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# EXTENSION OF LOCALLY UNIFORMLY EQUIVALENT METRICS

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Hausdorff showed that if $A$ is a closed subset of a metric space $(X, d)$ and if $\varrho$ is a metric on $A$ which is topologically equivalent to $d|A \times A$, then there is a metric on $X$ that extends $\varrho$ and is topologically equivalent to $d$ ([1], II, Theorem 3.2). The word "topologically" can be replaced either by "Lipschitz" [5] or by "locally Lipschitz" [3] or, under some additional assumptions, by "uniformly" [5] (for Lipschitz equivalent and uniformly equivalent metrics the closedness of $A$ is irrelevant). The purpose of this paper is to show that "locally uniformly" applies as well. The results and the proofs are similar to those for locally Lipschitz equivalent metrics in [3].

A map $f\colon S \to T$ between uniform spaces is called *locally uniformly continuous* if each point of $S$ has a neighborhood on which $f$ is uniformly continuous. If $f$ is bijective and both $f$ and $f^{-1}$ are locally uniformly continuous, then $f$ is called a *locally uniform homeomorphism*. Two metrics $d_1$ and $d_2$ on a set $D$ are said to be *locally uniformly equivalent* if the identity map id: $(D, d_1) \to (D, d_2)$ is a locally uniform homeomorphism. A metric on a subset $A$ of a metric space $(X, d)$ is said to be *locally uniformly compatible* if it is locally uniformly equivalent to $d|A \times A$.

LEMMA. *Let $(X, d)$ be a metric space, let $A \subset X$ be closed, and let $\varrho$ be a locally uniformly compatible metric on $A$ such that $\varrho \leqslant d|A \times A$. Let $e(x, y)$ be the minimum of $d(x, y)$ and $\inf\{d(x, a)+\varrho(a, b)+d(b, y) \mid a, b \in A\}$ for $x, y \in X$. Then $e$ is a locally uniformly compatible metric on $X$ extending $\varrho$.*

Proof. By [2], p. 517, $e$ is a metric on $X$ that extends $\varrho$ and is topologically equivalent to $d$. To make the proof independent of the last fact, we remark that $A$ is closed in $(X, e)$ because $e(x, A) = d(x, A)$ for each $x \in X$. Observe that $e \leqslant d$ and that $e(x, y) = d(x, y)$ for all $x, y \in X$ such that $e(x, A) \geqslant \varepsilon$, $e(y, A) \geqslant \varepsilon$, and $e(x, y) < 2\varepsilon$ for some $\varepsilon > 0$. Hence it suffices to prove that each $p \in A$ has a neighborhood $U$ in $(X, e)$ such that id: $(U, e|U \times U) \to (X, d)$ is uniformly continuous. There is an $r > 0$ such that if $V = \{x \in A \mid \varrho(x, p) < 2r\}$, then the identity map id: $(V, \varrho | V \times V) \to (X, d)$ is uniformly continuous. We show that one can choose

$$U = \{x \in X \mid e(x, p) < r\}.$$

Let $\varepsilon > 0$. Choose $\delta > 0$ with $\delta < \min(r, \varepsilon/2)$ such that $a, b \in V$ and $\varrho(a, b) < \delta$ imply $d(a, b) < \varepsilon/2$. Suppose that $x, y \in U$ and $e(x, y) < \delta$. To prove $d(x, y) < \varepsilon$, we may assume $e(x, y) \neq d(x, y)$. Then there are $a, b \in A$ such that $d(x, a) + \varrho(a, b) + d(b, y) < \delta$. This implies

$$\varrho(a, p) = e(a, p) \leqslant e(a, x) + e(x, p) < d(a, x) + r < \delta + r < 2r,$$

whence $a \in V$; similarly $b \in V$. Since $\varrho(a, b) < \delta$, we get

$$d(x, y) \leqslant d(x, a) + d(a, b) + d(b, y) < \delta + \varepsilon/2 < \varepsilon.$$

THEOREM 1. *Let $\varrho$ be a locally uniformly compatible metric on a closed subset $A$ of a metric space $(X, d)$. Then there is a locally uniformly compatible metric on $X$ extending $\varrho$.*

Proof. Let $m(A)$ be the space of all bounded real functions on $A$ (with the sup norm). There is an isometric embedding $f: (A, \varrho) \to m(A)$ (see [1], II, Proposition 1.1). Since $f$ is locally uniformly continuous with respect to $d$, by [6], Theorem 1, $f$ has a locally uniformly continuous extension $\bar{f}: X \to m(A)$. Define a metric $d_1$ on $X$ by

$$d_1(x, y) = d(x, y) + \|\bar{f}(x) - \bar{f}(y)\|.$$

Then $d_1$ is locally uniformly compatible and $\varrho \leqslant d_1 | A \times A$. Thus an application of the Lemma completes the proof.

Next we study the case of a non-closed $A$.

THEOREM 2. *Let $\varrho$ be a locally uniformly compatible metric on a subset $A$ of a metric space $(X, d)$. Then $\varrho$ has an extension to a locally uniformly compatible metric on some neighborhood of $A$.*

Proof. From the Lemma in [4] it follows easily that $\varrho$ has an extension to a locally uniformly compatible metric $\varrho_1$ on an open neighborhood $U$ of $A$ in $\bar{A}$. Now $U = V \cap \bar{A}$ for some open neighborhood $V$ of $A$ in $X$. Since $U$ is closed in $V$, by Theorem 1 there is a locally uniformly compatible metric $\varrho_2$ on $V$ extending $\varrho_1$, and thus $\varrho$.

Remark. In Theorems 1 and 2 the extension of $\varrho$ can be chosen to be complete (respectively, totally bounded) if $\varrho$ is complete (respectively, totally bounded) and $d$ is locally complete (respectively, separable and locally totally bounded). These results can be proved as similar results in [3].

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