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# HOMEOMORPHISMS OF THE SPHERE AND A GENERALIZATION OF THE WHYBURN CONJECTURE FOR COMPACT CONNECTED MANIFOLDS WITH BOUNDARY

BY

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In this paper it is shown that every orientation-preserving homeomorphism of the two-sphere $S^2$ which fixes a point $p \in S^2$ is isotopic to the identity map via an isotopy which keeps $p$ fixed at each stage.

It is also shown that if $g$: $D^n \to D^n$ is a homeomorphism from the unit $n$-ball to itself which fixes the $(n-1)$-sphere $S^{n-1} = \partial D^n$ pointwise, then $g$ is isotopic to the identity under an isotopy which fixes $S^{n-1}$ pointwise.

Moreover, it is shown that if $f$ is a continuous map of the closed $n$-disk $D^n$ into itself, which is a homeomorphism on $\partial D^n = S^{n-1}$ and a local homeomorphism on the interior of $D^n$, then $f$ is a homeomorphism of $D^n$. Finally, a generalization of the Whyburn Conjecture about the homeomorphisms of compact connected manifolds with boundary is given.

THEOREM 1. *Every orientation-preserving homeomorphism of the two-sphere $S^2$ which fixes a point $p \in S^2$ is isotopic to the identity map via an isotopy which keeps $p$ fixed at each stage.*

Proof. Suppose that $p \in S^2$ and that $f$: $S^2 \to S^2$ is an orientation-preserving homeomorphism which fixes $p$.

It has to be proved that there is an isotopy between $f$ and the identity which fixes $p$ at each stage.

For a simple closed curve $T$ in $S^2 - \{p\}$, denote by $\operatorname{int} T$ the component of $S^2 - T$ which contains $p$, and by $\operatorname{ext} T$ the other component.

Choose simple closed curves $T_1$ and $T_2$ in $S^2 - \{p\}$ such that $T_1 \cup f(T_1) \subset \operatorname{int} T_2$. Let $Q_1$, $Q_2$, and $Q_3$ be disjoint arcs with

$$Q_1 \cup Q_2 \cup Q_3 \subset \operatorname{int} T_2 - (T_1 \cup \operatorname{int} T_1),$$

each of the arcs joining the simple closed curves $T_1$ and $T_2$ (Fig. 1).

Let $\{x_i\} = Q_i \cap T_1$ and $\{y_i\} = Q_i \cap T_2$ $(i = 1, 2, 3)$. Since $f$ is orientation preserving and fixes $p$, there are disjoint arcs $K_1$, $K_2$ and $K_3$ joining $f(x_1)$ and $y_1$, $f(x_2)$ and $y_2$, $f(x_3)$ and $y_3$ (Fig. 2), respectively, such that

$$K_i \subset \operatorname{int} T_2 - [f(T_1) \cup \operatorname{int} f(T_1)].$$

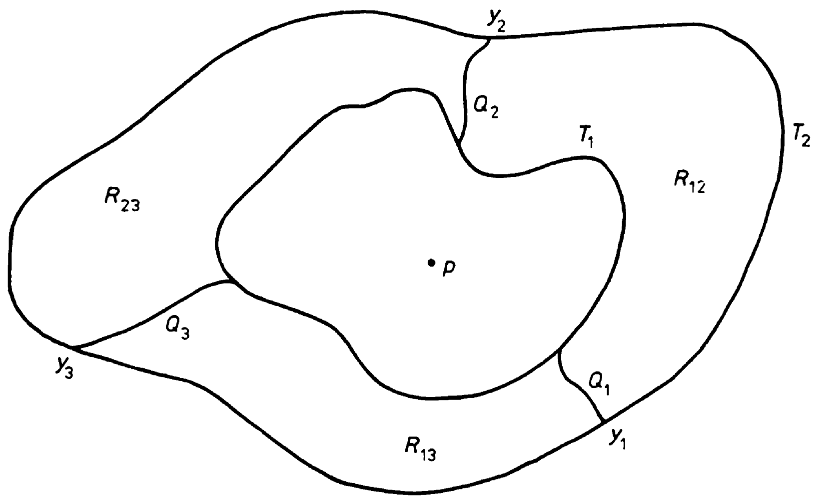


Fig. 1

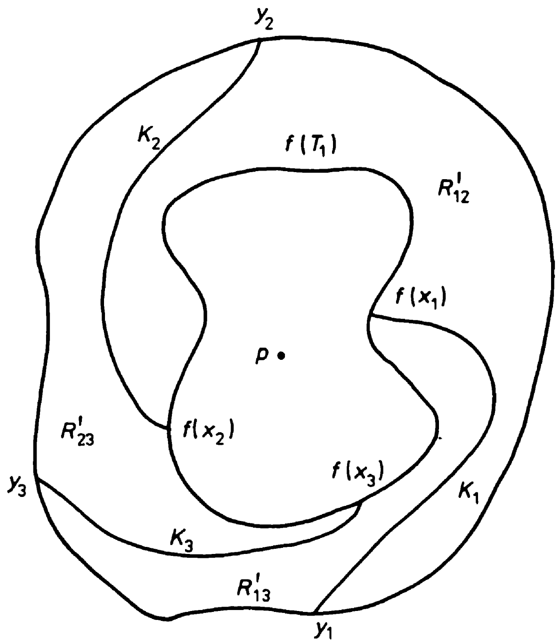


Fig. 2

Let $R_{ij}$ $(i \neq j)$ be the component of

$$\operatorname{int} T_2 - [Q_1 \cup Q_2 \cup Q_3 \cup T_1 \cup \operatorname{int} T_1],$$

which contains $Q_i \cup Q_j$ in its boundary.

Let $R'_{ij}$ be the component of

$$\operatorname{int} T_2 - [K_1 \cup K_2 \cup K_3 \cup f(T_1) \cup \operatorname{int} f(T_1)],$$

which contains $K_i \cup K_j$ in its boundary.

Now, we construct a homeomorphism $g$: $S^2 \to S^2$ having the following properties:

(i) $g|(T_1 \cup \operatorname{int} T_1) = f|(T_1 \cup \operatorname{int} T_1)$,

(ii) $g|(T_2 \cup \text{ext } T_2) = \text{id}|(T_2 \cup \text{ext } T_2)$ (id = identity),

(iii) $g(R_{ij}) = R'_{ij}$.

Indeed, (i) and (ii) define $g|\text{bd}\, Q_i$ ($i = 1, 2, 3$). This partial map can be extended to take $Q_i$ to $K_i$ homeomorphically. This defines $g|\text{bd}\, R_{ij}$ for each $i \neq j$.

Now we need the following lemma:

LEMMA. *If $T_1$ and $T_2$ are simple closed curves in the plane $E^2$ and $h$: $T_1 \to T_2$ is any homeomorphism, then there is an extension of $h$ which takes $T_1 \cup \text{int } T_1$ homeomorphically onto $T_2 \cup \text{int } T_2$.*

By this lemma, $g|\text{bd}\, R_{ij}$: $\text{bd}\, R_{ij} \to \text{bd}\, R'_{ij}$ can be extented to a homeomorphism $g|\bar{R}_{ij}$: $\bar{R}_{ij} \to \bar{R}'_{ij}$, where $\bar{R}$ denotes the closure of the region $R$.

Now, by (i), the homeomorphisms $f$ and $g$ are isotopic under an isotopy which moves points only in ext $T_1$. By (ii), the homeomorphism $g$ and the identity are isotopic under an isotopy which moves points only in int $T_2$. Consequently, since $g(p) = p$, the latter isotopy may be chosen to fix $p$. The composition of the two isotopies gives the desired isotopy between $f$ and the identity map.

THEOREM 2. *Let $g$: $D^n \to D^n$ be a homeomorphism from the unit n-ball to itself which fixes the $(n-1)$-sphere $S^{n-1} = \partial D^n$ pointwise; then g is isotopic to the identity under an isotopy which fixes $S^{n-1}$ pointwise.*

Proof. Each point $x \in E^n - \{0\}$ has a unique "polar" representation $x = (r, \theta)$, where $r = |x|$, and $\theta = x/|x| \in S^{n-1}$. Suppose that

$$g(r, \theta) = [M(r, \theta), N(r, \theta)], \quad r \leqslant 1.$$

Then $M(1, \theta) = 1$ and $N(1, \theta) = \theta$ because $g|S^{n-1} = \text{id}$. Now, extend $g = (M, \theta)$ to all of $E^n$ by the identity. Define the isotopy $g_t$: $E^n \to E^n$ ($0 < t \leqslant 1$, $r \leqslant 1$) as follows:

$$g_t(r, \theta) = [t \cdot M(r/t, \theta), N(r/t, \theta)].$$

We observe that $g_1 = g$, $g_t(r, \theta) = (r, \theta)$ for $r > t$, and for a fixed $r \leqslant t$ the map $g_t$ is the map $g$ with a scale factor.

Thus, the effect of the isotopy is to push the distorted region of $E^n$ into an $n$-ball of smaller and smaller radius.

The maps $g_t$ approach the identity map continuously as $t \to 0$. Hence $g_0$ may be defined to be the identity and $g_t$ ($0 \leqslant t \leqslant 1$) is the desired isotopy.

THEOREM 3. *Let f be a continuous map of the closed n-disk $D^n$ into itself, which is a homeomorphism on $\partial D^n = S^{n-1}$ and a local homeomorphism on the interior of $D^n$. Then f is a homeomorphism of $D^n$.*

Proof. Since $f$ is a local homeomorphism on int $D^n$, $f$ is an immersion on int $D^n$, and so $f^{-1}f(x)$ is a finite set.

Let $S(f) = \{x \in D^n: (\exists y \neq x)\, f(y) = f(x)\}$; in other words, $S(f)$ contains all points in $D^n$ for which there are other points having the same image. Claim that $S(f) = \emptyset$; in other words, $f$ is 1-1.

We prove that $S(f)$ is both an open and closed subset of $D^n$, and so $S(f) = \emptyset$ or $D^n$. However, $S(f)$ cannot be $D^n$ since $f \mid \partial D^n$ is a homeomorphism. Thus, $S(f) = \emptyset$, and so $f$ is 1-1. Hence $f$ is a homeomorphism.

The part (b) of the following theorem is a generalization of the Whyburn Conjecture.

THEOREM 4. (a) *Let $M$ be a compact connected manifold with $\partial M \neq \emptyset$ and let $f$ be a discrete*(1) *open map $f\colon M \to M$, which is surjective and such that $f(\partial M) = \partial M$, $f(\operatorname{int} M) = \operatorname{int} M$, and $f \mid \partial M$ is a homeomorphism. Then $f\colon M \to M$ is a homeomorphism.*

(b) *Let $M$ be a compact connected manifold with boundary equal to $S^n$, and let $f$ be a continuous, discrete, open map of $M$ onto itself such that $f(\partial M) \cap f(\operatorname{int} M) = \emptyset$ and $f \mid \partial M$ is a homeomorphism. Then $f$ is a homeomorphism.*

THEOREM 5. (a) *Let $f$ be a continuous map of an $n$-dimensional disk $C^n$ onto an $n$-dimensional disk $D^n$, where $C^n$ and $D^n$ lie on $S^n = \{x \in E^{n+1}: |x| = 1\}$, such that $f(\partial C^n) \cap f(\operatorname{int} C^n) = \emptyset$ and $f \mid \partial C^n$ is a homeomorphism. Then $f$ is a homeomorphism.*

(b) *Let $f\colon D^n \to D^n$ be a continuous, finitely-to-one, open map of $D^n$ onto $D^n$ such that $f(\partial D^n) \cap f(\operatorname{int} D^n) = \emptyset$ and $f \mid \partial D^n$ is a homeomorphism. Then $f$ is a homeomorphism.*

Remark. Let $f\colon D^n \to D^n$ be a continuous map such that $f(\partial D^n) \cap f(\operatorname{int} D^n) = \emptyset$ and $f \mid \partial D^n$ is a homeomorphism. Then $f(\partial D^n) = \partial D^n$.

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(1) Note "discrete" means that each point-inverse is discrete; in other words, it consists of isolated points.

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