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ON SUBSPACES OF LOCALLY CONVEX SPACES WITH UNCONDITIONAL SCHAUDER BASES

BY

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It was shown by James ([7]; see also [12]) that if Y is a Banach space with an unconditional basis, then

(a) if no subspace of Y is isomorphic to c_0, then Y is weakly sequentially complete;

(b) if no subspace of Y is isomorphic to l_1, then every bounded subset of Y is *weakly sequentially precompact* (i.e., every bounded sequence in Y has a weak Cauchy subsequence).

Later, Bessaga and Pełczyński [1] proved that these results remain also valid when Y is merely a closed subspace of a Banach space with an unconditional basis.

The purpose of this note is to prove some analogues of the results of Bessaga and Pełczyński for subspaces of locally convex spaces with unconditional bases. For similar extensions of the original results of James, see [9], [14], and [15] (cf. also the recent book of Jarchow [8]). We hope that our proof of part (a) of the Theorem below may be of some interest also in the case of Banach spaces.

Following De Grande-De Kimpe [3], by a G-*space* we understand a (Hausdorff) locally convex space X whose topology is the Mackey topology $\tau(X, X')$ and whose (topological) dual space X' is $\sigma(X', X)$-sequentially complete. The reader is referred to [2], [3], and [10] for more information about this class of spaces.

A sequence (x_n) in a locally convex space X is called a *Schauder basis* of X if there exists a sequence (f_n) in X' which is biorthogonal to (x_n) and such that

$$x = \sum_{i=1}^{\infty} f_i(x)\, x_i \quad \text{for every } x \in X.$$

A Schauder basis of a G-space is *equicontinuous*, i.e., the partial sum operators associated with the basis are equicontinuous ([3], Proposition 3.4).

Similarly, if (x_n) is an unconditional Schauder basis of a G-space X, then the family of all projections

$$P_A x = \sum_{i \in A} f_i(x) x_i,$$

where A is a finite subset of $N = \{1, 2, \ldots\}$, is equicontinuous. If a basis has this property, then we say that it is *unconditionally equicontinuous* (u.e.). It is easily seen that if (x_n) is an equicontinuous (resp., u.e.) basis of a locally convex space X, then (x_n) is also an equicontinuous (resp., u.e., and hence unconditional) basis for the completion $\tilde{X}$ of X. As an immediate consequence of this fact we infer that a sequentially complete space with an equicontinuous basis is necessarily complete. It is also clear that if (z_n) is a block sequence of an equicontinuous basis (x_n), then (z_n) is an equicontinuous basic sequence, i.e., an equicontinuous basis for the closed linear span $[z_n]$ of (z_n).

THEOREM. *Let X be a complete G-space with an unconditional Schauder basis and let Y be a closed linear subspace of X. Then*:

(a) *If Y is metrizable and has no subspace isomorphic to c_0, then Y is weakly sequentially complete.*

(b) *If Y has no subspace isomorphic to l_1, then every bounded subset of Y is weakly sequentially precompact.*

Proof. Let (x_n) be an unconditional Schauder basis of X and (f_n) the associated sequence of biorthogonal functionals. Thus, for each x in X,

$$x = \sum_{i=1}^{\infty} f_i(x) x_i,$$

where the series converges unconditionally, and hence is subseries summable (cf. [8], Corollary 14.6.6).

(a) With every x in X we may therefore associate a countably additive set function

$$x(\cdot)\colon \mathcal{P}(N) \to X,$$

where $\mathcal{P}(N)$ is the power set of N, by defining

$$x(A) = \sum_{i \in A} f_i(x) x_i \quad \text{for } A \subset N.$$

Next, by the weak-star sequential completeness of X', we easily see that if $f \in X'$ and $A \subset N$, then the linear functional f_A defined on X by

$$f_A(x) = f\big(x(A)\big) = \lim_{n\to\infty} \Big(\sum_{\substack{i \in A \\ i \leqslant n}} f(x_i) f_i\Big)(x)$$

is continuous. Clearly, if $x \in X$, $f \in X'$, then

$$fx(\cdot)\colon\ A \to f_A(x) = f(x(A))$$

is a countably additive scalar-valued measure on $\mathscr{P}(N)$.

Let (y_n) be a weak Cauchy sequence in Y. Then, for each $f \in X'$, the sequence of measures $(fy_n(\cdot))$ is setwise convergent on $\mathscr{P}(N)$. By the Nikodym theorem ([6], Corollary III.7.4), the limit set function, say m_f, is countably additive. Moreover, if

$$t_i = \lim_{n\to\infty} f_i(y_n) \quad \text{for every } i \in N,$$

then

$$\sum_{i=1}^{\infty} |f(t_i x_i)| < \infty \quad \text{and} \quad m_f(A) = \sum_{i\in A} f(t_i x_i)$$

for all $f \in X'$ and $A \subset N$.

Suppose the sequence (y_n) does not converge weakly. Then the series $\sum_{i=1}^{\infty} t_i x_i$ does not converge in X (otherwise, its sum would be the weak limit of (y_n)). It follows that there exist a continuous seminorm p on X and a sequence of integers $0 = n_0 < n_1 < \ldots$ such that if

$$z_j = \sum_{i\in A_j} t_i x_i, \quad \text{where } A = \{n_{j-1}+1, \ldots, n_j\},$$

then

(1) $$p(z_j) \geqslant 1 \quad \text{for every } j \in N.$$

Clearly,

(2) $$\sum_{j=1}^{\infty} |f(z_j)| < \infty \quad \text{for every } f \in X'.$$

We shall now prove that the subspace $Z = [z_j]$ is isomorphic to c_0.

Since X is complete, conditions (1) and (2) imply that

(3) $$\sum_{j=1}^{\infty} s_j z_j \text{ converges if and only if } (s_j) \in c_0.$$

Let h_j be a linear functional on $E_j = [x_i\colon i \in A_j]$ such that

$$h_j(z_j) = 1 \quad \text{and} \quad |h_j(x)| \leqslant p(x) \text{ for every } x \in E_j.$$

Define $g_j \in X'$ by

$$g_j(x) = h_j(P_{A_j} x).$$

Since

$$|g_j(x)| \leqslant p(P_{A_j}x) \to 0 \text{ as } j \to \infty \quad \text{for every } x \in X,$$

using (3) we may define a linear operator P on X by

$$Px = \sum_{j=1}^{\infty} g_j(x) z_j.$$

From the Banach–Steinhaus theorem for G-spaces [2] it follows that P is a continuous projection of X onto Z. Hence Z is a G-space. Now the linear map T: $c_0 \to Z$ defined by

$$T(s_i) = \sum_{j=1}^{\infty} s_j z_j$$

is clearly continuous, one-to-one and onto. From the closed graph theorem for G-spaces [10] (or from the Banach–Steinhaus theorem for G-spaces [2] again) we infer easily that T^{-1} is continuous. Thus T is an isomorphism of c_0 onto Z (and $Te_n = z_n$, where e_n is the n-th unit vector in c_0).

Now, if no subspace of Y is isomorphic to c_0, then Y and Z are *totally incomparable*, i.e., they have no isomorphic infinite-dimensional subspaces (use [12], Proposition 2.a.2). Since then $\dim(Y \cap Z) < \infty$, we may assume $Y \cap Z = \{0\}$. Then, by the generalized Gurariĭ–Rosenthal theorem proved in [5], the subspace $Y+Z$ is the topological direct sum of Y and Z. (It is the only point where the metrizability of Y is needed.)

Since (y_n) does not converge weakly to 0, there is $g \in Y'$ such that

$$\lim_{n\to\infty} g(y_n) \neq 0.$$

We extend g to $Y+Z$ so that $g|_Z = 0$, and next, using the Hahn–Banach theorem, to a continuous linear functional f on X. Then

$$m_f(N) = \lim_{n\to\infty} fy_n(N) = \lim_{n\to\infty} g(y_n) \neq 0.$$

However, on the other hand,

$$m_f(N) = \sum_{j=1}^{\infty} f(z_j) = 0,$$

and thus we have obtained a contradiction. It follows that Y must contain an isomorphic copy of c_0.

(b) (Cf. [9], Proof of Theorem 2.2.) As easily seen, it is enough to prove that if (y_n) is a bounded sequence in Y such that

$$\lim_{n\to\infty} f_i(y_n) = 0 \quad \text{for every } i \in N, \tag{4}$$

then $y_n \to 0$ weakly. Suppose it is not so. Then we may assume that there exists $f \in X'$ such that

$$3a = \inf_n |f(y_n)| > 0.$$

Let

$$q(x) = \sum_{n=1}^{\infty} |f_n(x) f(x_n)| \quad \text{for every } x \in X;$$

then q is easily seen to be a continuous seminorm on X. Using (4) we pass to a subsequence of (y_n), which we still denote by (y_n), and find a sequence of integers $0 = m_0 < m_1 < \ldots$ such that if

$$z_n = \sum_{i=m_{n-1}+1}^{m_n} f_i(y_n) x_i,$$

then

$$|f(y_n - z_n)| \leqslant q(y_n - z_n) \leqslant a \quad \text{for every } n \in N. \tag{5}$$

Hence $|f(z_n)| \geqslant 2a$ for every $n \in N$. Since the sequence (y_n) is bounded, so are the sequences (z_n) and $(y_n - z_n)$. Hence, if $t = (t_n) \in l_1$, then the series $\sum t_n y_n$, $\sum t_n z_n$, and $\sum t_n(y_n - z_n)$ converge in X and, as easily seen,

$$\begin{aligned} &q(\textstyle\sum t_n z_n) \geqslant \sum |f(z_n)|\,|t_n| \geqslant 2a\,\|t\|, \\ &q(\textstyle\sum t_n(y_n - z_n)) \leqslant a\,\|t\|. \end{aligned} \tag{6}$$

The linear map $A\colon l_1 \to Y$ defined by

$$At = \sum_{n=1}^{\infty} t_n y_n$$

is clearly continuous. Since (6) implies $q(At) \geqslant a\,\|t\|$, A is an isomorphism, and thus Y has a subspace isomorphic to l_1. A contradiction.

COROLLARY. *Suppose X is as in the Theorem and let Y be a closed metrizable subspace of X. Then Y is reflexive if and only if Y contains no subspace isomorphic to c_0 or l_1.*

Proof. If Y has no subspace isomorphic to c_0 or l_1, then from the Theorem it follows that every bounded subset of Y is relatively weakly sequentially compact. Hence Y is reflexive (cf. [11], 24.2(7)).

Remarks. 1. The author does not know whether the metrizability of Y is essential in the part (a) of the Theorem. Note that, by the extension of a famous result of Rosenthal proved in [13], if Y is a Fréchet space containing no subspace isomorphic to l_1, then every bounded subset of Y is weakly sequentially precompact. That is, in this case, we do not have to assume the existence of $X \supset Y$ with an unconditional basis.

2. The following two results answer some natural questions concerning spaces with unconditional bases:

(A) *Every locally convex space X with an equicontinuous (u.e.) basis (x_n) is isomorphic to a subspace of a product of Banach spaces each of which has an (unconditional) basis.*

(B) *If Y is a metrizable (normed) subspace of a locally convex space with a u.e. basis, then Y is isomorphic to a subspace of a Fréchet (Banach) space with an unconditional basis.*

Sketch of the proofs. (A) Let (S_n) be the sequence of partial sum operators associated with the basis (x_n). For each continuous seminorm p on X let

$$\hat{p}(x) = \sup_n p(S_n x);$$

then $\hat{p}$ is a continuous seminorm on X and $p \leqslant \hat{p}$. Furthermore, let $X_p = X/\hat{p}^{-1}(0)$ and let $Q_p\colon X \to X_p$ be the quotient map. Then the sequence obtained from $(Q_p x_n)_{n\in N}$ by removing the zero terms is a basis of the Banach space $\tilde{X}_p$ which is the completion of the normed space $(X_p, \bar{p})$, where $\bar{p}(Q_p x) = \hat{p}(x)$ for $x \in X$. Suppose now that P is a family of seminorms determining the topology of X; then also $\{\hat{p}\colon p \in P\}$ has this property. It is now easily seen that the map $x \to (Q_p x)_{p\in P}$ is a required embedding of X into a product of Banach spaces with bases. The other part of (A) has a similar proof.

(B) follows immediately from (A) and from the fact that a metrizable (normed) subspace of a product is "essentially contained" in a subproduct of countably (finitely) many factors (cf. [4], Proposition 3).

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