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ON A CONJECTURE OF NARKIEWICZ ABOUT FUNCTIONS WITH NON-DECREASING NORMAL ORDER

BY

P. D. T. A. ELLIOTT (BOULDER, COLORADO)

An arithmetic function $f(n)$ is said to have a non-decreasing *normal order* $g(n)$ if, for each $\varepsilon > 0$, those integers n for which the inequality

$$|f(n)-g(n)| \geqslant \varepsilon g(n) \geqslant 0$$

is satisfied have asymptotic density zero.

A well-known example is $f(n) = \omega(n)$, the number of distinct prime divisors of the integer n. In this case we may set

$$g(n) = \begin{cases} \log\log n & \text{if } n > 9, \\ 0 & \text{if } 1 \leqslant n \leqslant 8. \end{cases}$$

This result was established by Hardy and Ramanujan [3]. We may express it in the form "almost all integers n have about $\log\log n$ distinct prime factors".

In his paper [5] Narkiewicz made the following conjecture (P 923):

Assume that the function $f(n)$ with a non-decreasing normal order is of the form

$$f(n) = \sum_{p|n} f(p) \qquad (p \text{ prime}),$$

where $f(p)$ is non-negative, and increases with the prime argument p. In particular, $f(n)$ is strongly additive. Then for each fixed $\varepsilon > 0$ the bound

$$f(p) = O\big((\log p)^{1+\varepsilon}\big)$$

holds for all primes p. He could establish this conjecture if the function $f(p)/\log p$ were decreasing. In particular, he showed the result to be false if ε is replaced by zero.

We shall here prove his conjecture to be correct.

Remarks. It proves possible to establish a form of Narkiewicz's conjecture even if we do not assume that $f(p)$ increases with p.

It was shown by Birch [1] that the only multiplicative functions which have non-decreasing normal orders are those of the form n^c, where c is a constant.

Without loss of generality we may assume that $g(n) \geqslant 0$ holds for every positive integer n. For example, we may replace $f(n)$ and $g(n)$ by $f(n)+A\log n$ and $g(n)+A\log n$, respectively, with a suitably chosen constant A.

THEOREM. *Let the additive function $f(n)$ have a non-decreasing normal order $g(n)$. Then for each fixed $\varepsilon > 0$ there is a constant $c(\varepsilon)$ such that for all integers $n \geqslant 2$ the inequality*

$$|g(n)| \leqslant c(\varepsilon)(\log n)^{1+\varepsilon}$$

is satisfied. Moreover, if $\varepsilon_0 > 0$ and x is sufficiently large, the bound $|f(p)| \leqslant D(\log x)^{1+\varepsilon}$ holds for every prime p not exceeding x save possibly for a set of primes q which satisfy

$$\sum_{q \leqslant x} \frac{1}{q} \leqslant \varepsilon_0.$$

Remark. In the statement of this theorem the function $f(n)$ is only assumed to be additive. The number ε_0 is considered fixed, also the number D, which depends upon ε_0 and ε; and x is to be large enough depending upon ε_0 and ε.

Deduction of Narkiewicz's conjecture. Assume that $f(p)$ is non-negative, and increases with p. Then, if p is sufficiently large, we can find a real number x such that $\frac{1}{2}x < p \leqslant x$. We apply the Theorem with $\varepsilon_0 = \frac{1}{2}\log 2$, any fixed ε. Since for x large (cf. [4], p. 351)

$$\sum_{\substack{x < l \leqslant x^2 \\ l \text{ prime}}} \frac{1}{l} = \log\left(\frac{\log x^2}{\log x}\right) + O\left(\frac{1}{\log x}\right) > \varepsilon_0,$$

there is a prime l in the interval $(x, x^2]$ for which the bound

$$|f(l)| \leqslant D(\log l)^{1+\varepsilon}$$

is assured.

Hence

$$0 \leqslant f(p) \leqslant f(l) \leqslant D(\log x^2)^{1+\varepsilon} \leqslant E(\log p)^{1+\varepsilon}$$

for a suitably chosen constant E.

This establishes Narkiewicz's conjecture.

In order to prove the Theorem we need the following result of large-sieve type, a proof of which may be found in the author's paper [2]. In this, and in the sequel, the symbols p, q and l will denote rational prime numbers.

LEMMA 1. *There is an absolute constant c_1 such that the inequality*

$$\sum_{p\leqslant x} p\,\Big|\sum_{\substack{n\leqslant x\\ p\| n}} a_n - p^{-1}\sum_{n\leqslant x} a_n\Big|^2 \leqslant c_1 x \sum_{n\leqslant x} |a_n|^2$$

holds uniformly for all real $x\geqslant 1$, and for complex numbers a_n ($n = 1, 2, \ldots \ldots, [x]$).

Remark. The notation $p\|n$ means that p divides n, p^2 does not.

LEMMA 2. *Let $b_1 < b_2 < \ldots$ be a sequence of integers of asymptotic density one. Let w and x be real numbers, $w\geqslant 3$, $2\leqslant x\leqslant w(\log w)^{-1}$. Let $\varepsilon > 0$ be fixed. Then for each prime p not exceeding x, save possibly for a set of primes q which satisfy*

$$\sum_{q\leqslant x}\frac{1}{q}\leqslant \varepsilon,$$

the equation $b_i = pb_j$ ($p\nmid b_j$) is soluble with an integer b_i in the range $\frac{1}{2}w < b_i \leqslant w$, provided only that w exceeds a certain value w_0 which does not depend upon x.

Remark. The method of proof of this lemma has considerable flexibility in application.

Proof. Let $r_1 < r_2 < \ldots$ be those integers r in the interval $\frac{1}{2}w < r \leqslant w$ which are not members of the sequence b_i. Let δ be a positive number which is to be chosen presently.

We apply Lemma 1 with

$$a_n = \begin{cases} 1 & \text{if } n = r_i \text{ for some } i,\\ 0 & \text{otherwise.}\end{cases}$$

Consider those primes q in the range $2\leqslant q\leqslant x$ ($x\leqslant w$) for which the inequality

$$\Big|\sum_{q\| n} a_n - q^{-1}\sum a_n\Big| < \delta w q^{-1}$$

fails. According to Lemma 1 these primes may be estimated by

$$\sum_{q\leqslant x}\frac{1}{q}\leqslant \frac{1}{(\delta w)^2}c_1 w\sum_{r_i\leqslant w} 1.$$

Since the b_i have density 1, for all sufficiently large values of w the number of r_i not exceeding w is at most $\varepsilon^2 w$. We choose $\delta = (c_1\varepsilon)^{1/2}$ so that

$$\sum_{q\leqslant x}\frac{1}{q}\leqslant \varepsilon.$$

We shall now show that for large enough w, and each remaining prime p not exceeding x, the equation $b_i = pb_j$ $(p \nmid b_j)$ is soluble. In fact, the number of b_i which lie in the range $\frac{1}{2}w < b_i \leqslant w$ and which are exactly divisible by p is at least

$$\sum_{\substack{w/2<m\leqslant w \\ p\|m}} 1 - \sum_{\substack{w/2<r_i\leqslant w \\ p\|r_i}} 1 \geqslant \frac{w}{2p} - \frac{w}{2p^2} - 4 - \frac{w}{p}(\varepsilon^2+\delta) > \frac{w}{5p}$$

provided that ε is sufficiently (absolutely) small, and w is sufficiently large.

Thus there are at least $w/5p$ integers $p^{-1}b_i$ in the interval $(w/2p, w/p]$.

This same interval contains

$$\sum_{w/2p<b_j\leqslant w/p} 1 > \frac{w}{2p}(1-2\varepsilon^2)+O(1) > \frac{2w}{5p}$$

members of the sequence of b_j, once again assuming that w is sufficiently large. Since the $p^{-1}b_i$ and b_j comprise between them more than $3w/5p$ integers in an interval of length $w/2p$, we must have

$$p^{-1}b_i = b_j \quad (p\|b_i)$$

for some integer b_i in the range $\frac{1}{2}w < b_i \leqslant w$. This is what we wished to prove.

Proof of the Theorem. Define the function $g(x)$ by linear interpolation, so that $g(x) = g(n)$ if x has the integral value n $(n \geqslant 1)$.

Let w and x be real numbers, $w \geqslant 3$, $2 \leqslant x \leqslant w/\log w$. In the following argument these may be thought of as being large.

We apply Lemma 2 with the b_j chosen to be those integers n for which the inequality $|f(n)-g(n)| \leqslant \varepsilon g(n)$ is satisfied $(g(n) \geqslant 0,\ \varepsilon$ fixed, $0 < 4\varepsilon < 1)$. Then from each equation of the form

$$b_i = pb_j \quad (\frac{1}{2}w < b_i \leqslant w,\ p \nmid b_j),$$

and the hypothesis that $f(n)$ is additive, we deduce that

$$f(b_i) = f(p)+f(b_j)$$

and

$$g(b_i)(1+\theta_1\varepsilon) = f(p)+g(b_j)(1+\theta_2\varepsilon), \tag{1}$$

where $|\theta_k| \leqslant 1$ $(k = 1, 2)$.

We may do this for each prime p not exceeding x, save possibly for a set of primes q which satisfy, say,

$$\sum_{q\leqslant x} \frac{1}{q} \leqslant \varepsilon_0.$$

Here ε_0 may be chosen arbitrarily small if w is chosen sufficiently large.

We apply this argument with $w = x$, and then $w = x^{1/2+\varepsilon}$ in turn, and find solutions of equation (1) with the same value of the prime p for each prime p in the range $x^{1/2-\varepsilon} < p \leqslant x^{1/2}$, save possibly for a set of primes q_1 which satisfy

$$\sum \frac{1}{q_1} \leqslant 2\varepsilon_0.$$

Since

$$\sum_{\substack{x^{1/2-\varepsilon}<l\leqslant x^{1/2}\\ l \text{ prime}}} \frac{1}{l} = \log\left(\frac{1}{1-2\varepsilon}\right) + O\left(\frac{1}{\log x}\right) > 2\varepsilon_0,$$

if we choose ε_1 to be $B\varepsilon$ with a suitable constant B, we conclude that the simultaneous solution of equation (1) for $w = x$, $x^{1/2+\varepsilon}$ and at least one prime p in the range $x^{1/2-\varepsilon} < p \leqslant x$ is possible.

Let us denote the second solution by

$$(2) \qquad g(b_r)(1+\theta_3\varepsilon) = f(p) + g(b_s)(1+\theta_4\varepsilon) \qquad (\tfrac{1}{2}x^{1/2+\varepsilon} < b_r \leqslant x^{1/2+\varepsilon})$$

with $|\theta_k| \leqslant 1$ $(k = 3, 4)$.

Eliminating $f(p)$ from equations (1) and (2), and making use of the fact that $g(b_s)$ is non-negative, we see that

$$g(b_i) \leqslant \frac{1+\varepsilon}{1-\varepsilon}\{g(b_j) + g(b_r)\}.$$

Since $g(x)$ is non-decreasing (a property which we use here for the first time in the proof of the Theorem),

$$g\left(\frac{x}{2}\right) \leqslant g(b_i) \leqslant \frac{1+\varepsilon}{1-\varepsilon}\left\{g\left(\frac{x}{p}\right) + g(x^{1/2+\varepsilon})\right\} \leqslant 2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)g(x^{1/2+\varepsilon}).$$

We can express this result more simply, by adjusting the value of ε to be sufficiently small, as

$$g(x) \leqslant (2+K\varepsilon)g(x^{1/2+\varepsilon})$$

for some constant K, for all $x \geqslant x_1 \geqslant 2$.

We apply this inequality iteratively:

$$g(x) \leqslant (2+K\varepsilon)^k g(x^{(1/2+\varepsilon)^k}) \qquad (k = 1, 2, \ldots),$$

and choose the integer k to be the largest one which is consistent with the requirement that $x^{(1/2+\varepsilon)^{k-1}} \geqslant x_1$. Hence

$$g(x) \leqslant c_2 \exp\left(\frac{\log\log x}{\log(2/(1+2\varepsilon))}\log(2+K\varepsilon)\right) \leqslant c_2(\log x)^{1+\mu},$$

where

$$\mu = \frac{1}{\log\left(2/(1+2\varepsilon)\right)} \log\left\{1+\left(\frac{K}{2}+2\right)\varepsilon+K\varepsilon^2\right\}$$

may be made arbitrarily small if ε is chosen near enough to zero.

This proves that

$$g(x) = O\left((\log x)^{1+\varepsilon_1}\right) \quad \text{for any fixed } \varepsilon_1 > 0.$$

From equation (1) we see that for most primes p (with a sufficiently thin exceptional set)

$$|f(p)| \leqslant 2g(b_i)+2g(b_j) \leqslant 4g(x) = O\left((\log x)^{1+\varepsilon_1}\right).$$

This completes the proof of the Theorem, and so of Narkiewicz's conjecture.

REFERENCES

[1] B. J. Birch, *Multiplicative functions with non-decreasing normal order*, The Journal of the London Mathematical Society 42 (1967), p. 149-151.

[2] P. D. T. A. Elliott, *On connections between the Turán-Kubilius inequality and the large-sieve: Some applications*, Proceedings of Symposium of the American Mathematical Society 24 (1972), p. 77-82.

[3] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n*, Quarterly Journal of Pure and Applied Mathematics 48 (1917), p. 76-92.

[4] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4-th edition, Oxford 1960.

[5] W. Narkiewicz, *On additive functions with a non-decreasing normal order*, Colloquium Mathematicum 32 (1974), p. 137-142.

BOULDER, COLORADO

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