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ANOTHER GENERALIZATION OF THE DUGUNDJI EXTENSION THEOREM

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For any spaces X and Y, we let $C(X, Y)$ denote the set of continuous functions from X to Y. The most general version of a result of Dugundji on the extension of continuous functions is the following

THEOREM 1. *Let A be a closed subspace of a linearly stratifiable space X and let E be a locally convex linear topological space. Then there exists a linear function ψ: $C(A, E) \to C(X, E)$ such that*

(a) *$\psi(f)|A = f$ and $\psi(f)(X)$ is contained in the convex hull of $f(A)$;*

(b) *ψ is continuous with respect to the following topologies on both spaces: pointwise convergence topology, uniform convergence topology and compact-open topology;*

(c) *$\|\psi\| = 1$ if E is a normed linear space.*

Dugundji [4] proved parts (a) and (c) for metrizable spaces X. Michael [7] proved part (b) for metrizable spaces. Borges [2] generalized the preceding results to linearly stratifiable spaces X (linearly stratifiable spaces are discussed in [9]).

Expanding the terminology of Kulpa [6], given any collection $\mathscr{C}$ of pairs (X, A) of spaces, with A a closed subspace of X, we say that $\mathscr{C}$ satisfies the *Dugundji* (*Dugundji–Michael*) *Extension Theorem* if there exists a linear function ψ: $C(A, E) \to C(X, E)$ which satisfies conditions (a) and (c) (conditions (a), (b) and (c)) of Theorem 1 for each (X, A) in $\mathscr{C}$. In this language, Theorem 1 says that

$$\mathscr{C} = \{(X, A)|\ X \text{ is linearly stratifiable}\}$$

satisfies the Dugundji–Michael Extension Theorem.

In 1976, Kulpa [6] reported that Banilower [1] had proved that

$$\mathscr{G} = \{(X, A)|\ X \text{ is GO-space}\}$$

satisfies the Dugundji Extension Theorem (recall that GO-spaces are the subspaces of linearly ordered spaces), but it appears that [1] was never published. We therefore assume that this remains an open question. In [5] it is proved that the Dugundji–Michael Extension Theorem is indeed not valid for the Michael line (i.e., the real line with the irrationals discretefied), which is a GO-space. In [6], it is proved that the Dugundji Extension Theorem is valid for spaces X with countable C.A. systems; it is also proved that

$$\mathscr{C}_\omega = \{(X, A) \mid X \text{ has a well-ordered C.A. system and the boundary of } A \text{ is compact}\}$$

satisfies the Dugundji Extension Theorem.

Our purpose is to show that the added condition that the boundary of A be compact permits a full and wide-ranging generalization of Theorem 1.

Let us first recall that a space $(X, \mathscr{T})$ is said to be *submetrizable* if $\mathscr{T}$ contains a metrizable topology on X (equivalently, there exists a continuous bijection j: $(X, \mathscr{T}) \to (X, \varrho)$ with ϱ being a metric on X). Note that every paracompact space with a G_δ-diagonal is submetrizable (see Lemma 8.2 of [3]).

THEOREM 2. *$\mathscr{G}_m = \{(X, A) \mid X$ is submetrizable and the boundary of A is compact$\}$ satisfies the Dugundji–Michael Extension Theorem.*

Proof. First note that, letting bdry A denote the boundary of A, it suffices to produce a linear function

$$\gamma\colon C(\operatorname{bdry} A, E) \to C(X - A^\circ, E)$$

which satisfies conditions (a), (b) and (c) of Theorem 1, for then we can define the required ψ: $C(A, E) \to C(X, E)$ by letting

$$\psi(f)(x) = \begin{cases} \gamma(f \mid \operatorname{bdry} A)(x) & \text{if } x \in X - A^\circ, \\ f(x) & \text{if } x \in A^\circ. \end{cases}$$

(Of course, if bdry $A = \emptyset$, then the existence of ψ is trivial: Send $X - A$ to any fixed point in E for all $f \in C(A, E)$.) Therefore, without loss of generality, we assume that A is compact.

Let us now consider the diagram

$$\begin{tikzcd} (X,\tau) \arrow[r, "j"] & (X,\varrho) \arrow[d, "\tilde{f}"] \\ A \arrow[u, hook] \arrow[ur, "j'"] \arrow[r, "f"'] & E \end{tikzcd}$$

where ϱ is a metric on X, $j' = j \mid A$ is a homeomorphism, because A is (assumed) compact, and $\tilde{f} = \psi(f \circ (j')^{-1})$ is the extension of f according to Theorem 1. Finally, letting $\psi(f) = \tilde{f} \circ j$, we immediately infer that ψ satisfies all the requirements. This completes the proof.

Note that Theorem 2 applies to such spaces as the Niemitzky plane (which does not have a well-ordered C.A. system, since it is not even normal) and the Michael line which is a hereditarily paracompact GO-space (therefore, the condition that bdry A be compact is not superfluous even when X has rather strong topological properties).

THEOREM 3. *$\mathscr{C}_{\mathscr{N}} = \{(X, A) |$ X is collectionwise normal and* bdry A *is metrizable$\}$ satisfies the Dugundji–Michael Extension Theorem.*

Proof. Let $(X, A) \in \mathscr{C}_{\mathscr{N}}$ and let $\mathscr{T}$ be the topology on X. By the argument in the proof of Theorem 2, without loss of generality, we may assume that A is metrizable. Let d be a compatible metric for $(A, \mathscr{T}|A)$. By Theorem 5.2 of [8], there exists a pseudometric ϱ on X, whose topology is coarser than $\mathscr{T}$, such that $\varrho|A \times A = d$. Let j: $(X, \mathscr{T}) \to (X, \varrho)$ be the identity function and note that j is continuous.

Now, define μ: $C(A, E) \to C(X, E)$ by letting $\mu(f) = j\psi(f)$, such that ψ satisfies all the conditions of Theorem 1 for $((X, \varrho), (A, d))$. (The proof of Theorem 1 for metrizable spaces X works just as well for pseudometrizable spaces X.) It immediately follows that $\mu(f)$ satisfies all the conditions of Theorem 1 for $((X, \mathscr{T}), (A, \mathscr{T}|A))$. This completes the proof.

THEOREM 4. *$\mathscr{N}_{\varrho} = \{(X, A) |$ X is normal and* bdry A *is separable metrizable$\}$ satisfies the Dugundji–Michael Extension Theorem.*

The proof is similar to the proof of Theorem 3 because of Corollary 4.8 of [8].

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