

Therefore, we can restrict our consideration to the maximal periodic subgroup $H \subset G$ and suppose $\nu(\boldsymbol{Ind}_l H) = \infty$. Let $H = \sum H_{p_i}$ be a decomposition of H into primary p_i-groups H_{p_i}. If there are infinitely many summands H_{p_i} in the decomposition, then $r_0(G) = \infty$ and the theorem follows. If not, the theorem is an immediate consequence of the formula:

$$\nu\big(\boldsymbol{Ind}_l(H_{p_i} \times H_{p_j})\big) = \nu\big(\boldsymbol{Ind}_l(H_{p_i})\big) + \nu\big(\boldsymbol{Ind}_l(H_{p_j})\big).$$

Now we pass to the family $\boldsymbol{Ind}(G)$ and our aim is to prove that it has a rank and, consequently, it does not have property (ω).

It is sufficient to show that

(v) *If G is a bounded group, then the family $\boldsymbol{Ind}(G)$ has a rank.*

Let I be a maximal subset of linearly independent elements of maximal order in G. Since G is a bounded group, there are only finitely many direct summands in the decomposition

$$G = H_{p_1} + \ldots + H_{p_r},$$

where $p_i \neq p_j$ for $i \neq j$ and H_{p_j} is a p_j-group. Since every element a of I has maximal order, $a = a_{p_1} + \ldots + a_{p_r}$, where $a_{p_j} \in H_{p_j}$ and a_{p_j} has maximal order in H_{p_j}. It is easy to verify that $I_{p_j} = \{a_{p_j} : a \in I\}$ is linearly independent in H_{p_j}. In fact, $k_1 a^1_{p_j} + \ldots + k_n a^n_{p_j} = 0$ and $k_i a^i_{p_j} \neq 0$ implies $Ak_1 a^1 + \ldots + Ak_n a^n = 0$, where $A =$ maximal order of $\sum_{i \neq j} H_{p_i}$ and $Ak_i a^i \neq 0$.

Let p_j^n be the maximal order of H_{p_j}. Then the set $p_j^{n-1} I_{p_j}$ is linearly independent and, of course, $|p_j^{n-1} I_{p_j}| = |I_{p_j}|$. Since $p_j^{n-1} H_{p_j}$ is a linear space over Z_{p_j}, we have

$$|I_{p_j}| = |p_j^{n-1} I_{p_j}| \leqslant \dim p_j^{n-1} H_{p_j}.$$

On the other hand, since I is maximal, we have, for some p_j, $|I_{p_j}| = |p_j^{n-1} I_{p_j}| = \dim p_j^{n-1} H_{p_j}$. Thus

$$|I| = \min_j \dim p_j^{n-1} H_{p_j},$$

which completes the proof of (v).

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