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GROUPS ACTING ON COVERING SPACES

BY

T. JAKUBOWSKI (WROCŁAW)

1. Introduction. In [1] Bredon proved that if a connected Lie group G acts effectively on a connected, locally connected and pathwise connected space X, then the unique covering group $\tilde{G}$ of G acts effectively on some covering space $\tilde{X}$ of X.

In this paper we consider the more general problem:

Given a G-space X and a covering group $\tilde{G}$ of the topological group G, under what (necessary and sufficient) conditions does $\tilde{G}$ act on some covering space $\tilde{X}$ of X?

The answer is provided for X connected, locally pathwise connected and locally 1-connected, and this yields a generalization of Bredon's theorem to topological groups. (Bredon's remark that his theorem remains true for a topological group having an open component of the unit is invalid: the group Z_3 acts effectively on $S^1 = \{z \in C\colon |z| = 1\}$ by the rotations as the 3-rd roots of the unit but no covering group of Z_3 acts effectively on the universal covering space R^1 of S^1).

2. Actions of pathgroup. Denote by $\mathfrak{A}$ the class of all connected, locally pathwise connected and locally 1-connected topological spaces.

For $X \in \mathfrak{A}$, let $P(X, x_0)$ be the space of all paths starting at the point $x_0 \in X$. This space provided with compact-open topology is a total space of the fibration $\big(P(X, x_0), a_X, X\big)$, where $a_X\colon P(X, x_0) \to X$ is a continuous map given by $a_X(\omega) = \omega(1)$ (see [2]).

Every continuous map $f\colon X \to Y$ induces the unique continuous map $P(f)\colon P(X, x_0) \to P\big(Y, f(x_0)\big)$ such that the diagram

(1) $$\begin{array}{ccc} P(X, x_0) & \xrightarrow{P(f)} & P\big(Y, f(x_0)\big) \\ {\scriptstyle a_X}\big\downarrow & & \big\downarrow{\scriptstyle a_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

commutes.

If G is a topological group in the class $\mathfrak{A}$, then the group rule in G induces a continuous rule in the space $P(G)$ of all paths starting at the unit $e \in G$. The space $P(G)$ with this rule forms a topological group and a_G is a continuous homomorphism.

Let a topological group $G \in \mathfrak{A}$ act on the space $X \in \mathfrak{A}$ (for simplicity, we consider only left actions) and let $\nu\colon G \times X \to X$ be this action. If ν is a free action (i.e., if $\nu(g, x) = x$, then $g = e$) and $\tau_\nu\big((g, x), x\big) = g$ is a continuous map of the space $X^* = \big\{\big(\nu(g, x), x\big)\colon x \in X \text{ and } g \in G\big\} \subset X \times X$ onto G, then ν is a principal action.

Proposition 1. *If X is a left G-space, then $P(X, x_0)$ is a left $P(G)$-space and the diagram*

$$\begin{array}{ccc} P(G) \times P(X, x_0) & \xrightarrow{P(\nu)} & P(X, x_0) \\ {\scriptstyle a_G \times a_X}\big\downarrow & & \big\downarrow{\scriptstyle a_X} \\ G \times X & \xrightarrow{\quad \nu \quad} & X \end{array} \tag{2}$$

commutes. Moreover,

(i) *if ν is a principal action, then so is $P(\nu)$,*

(ii) *if ν is an effective action, then so is $P(\nu)$.*

Proof. Commutativity of diagram (2) follows from that of (1), since

$$P\big(G \times X, (e, x_0)\big) = P(G) \times P(X, x_0).$$

The equations

$$P(\nu)\big(\gamma_1, P(\nu)(\gamma_2, \omega)\big) = P(\nu)(\gamma_1\gamma_2, \omega) \quad \text{and} \quad P(\nu)(\tilde{e}, \omega) = \omega,$$

where $\gamma_1, \gamma_2 \in P(G)$, $\omega \in P(X, x_0)$, and $\tilde{e}$ is a constant path at e, follow from that for ν.

Now let ν be a principal action. We show first that $P(\nu)$ is a free action. Note that if $P(\nu)(\gamma, \omega) = \omega$, then $\nu\big(\gamma(t), \omega(t)\big) = \omega(t)$ for all $t \in [0, 1]$. Since G acts freely on X by assumption, $\gamma(t) = e$ for all t. Hence $\gamma = \tilde{e}$. Since the action ν is principal, we have the continuous map

$$\tau_\nu\colon X^* \to G.$$

Thus $P\big(X^*, (x_0, x_0)\big) = \big(P(X, x_0)\big)^*$, and so the map $\tau_{P(\nu)}$ coincides with the continuous map induced by τ_ν. Hence $P(\nu)$ is a principal action.

Finally, let ν be an effective action. Assume that $P(\nu)(\gamma, \omega) = \omega$ for all $\omega \in P(X, x_0)$. Fix the point $t_0 \in (0, 1]$. Then for every $x \in X$ there exists $\omega_x \in P(X, x_0)$ such that $\omega_x(t_0) = x$. Hence

$$x = \omega_x(t_0) = P(\nu)(\gamma, \omega_x)(t_0) = \nu\big(\gamma(t_0), \omega_x(t_0)\big) = \nu\big(\gamma(t_0), x\big) \quad \text{for all } x \in X.$$

Since ν is an effective action, we have $\gamma(t_0) = e$. Thus $\gamma(t_0) = e$ for all $t_0 \in [0, 1]$ and $P(\nu)$ is an effective action.

In the proofs of next theorems we shall use the following

LEMMA 1. *Let* $\varrho_1, \varrho_2 \subset X \times X$ *and* $\varrho_1', \varrho_2' \subset Y \times Y$ *be equivalence relations on topological spaces* X *and* Y, *respectively. Let* $f\colon X \to Y$ *be a continuous map and let*

(i) $\varrho_1 \subset \varrho_2$ *and* $\varrho_1' \subset \varrho_2'$,

(ii) $(f \times f)(\varrho_i) \subset \varrho_i'$ *for* $i = 1, 2$.

Then there exist unique continuous maps f_1 *and* f_2 *such that the diagram*

(3)
$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd] & \\
& X/\varrho_1 \arrow[rr, "f_1"] \arrow[dl] & & Y/\varrho_1' \arrow[dl] \\
X/\varrho_2 \arrow[rr, "f_2"] & & Y/\varrho_2 &
\end{tikzcd}$$

commutes.

The proof is obvious.

PROPOSITION 2. *Let* X_π *be the covering space of* X *determined by a subgroup* π *of* $\pi_1(X, x_0)$. *If* $\nu\colon G \times X \to X$ *is a continuous action of* G *on* X, *then there exists only one continuous action* $P_\pi\colon P(G) \times X_\pi \to X_\pi$ *such that the diagram*

(4)
$$\begin{tikzcd}
P(G) \times P(X, x_0) \arrow[rr, "P(\nu)"] \arrow[dr, "\alpha_G \times \alpha_X"] \arrow[dd, "1 \times \alpha_X^\pi"'] & & P(X, x_0) \arrow[dr, "\alpha_X"] \arrow[dd, "\alpha_X^\pi"] & \\
& G \times X \arrow[rr, "\nu"] & & X \\
P(G) \times X_\pi \arrow[rr, "P_\pi"] \arrow[ur, "\alpha_G \times p_\pi"] & & X_\pi \arrow[ur, "p_\pi"'] &
\end{tikzcd}$$

commutes.

Proof. The homotopy class of a loop ω will be denoted by $[\omega]$. Define two relations, ϱ on $P(G) \times P(X, x_0)$ and ϱ' on $P(X, x_0)$, as follows:

$(\gamma, \omega)\, \varrho\, (\gamma', \omega')$ if and only if $\omega(1) = \omega'(1)$, $[\omega \circ \omega'^{-1}] \in \pi$ and $\gamma = \gamma'$;

$\omega \varrho' \omega'$ if and only if $\omega(1) = \omega'(1)$ and $[\omega \circ \omega'^{-1}] \in \pi$.

We show that if $(\gamma_1, \omega_1)\, \varrho\, (\gamma_2, \omega_2)$, then

$$\big(P(\nu)(\gamma_1, \omega_1)\big)\, \varrho'\, \big(P(\nu)(\gamma_2, \omega_2)\big).$$

If $(\gamma_1, \omega_1)\, \varrho (\gamma_2, \omega_2)$, then

$$\gamma_1 = \gamma_2, \qquad \omega_1(1) = \omega_2(1) \qquad \text{and} \qquad [\omega_1 \circ \omega_2^{-1}] \in \pi.$$

Thus

$$P(\nu)(\gamma_2, \omega_2)(1) = \nu\big(\gamma_2(1), \omega_2(1)\big) = \nu\big(\gamma_1(1), \omega_1(1)\big) = P(\nu)(\gamma_1, \omega_1)(1).$$

To show that $\big[P(\nu)(\gamma_1, \omega_1)\circ\big(P(\nu)(\gamma_2, \omega_2)\big)^{-1}\big] \in \pi$, we put

$$\omega_i'(t) = \begin{cases}\omega_i(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2},\\ x & \text{if } \frac{1}{2} \leqslant t \leqslant 1\end{cases}$$

for $i = 1, 2$, where $x = \omega_1(1) = \omega_2(1)$, and

$$\gamma'(t) = \begin{cases}e & \text{if } 0 \leqslant t \leqslant \frac{1}{2},\\ \gamma(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1.\end{cases}$$

Note that ω_i', ω_i are homotopic rel $\{0, 1\}$ for $i = 1, 2$, and γ, γ' are homotopic rel $\{0, 1\}$. Hence

$$P(\nu)(\gamma, \omega_1)\circ\big(P(\nu)(\gamma, \omega_2)\big)^{-1} \quad \text{and} \quad P(\nu)(\gamma', \omega_1')\circ\big(P(\nu)(\gamma', \omega_2')\big)^{-1}$$

are homotopic as loops. For $0 \leqslant t \leqslant \frac{1}{2}$ we have

$$P(\nu)(\gamma', \omega_i')(t) = \nu\big(\gamma'(t), \omega_i'(t)\big) = \nu\big(e, \omega_i'(t)\big) = \omega_i'(t) = \omega_i(2t),$$

and for $\frac{1}{2} \leqslant t \leqslant 1$ we obtain

$$P(\nu)(\gamma', \omega_i')(t) = \nu\big(\gamma'(t), \omega_i'(t)\big) = \nu\big(\gamma(2t-1), x\big).$$

If we denote by ω the path $\nu\big(\gamma(t), x\big)$, then

$$P(\nu)(\gamma', \omega_i') = \omega_i\circ\omega$$

and

$$P(\nu)(\gamma', \omega_1')\circ\big(P(\nu)(\gamma', \omega_2')\big)^{-1} = (\omega_1\circ\omega)\circ(\omega_2\circ\omega)^{-1}.$$

The map

$$h(t, v) = (\omega_1\circ\omega_v)\circ(\omega_2\circ\omega_v)^{-1}(t),$$

where $\omega_v(t) = \omega\big((1-v)t\big)$, is a homotopy of the loops $(\omega_1\circ\omega)\circ(\omega_2\circ\omega)^{-1}$ and $\omega_1\circ\omega_2^{-1}$. Thus

$$\big[P(\nu)(\gamma, \omega_1)\circ\big(P(\nu)(\gamma, \omega_2)\big)^{-1}\big] = [(\omega_1\circ\omega)\circ(\omega_2\circ\omega)^{-1}] = [\omega_1\circ\omega_2^{-1}] \in \pi.$$

Since

$$P(G)\times X_\pi = \big(P(G)\times P(X, x_0)\big)/\varrho \quad \text{and} \quad X_\pi = P(X, x_0)/\varrho',$$

$P(\nu)$ induces a continuous map $P_\pi\colon P(G)\times X_\pi \to X$ which is an action. Commutativity of diagram (4) follows from Lemma 1.

3. Groups acting on covering spaces. Let $\nu\colon G\times X \to X$ be an action of G on X and let $p_\pi\colon X_\pi \to X$ be a covering determined by a subgroup $\pi \subset \pi_1(X, x_0)$. We will find all groups $\tilde{G}$ covering the group G which are acting on X_π in a way such that the diagram

$$\text{(5)}\qquad \begin{array}{ccc} \tilde{G}\times X_\pi & \xrightarrow{\tilde{\nu}} & X_\pi \\ {\scriptstyle h\times p_\pi}\downarrow & & \downarrow{\scriptstyle p_\pi} \\ G\times X & \xrightarrow{\nu} & X \end{array}$$

commutes, $h\colon \tilde{G}\to G$ being a covering homomorphism and $\tilde{\nu}\colon \tilde{G}\times X_\pi\to X_\pi$ an action.

Let $\tilde{\pi} = (\nu_{x_0})_{\#}^{-1}(\pi)$, where $(\nu_{x_0})_{\#}\colon \pi_1(G, e)\to\pi_1(X, x_0)$, be a homomorphism of fundamental groups induced by the map $\nu_{x_0}(g) = \nu(g, x_0)$. Denote by $a_G^{\tilde{\pi}}\colon P(G)\to G_{\tilde{\pi}}$ the canonical homomorphism.

LEMMA 2. *If $P_\pi\colon P(G)\times X_\pi\to X_\pi$ is the action induced by ν, then $P_\pi(\gamma, y) = y$ for all $y\in X$ and all $\gamma\in\operatorname{Ker} a_G^{\tilde{\pi}}$.*

Proof. Let $\omega\in P(X, x_0)$ represent $y\in X_\pi$ and let $\gamma\in\operatorname{Ker} a_G^{\tilde{\pi}}$. Put

$$\gamma'(t) = \begin{cases}\gamma(2t) & \text{if } 0\leqslant t\leqslant\frac{1}{2},\\ e & \text{if } \frac{1}{2}\leqslant t\leqslant 1,\end{cases}$$

and

$$\omega'(t) = \begin{cases}x_0 & \text{if } 0\leqslant t\leqslant\frac{1}{2},\\ \omega(2t-1) & \text{if } \frac{1}{2}\leqslant t\leqslant 1.\end{cases}$$

The paths $P(\nu)(\gamma, \omega)$ and $P(\nu)(\gamma', \omega')$ are homotopic rel$\{0, 1\}$. For $0\leqslant t\leqslant\frac{1}{2}$,

$$P(\nu)(\gamma', \omega')(t) = \nu\big(\gamma'(t), \omega'(t)\big) = \nu\big(\gamma(2t), x_0\big) = \nu_{x_0}(\gamma)(2t),$$

and, for $\frac{1}{2}\leqslant t\leqslant 1$,

$$P(\nu)(\gamma', \omega')(t) = \nu\big(\gamma'(t), \omega'(t)\big) = \nu\big(e, \omega(2t-1)\big) = \omega(2t-1).$$

Therefore, the path $P(\nu)(\gamma', \omega')$ is a juxtaposition $\nu_{x_0}(\gamma)\circ\omega$. Hence

$$\begin{aligned}[P(\nu)(\gamma, \omega)\circ\omega^{-1}] &= [P(\nu)(\gamma', \omega')\circ\omega^{-1}] = \big[(\nu_{x_0}(\gamma)\circ\omega)\circ\omega^{-1}\big]\\ &= [\nu_{x_0}(\gamma)] = (\nu_{x_0})_{\#}([\gamma])\in\pi.\end{aligned}$$

Thus the path $P(\nu)(\gamma, \omega)$ represents the point y. By diagram (4),

$$P_\pi(\gamma, y) = P_\pi\circ(1\times a_X^\pi)(\gamma, \omega) = a_X^\pi P(\nu)(\gamma, \omega) = y.$$

COROLLARY 1. *For every subgroup $\pi'\subset\tilde{\pi} = (\nu_{x_0})_{\#}^{-1}(\pi)$ the covering group $G_{\pi'}$ of G acts on X_π in a way such that diagram* (5) *commutes.*

Proof. Since $P_\pi(\gamma, x) = x$ for every $\gamma\in\operatorname{Ker} a_G^{\tilde{\pi}}$ and every $x\in X_\pi$, then the factor group $P(G)/\operatorname{Ker} a_G^{\tilde{\pi}} = G_\pi$ acts on X_π. Let ν_π be this action. If $\pi'\subset\tilde{\pi}$, then the group $G_{\pi'}$ covers $G_{\tilde{\pi}}$, and if $h_{\tilde{\pi}}^{\pi'}\colon G_{\pi'}\to G_\pi$ is a covering homomorphism, then the mapping

$$\nu_\pi\circ(h_{\tilde{\pi}}^{\pi'}\times 1)\colon G_{\pi'}\times X_\pi\to X_\pi$$

is an action. Commutativity of diagram (5) follows from that of diagram (4).

THEOREM 1. *The covering group $\tilde{G}$ of G acts on X_π in a way such that diagram* (5) *commutes if and only if $\tilde{G} = G_{\pi'}$ for some subgroup $\pi' \subset (\nu_{x_0})_{\#}^{-1}(\pi)$.*

Proof. Let $G_{\pi'}$ act on X_π in a way such that diagram (5) commutes. Put $\tilde{\nu}_{x_\pi}(g') = \tilde{\nu}(g', x_\pi)$. Since

$$P_\pi \circ \nu_{x_\pi} = \nu_{x_0} \circ h \quad \text{and} \quad (\tilde{\nu}_{x_\pi})_{\#}\big(\pi_1(G_{\pi'})\big) \subset \pi_1(X_\pi, x_\pi),$$

we have

$$\begin{aligned}(\nu_{x_0})_{\#}(\pi') &= (\nu_{x_0})_{\#}\Big(h_{\#}\big(\pi_1(G_{\pi'})\big)\Big) = (p_\pi)_{\#} \circ (\tilde{\nu}_{x_\pi})_{\#}\big(\pi_1(G_{\pi'})\big) \\ &\subset (p_\pi)_{\#}\big(\pi_1(X_\pi, x_\pi)\big) = \pi.\end{aligned}$$

Hence the condition $\pi' \subset (\nu_{x_0})_{\#}^{-1}(\pi)$ is necessary. The sufficiency follows from Corollary 1.

Note that, on some covering spaces X_π of a G-space X, several covering groups of G may act. If $G_{\tilde{\pi}}$ acts on X_π and $\pi' \subset \tilde{\pi}$, then also the covering group $G_{\pi'}$ acts on X_π. But if we consider only effective or principal actions, then exactly one covering group $G_{\tilde{\pi}}$ acts effectively or principally on a covering space X_π.

We start with the following

LEMMA 3. *Let $\nu\colon G \times X \to X$ be an effective action and let $\pi \subset \pi_1(X, x_0)$ be any subgroup. If $P_\pi(\gamma, y) = y$ for all $y \in X_\pi$, then $\gamma \in \operatorname{Ker} a_G^{\tilde{\pi}} \subset P(G)$.*

Proof. By diagram (4) we have

$$p_\pi\big(P_\pi(\gamma, y)\big) = \nu\big(a_G(\gamma), p_\pi(y)\big) = \nu\big(\gamma(1), p_\pi(y)\big).$$

Assume that $P_\pi(\gamma, y) = y$ for all $y \in X_\pi$; then $\nu\big(\gamma(1), p_\pi(y)\big) = p_\pi(y)$. Since G acts effectively on X, $\gamma(1) = e$. Let the path $\omega \in P(X, x_0)$ represent y. Since

$$y = P_\pi(\gamma, y) = P_\pi \circ (1 \times a_X^\pi)(\gamma, \omega) = a_X^\pi\big(P(\nu)(\gamma, \omega)\big),$$

we have $[P(\nu)(\gamma, \omega) \circ \omega^{-1}] \in \pi$. Let γ' and ω' be defined as in the proof of Lemma 2. The paths $P(\nu)(\gamma, \omega)$ and $P(\nu)(\gamma', \omega')$ are homotopic rel $\{0, 1\}$. On the other hand,

$$P(\nu)(\gamma', \omega')(t) = \begin{cases} \nu_{x_0}(\gamma)(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \omega(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Thus

$$P(\nu)(\gamma', \omega') \circ \omega^{-1} = \big(\nu_{x_0}(\gamma) \circ \omega\big) \circ \omega^{-1}$$

and

$$\begin{aligned}(\nu_{x_0})_{\#}([\gamma]) &= [\nu_{x_0}(\gamma)] = \big[\big(\nu_{x_0}(\gamma) \circ \omega\big) \circ \omega^{-1}\big] = [P(\nu)(\gamma', \omega') \circ \omega^{-1}] \\ &= [P(\nu)(\gamma, \omega) \circ \omega^{-1}] \in \pi.\end{aligned}$$

Hence $[\gamma] \in \pi$. Since $G_{\tilde{\pi}} = P(G)/\operatorname{Ker} a_G^{\tilde{\pi}}$, we have $\gamma \in \operatorname{Ker} a_G^{\tilde{\pi}}$.

The following Corollary is a generalization of Bredon's result from [1].

COROLLARY 2. *If* $v\colon G \times X \to X$ *is an effective action and* $\pi \subset \pi_1(X, x_0)$ *is any subgroup, then only the group* $G_{\tilde{\pi}}$, *where* $\tilde{\pi} = (v_{x_0})_{\#}^{-1}(\pi)$, *acts effectively on* X_π *is a way such that diagram* (5) *commutes.*

Proof. By Lemma 2 the kernel of ineffectiveness of the action $P_\pi\colon P(G) \times X_\pi \to X_\pi$ is $\operatorname{Ker} a_G^{\tilde{\pi}}$. Hence the group $G_{\tilde{\pi}} = P(G)/\operatorname{Ker} a_G^{\tilde{\pi}}$ acts effectively on X_π and the corollary follows from Theorem 1.

THEOREM 2. *Let* $v\colon G \times X \to X$ *be a principal action and let* $\tilde{\pi} = (v_{x_0})_{\#}^{-1}(\pi)$, *where* $\pi \subset \pi_1(X, x_0)$ *is any subgroup. Then only the group* $G_{\tilde{\pi}}$ *acts principally on* X_π *in a way such that diagram* (5) *commutes.*

Proof. By Theorem 1, $G_{\tilde{\pi}}$ acts on X_π and diagram (5) commutes. We have to show that this action v_π is principal. We show first that v_π is a free action. Assume that $v_\pi(g, y_0) = y_0$ for some $y_0 \in X$ and $g \in G_{\tilde{\pi}}$. Then

$$p_\pi(y_0) = p_\pi\big(v_\pi(g, y_0)\big) = v\big(h_{\tilde{\pi}}(g), p_\pi(y_0)\big),$$

where $h_{\tilde{\pi}}\colon G_{\tilde{\pi}} \to G$ is a covering homomorphism. Since v is a free action, $h_{\tilde{\pi}}(g) = e$.

Let V be an open neighbourhood of y_0 such that $p_{\pi|V}\colon V \to p_\pi(V)$ is a homeomorphism and let $U \subset V$ be an open neighbourhood of y_0 such that $v_\pi(g, U) \subset V$ (such a U exists since v_π is a continuous action). For $y' \in U$ we have

$$p_{\pi|V}\big(v_\pi(g, y')\big) = v\big(h_{\tilde{\pi}}(g), p_{\pi|V}(y')\big) = p_{\pi|V}(y')$$

because $h_{\tilde{\pi}}(g) = e$. Since $p_{\pi|V}$ is a homeomorphism and $v_\pi(g, y') \in V$, we have $v_\pi(g, y') = y'$ for all $y' \in U$. If ω is a path starting at y_0, then, by the standard argument, we have $v_\pi\big(g, \omega(t)\big) = \omega(t)$ for all t. Hence $v_\pi(g, y) = y$ for all $y \in X_\pi$. Since v_π is an effective action by Corollary 2, $g = e$. Thus v_π is a free action.

Let

$$\big(P(X, x_0)\big)^* = \big\{\big(P(v)(\gamma, \omega), \omega\big)\colon \gamma \in P(G),\ \omega \in P(X, x_0)\big\}$$

and

$$X^* = \big\{\big(v(g, x), x\big);\ g \in G,\ x \in X\big\}.$$

Define the following four relations:

$\big(P(v)(\gamma, \omega), \omega\big)\ \varrho_1\ \big(P(v)(\gamma', \omega'), \omega'\big)$ if and only if $\omega(1) = \omega'(1)$, $\gamma(1) = \gamma'(1)$, $[\omega \circ \omega'^{-1}]$, $\big[P(v)(\gamma, \omega) \circ \big(P(v)(\gamma', \omega')\big)^{-1}\big] \in \pi$ and $[\gamma \circ \gamma'^{-1}] \in \tilde{\pi}$;

$\big(P(v)(\gamma, \omega), \omega\big)\ \varrho_2\ \big(P(v)(\gamma', \omega'), \omega'\big)$ if and only if $\omega(1) = \omega'(1)$ and $\gamma(1) = \gamma'(1)$;

$\gamma\ \varrho_1'\ \gamma'$ if and only if $\gamma(1) = \gamma'(1)$ and $[\gamma \circ \gamma'^{-1}] \in \tilde{\pi}$;

$\gamma\ \varrho_2'\ \gamma'$ if and only if $\gamma(1) = \gamma'(1)$.

Then $\varrho_1 \subset \varrho_2$ and $\varrho_1' \subset \varrho_2'$. Let $P(\tau)\colon \big(P(X, x_0)\big)^* \to P(G)$ be a continuous map induced by $\tau\colon X^* \to G$. Since $P(\tau)\big(P(\tau)(\gamma, \omega), \omega\big) = \gamma$, we have $P(\tau) \times P(\tau)(\varrho_i) \subset \varrho_i'$ for $i = 1, 2$. And since coset-spaces of the relations $\varrho_1, \varrho_2, \varrho_1', \varrho_2'$ are X^*_π, X^*, $G_{\tilde{\pi}}$, G, respectively, there exists a unique continuous map τ_π such that the diagram

$$\begin{tikzcd}
 & \big(P(X, x_0)\big)^* \arrow[rrr, "P(\tau)"] \arrow[dl, "a_{X^*_\pi}"'] \arrow[dd, "a_{X^*}"] & & & P(G) \arrow[dl, "a^{\tilde{\pi}}_G"'] \arrow[dd, "a_G"] \\
X^*_\pi \arrow[rrr, "\tau_\pi"] \arrow[dr, "p^*_\pi"'] & & & G_{\tilde{\pi}} \arrow[dr, "h_{\tilde{\pi}}"] & \\
 & X^* \arrow[rrr, "\tau"] & & & G
\end{tikzcd}$$

commutes. Observe that

$$\begin{aligned}
\tau_\pi\big((\tilde{g}, y), y\big) &= \tau_\pi \circ a_{X^*_\pi}\big(P(\nu)(\gamma, \omega), \omega\big) \\
&= a^{\tilde{\pi}}_G P(\tau)\big(P(\nu)(\gamma, \omega), \omega\big) = a^{\tilde{\pi}}_G(\gamma) = \tilde{g},
\end{aligned}$$

where γ and ω represent $\tilde{g}$ and y, respectively. Thus ν_π is a principal action. The proof is complete.

Every principal action $\nu\colon G \times X \to X$ determines the principal G-bundle

$$\xi_\nu = (X, q_\nu, X_\nu),$$

where X_ν is a coset-space of orbits and q_ν is a canonical map. The map q_ν is open, since $(q_\nu)^{-1}\big(q_\nu(U)\big) = p_X\big(\nu^{-1}(U)\big)$, where p_X is a projection of $G \times X$ onto X.

Let X_π be a covering space of X and let $\nu_\pi\colon G_{\tilde{\pi}} \times X_\pi \to X_\pi$ be the induced principal action. Consider the principal $G_{\tilde{\pi}}$-bundle

$$\xi_\pi = (X_\pi, q_\pi, X_{\nu_\pi})$$

determined by ν_π. The subgroup $\operatorname{Ker} h_{\tilde{\pi}} \subset G_{\tilde{\pi}}$, where $h_{\tilde{\pi}}\colon G_{\tilde{\pi}} \to G$ is a covering homomorphism, acts principally on the space X_π. Denote by $X^{\tilde{\pi}}_\pi$ the orbit-space of this action, and by $q^{\tilde{\pi}}_\pi\colon X_\pi \to X^{\tilde{\pi}}_\pi$ the canonical map. The factor-group $G_{\tilde{\pi}}/\operatorname{Ker} h_\pi = G$ acts freely on the orbit-space $X^{\tilde{\pi}}_\pi$. It is easy to see that this action is principal. Denote it by $\tilde{\nu}_\pi$. The orbit-space of the action G on $X^{\tilde{\pi}}_\pi$ is the space X_{ν_π}. Thus we have the new principal G-bundle

$$\eta_\pi = (X^{\tilde{\pi}}_\pi, p^{\tilde{\pi}}_\pi, X_{\nu_\pi}),$$

where $p^{\tilde{\pi}}_\pi\colon X^{\tilde{\pi}}_\pi \to X_{\nu_\pi}$ is a canonical map. Since $\operatorname{Ker} h_{\tilde{\pi}}$ is a discrete subgroup of $G_{\tilde{\pi}}$ and it is a totally discontinuous subgroup of the group $G(X_\pi | X)$ of covering transformations, there exists a covering map $p^{\tilde{\pi}}_\nu\colon X^{\tilde{\pi}}_\pi \to X$ such that $p_\pi = p^{\tilde{\pi}}_\nu q_\pi$ (see [4], Part 2, § 6). Commutativity of the diagram

$$
\begin{tikzcd}
G\times X_\pi^{\tilde\pi} \arrow[r, "\tilde v_\pi"] \arrow[d, "1_G\times p_v^{\tilde\pi}"'] & X_\pi^{\tilde\pi} \arrow[d, "p_v^{\tilde\pi}"] \\
G\times X \arrow[r, "v"] & X
\end{tikzcd} \tag{6}
$$

implies that there exists a unique continuous map $p^{\tilde\pi,\pi}\colon X_{v_\pi}\to X_v$ such that

$$p^{\tilde\pi,\pi}\circ p_\pi^{\tilde\pi} = p_v^{\tilde\pi}\circ q_v.$$

The pair $(p_v^{\tilde\pi}, p^{\tilde\pi,\pi})$ forms a G-bundle mapping.

THEOREM 3. *$p^{\tilde\pi,\pi}$ is a covering map.*

Proof. Let G_x denote the orbit of the point $x\in X$. If $y\in X_v$, then $q_v^{-1}(y) = G_x$ for some $x\in X$. Since $p_v^{\tilde\pi}$ is a covering map, there exists a neighbourhood $U\subset X$ of x such that $(p_v^{\tilde\pi})^{-1}(U)$ is a union of disjoint open sets $\{U_t\}$, and $p_v^{\tilde\pi}|_{U_t}\colon U_t\to U$ is a homeomorphism for each t. For $x'\in U$ the set $(p_v^{\tilde\pi})^{-1}(G_{x'})$ is a union of disjoint orbits G_{x_t}, where $x_t\in U_t$ by diagram (6). Thus $p_\pi^{\tilde\pi}\big(\bigcup_t U_t\big)$ is a union of disjoint open sets $p_\pi^{\tilde\pi}(U_t)$. Hence $p^{\tilde\pi,\pi}\colon p_\pi^{\tilde\pi}(U_t)\to q_v(U)$ is a homeomorphism. Since $q_v(U)$ is an open neighbourhood of y, $p^{\tilde\pi,\pi}$ is a covering map.

THEOREM 4. *The covering $p^{\tilde\pi,\pi}$ is proper (i.e., not a homeomorphism) if and only if there exists a proper subgroup $\pi_1\subset\pi_1(X, x_0)$ such that $(v_{x_0})_\#^{-1}(\pi_1) = \pi_1(G)$ and $\pi\subset\pi_1$.*

Proof. Necessity. If $p^{\tilde\pi,\pi}$ is a proper covering, then so is $p_v^{\tilde\pi}$. Hence $\pi_1 = (p_v^{\tilde\pi})_\#\big(\pi_1(X_\pi^{\tilde\pi}, x_\pi^{\tilde\pi})\big)$ is a proper subgroup of $\pi_1(X, x_0)$ and $\pi\subset\pi_1$. Since G acts principally on $X_\pi^{\tilde\pi}$, we have $(v_{x_0})_\#^{-1}(\pi_1) = \pi_1(G)$ by Theorem 2.

Sufficiency. $(p_v^{\tilde\pi})_\#\big(\pi_1(X_\pi^{\tilde\pi}, x_\pi^{\tilde\pi})\big)$ is an intersection of all subgroups $\pi_1\subset\pi_1(X, x_0)$ for which $\pi\subset\pi_1$ and $(v_{x_0})_\#^{-1}(\pi_1) = \pi_1(G)$.

REFERENCES

[1] G. E. Bredon, *Introduction to compact transformation group*, New York 1972.
[2] Sze-Tsen Hu, *Homotopy theory*, New York 1959.
[3] D. Husemoler, *Fibre bundles*, New York 1966.
[4] E. Spanier, *Algebraic topology*, New York 1966.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

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