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PRIMITIVE RECURSIVE NOTATIONS FOR INFINITARY FORMULAS

BY

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Kino and Takeuti defined [2] a constructive sublanguage (which we call $\mathrm{CL}_{\omega_1\omega_1}$) of $\mathrm{L}_{\omega_1\omega_1}$. The formulas of $\mathrm{CL}_{\omega_1\omega_1}$ are just those of $\mathrm{L}_{\omega_1\omega_1}$ which are assigned a numerical notation by a method closely related to that for assigning notations for ordinals; the corresponding constructive sublanguage $\mathrm{CL}_{\omega_1\omega}$ of $\mathrm{L}_{\omega_1\omega}$ was studied by Lopez-Escobar in [4]. Just, as with ordinal notations (cf. [3]), it is possible to restrict the functions used in constructing notations to be primitive recursive, and, as in the ordinal case, it turns out that essentially the same formulas of $\mathrm{L}_{\omega_1\omega_1}$ are assigned notations. The purpose of this note is to prove this remark, and to observe that the methods used here and in [3] easily extend to the class $\mathcal{E}^4$ of [1]. We assume familiarity with [2] and [3].

Let $N_{\omega_1\omega_1}$ be the set of notations generated by 2.1.1-2.4.2 of [2]. We will refer to the elements of $N_{\omega_1\omega_1}$ as (*general recursive*) *notations*. Let $N'_{\omega_1\omega_1}$ be the set of notations generated by the following rules:

($N'1$) Any number generated by 2.1.1-2.1.4 of [2] belongs to $N'_{\omega_1\omega_1}$.

($N'2$) If $z \in N'_{\omega_1\omega_1}$, then $2^7 \cdot 7^z \in N'_{\omega_1\omega_1}$.

($N'3$) If $\mathrm{In}(z)$ and, for each n, $\mathrm{pr}(z, \langle n \rangle) \in N'_{\omega_1\omega_1}$, then both $2^9 \cdot 7^z \in N'_{\omega_1\omega_1}$ and $2^{11} \cdot 7^z \in N'_{\omega_1\omega_1}$.

($N'4$) If $y \in N'_{\omega_1\omega_1}$ and if $\mathrm{In}(z)$, then both $2^{13} \cdot 7^z \cdot 11^y \in N'_{\omega_1\omega_1}$ and $2^{15} \cdot 7^z \cdot 11^y \in N'_{\omega_1\omega_1}$.

($N'5$) A number z is in $N'_{\omega_1\omega_1}$ if and only if its being so follows from finitely many applications of ($N'1$)-($N'4$).

We define maps $\| \ \|$ and $\| \ \|'$ on $N_{\omega_1\omega_1}$ and $N'_{\omega_1\omega_1}$, respectively, whose values are formulas of $\mathrm{L}_{\omega_1\omega_1}$ (cf. [4]). For the case of $\| \ \|'$, we proceed as follows:

(1) If, in the notation of [2], $z = \ulcorner \boldsymbol{a} = \boldsymbol{b} \urcorner$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are either individual variables or constants, then $\|z\|'$ is the formula $\boldsymbol{a} = \boldsymbol{b}$.

(2) If $z \in N'_{\omega_1\omega_1}$, then $\|2^7 \cdot 7^z\|'$ is the formula $\neg \|z\|'$.

(3) If $2^9\cdot 7^z \in N'_{\omega_1\omega_1}$ ($2^{11}\cdot 7^z \in N'_{\omega_1\omega_1}$) according to ($N'3$), then $\|2^9\cdot 7^z\|'$ ($\|2^{11}\cdot 7^z\|'$) is the formula $\bigvee_n \|\mathrm{pr}(z, \langle n\rangle)\|'$ ($\bigwedge_n \|\mathrm{pr}(z, \langle n\rangle)\|'$).

(4) If $y \in N'_{\omega_1<\omega_1}$ and $\mathrm{In}(z)$, and if $n_i = \mathrm{pr}(z, \langle i\rangle)$ for all i, then $\|2^{13}\cdot 7^z\cdot 11^y\|'$ is $\exists\langle v_{n_i}: i<\omega\rangle \|y\|'$ and $\|2^{15}\cdot 7^z\cdot 11^y\|'$ is $\forall\langle v_{n_i}: i<\omega\rangle\|y\|'$.

The map $\|\ \|$ on $N_{\omega_1\omega_1}$ is defined similarly.

We write $\mathrm{rk}(\mathfrak{A}) = \mathrm{n}(\mathfrak{A})$ and $\mathrm{qr}(\mathfrak{A}) = \mathrm{n}'(\mathfrak{A})$, where the functions n and n′ are defined as in [2]. Let us write $\Phi(z, x)$ for the general recursive function $U(\mu y T_1(y, x, z))$, let g be a general recursive Gödel number of the function $\lambda zcn\Phi(z, \mathrm{pr}(c, \langle n\rangle))$, and let h be a general recursive Gödel number of the identity function. Define a function ϱ as follows:

$$\varrho(z, c) = \begin{cases} c & \text{if } \exists k, l \in \{3, 5\}\, \exists i, j\ [c = 2^2\cdot 7^{k^{i+1}}\cdot 11^{l^{j+1}}], \\ 2^k\cdot 7^{\varrho(z,(c)_3)} & \text{if } c = 2^k\cdot 7^{(c)_3} \text{ and } k\in\{9, 11\}, \\ 2^k\cdot 7^{S_1^2(g,h,(c)_3)}\cdot 11^{\varrho(z,(c)_4)} & \text{if } c = 2^k\cdot 7^{(c)_3}\cdot 11^{(c)_4},\ k\in\{13, 15\}, \\ 0 & \text{otherwise}. \end{cases}$$

Since the S_n^m-function is primitive recursive, so is ϱ. Then, by Kleene's Recursion Theorem for primitive recursive functions (see [3], p. 75), there exists a number r which is an index of $\lambda c\varrho(r, c)$. Set $\varrho(c) = \varrho(r, c)$, so that $\varrho(c)$ is primitive recursive. Then the following theorem is easy to establish by induction on cases corresponding to ($N'1$)-($N'5$):

THEOREM I. *For each* $c \in N'_{\omega_1\omega_1}$, $\varrho(c) \in N_{\omega_1\omega_1}$, *and* $\|c\|'$ *is* $\|\varrho(c)\|$.

Thus, as one expected, $N_{\omega_1\omega_1}$ has notations for at least as many formulas as $N'_{\omega_1\omega_1}$ does. Next, let q be a Gödel number of the function $\lambda zci\Phi(z, \Phi(c, i))$, and set $s(z, c) = S_1^2(q, z, c)$, so that $s(z, c)$ is primitive recursive. Then define

$$\psi(z, c, 0) = \begin{cases} \ulcorner 0 = 1\urcorner & \text{if } (c)_0 = 9, \\ \ulcorner 0 = 0\urcorner & \text{otherwise}; \end{cases}$$

$$\psi(z, c, n+1) = \begin{cases} U\big((n)_{\mathrm{lh}(n)\dot{-}1} \dot{-} 1\big) & \text{if } \mathrm{Seq}(n) \,\&\, \mathrm{lh}(n) > 0\ \& \\ & \forall i_{i<\mathrm{lh}(n)} T_1\big(s(z, c), i, (n)_i \dot{-} 1\big), \\ \psi(z, c, n) & \text{otherwise}. \end{cases}$$

Then ψ is primitive recursive, say with an index e. If we set $d(z, c) = \mathrm{sb}_1^1(e, s(z, c))$ (cf. [3], p. 75), then $d(z, c)$ is primitive recursive. Now define $\eta(z, c, 0) = 0$ and $\eta(z, c, n+1)$ just like $\psi(z, c, n+1)$, so that η is also primitive recursive, say with an index t. Let ζ be a primitive recursive function such that if g is a Gödel number of $\lambda xF(x)$, then $\zeta(g)$ is a

Gödel number of $\lambda x\big(F(x)+1\big)$; also, let g be a Gödel number of the function $\lambda z f i \Phi\big(z, \Phi(f, i)\big)$. Now define β by

$$\beta(z, c) = \begin{cases} c & \text{if } c = 2^2\cdot 7^{3^{i+1}}\cdot 11^{3^{j+1}}, \\ 2^2\cdot 7^{3^{i+1}}\cdot 11^{5^{j+2}} & \text{if } c = 2^2\cdot 7^{3^{i+1}}\cdot 11^{5^{j+1}}, \\ 2^2\cdot 7^{5^{i+2}}\cdot 11^{3^{j+1}} & \text{if } c = 2^2\cdot 7^{5^{i+1}}\cdot 11^{3^{j+1}}, \\ 2^2\cdot 7^{5^{i+2}}\cdot 11^{5^{j+2}} & \text{if } c = 2^2\cdot 7^{5^{i+1}}\cdot 11^{5^{j+1}}, \\ 2^7\cdot 7^{\beta(z,(c)_3)} & \text{if } c = 2^7\cdot 7^{(c)_3}, \\ 2^k\cdot 7^{S_1^2(g,z,(c)_3)} & \text{if } c = 2^k\cdot 7^{(c)_3},\ k\in\{9, 11\}, \\ 2^k\cdot 7^{\zeta((c)_3)}\cdot 11^{\beta(z,(c)_4)} & \text{if } c = 2^k\cdot 7^{(c)_3}\cdot 11^{(c)_4},\ k\in\{13, 15\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then β is primitive recursive, and so, again by the Recursion Theorem for primitive recursive functions, there is a number b which is an index of $\lambda c\beta(b, c)$. If $\beta(c) = \beta(b, c)$, then β is primitive recursive. Now, for any formula $\mathfrak{A}$ of $L_{\omega_1\omega_1}$, let $\mathfrak{A}^*$ be the result of simultaneously raising the subscript on every variable occurring (free or bound) in $\mathfrak{A}$ by 1. Then, it is easy to verify, by induction, that if $c\in N_{\omega_1\omega_1}$, then $\|\beta(c)\| = \|c\|^*$. Note that the variable v_0 cannot occur in $\mathfrak{A}^*$.

Now define a primitive recursive function $\bar{\pi}$ as follows:

$$\bar{\pi}(z, c) = \begin{cases} c & \text{if } \exists k, l\in\{3, 5\}\exists i, j\ [c = 2^2\cdot 7^{k^{i+1}}\cdot 11^{l^{j+1}}], \\ 2^7\cdot 7^{\bar{\pi}(z,(c)_3)} & \text{if } c = 2^7\cdot 7^{(c)_3}, \\ 2^k\cdot 7^{d(z,c)} & \text{if } c = 2^k\cdot 7^{(c)_3},\ k\in\{9, 11\}, \\ 2^k\cdot 7^{\mathrm{sb}_1^1(t,(c)_3)}\cdot 11^{\bar{\pi}(z,(c)_4)} & \text{if } c = 2^k\cdot 7^{(c)_3}\cdot 11^{(c)_4},\ k\in\{13, 15\}, \\ 0 & \text{otherwise.} \end{cases}$$

We infer from the Primitive Recursive Recursion Theorem that there exists a number p which is an index for the function $\lambda c\bar{\pi}\big(p, \beta(c)\big)$; let $\pi(c) = \bar{\pi}\big(p, \beta(c)\big)$, so that π is primitive recursive.

THEOREM II. *For each* $c\in N_{\omega_1\omega_1}$, $\pi(c)\in N'_{\omega_1\omega_1}$ *and*

(i) $\mathrm{rk}\,\|\pi(c)\|' = \mathrm{rk}\,\|c\|$ *and* $\mathrm{qr}\,\|\pi(c)\|' = \mathrm{qr}\,\|c\|$;

(ii) $\|\pi(c)\|'$ *is valid in the natural numbers iff* $\|c\|^*$ *is.*

Proof. We argue by induction on the definition of $c\in N_{\omega_1\omega_1}$. The reasonings corresponding to cases ($N1$) and ($N2$) are straightforward. Suppose that $c = 2^9\cdot 7^z$, $\mathrm{In}(z)$, and $z_n = \Phi(z, n_O)\in N_{\omega_1\omega_1}$ for each n. Then $\pi(c) = 2^9\cdot 7^{d(p,\beta(c))}$, $\beta(c) = 2^9\cdot 7^{S_1^2(g,b,z)}$, and

$$d\big(p, \beta(c)\big) = \mathrm{sb}_1^1\big(e, s(p, 2^9\cdot 7^{S_1^2(g,b,z)})\big).$$

In a similar manner to the ψ in Section 11 of [3], we see that $d(p, \beta(c))$ is an index of the sequence obtained from the sequence $\pi(z_0), \pi(z_1), \ldots, \pi(z_n), \ldots$ by first prefixing a certain (finite) number of occurrences of $\ulcorner 0 = 1 \urcorner$, and then replacing each $\pi(z_n)$ by a certain (finite) number of instances of itself. Now, by induction, $\pi(z_n) \in N'_{\omega_1\omega_1}$, and so, clearly, $\pi(c) \in N'_{\omega_1\omega_1}$. From the remarks above, (i) and (ii) immediately follow. The case where $(c)_0 = 11$ is treated similarly.

Now suppose that $c = 2^{13} \cdot 7^z \cdot 11^y$, $y \in N_{\omega_1\omega_1}$, and In$(z)$. By induction, $\pi(y) \in N'_{\omega_1\omega_1}$, and we have

$$\pi(c) = 2^{13} \cdot 7^{\mathrm{sb}_1^1(t, (\beta(c))_3)} \cdot 11^{\pi(y)}.$$

Now $\beta(c) = 2^{13} \cdot 7^{\zeta(z)} \cdot 11^{\beta(y)}$, so

$$\pi(c) = 2^{13} \cdot 7^{\mathrm{sb}_1^1(t, \zeta(z))} \cdot 11^{\pi(y)}.$$

For each n, let $z_n = \Phi(z, n_O)$. Then $\zeta(z)$ is a Gödel number of the sequence $z_0+1, z_1+1, \ldots, z_n+1, \ldots$ Just, as above with ψ, since t is an index of η, we infer that $\mathrm{sb}_1^1(t, \zeta(z))$ is an index for the sequence obtained from the sequence $z_0+1, z_1+1, \ldots$ by first prefixing some (finite) number of occurrences of 0, and then replacing each occurrence of z_n+1 by some (finite) number of occurrences of itself. Thus we see that $\|c\|$ is $\exists v_{z_0} \ldots v_{z_n} \ldots \|y\|$, $\pi(c) \in N'_{\omega_1\omega_1}$ and $\|\pi(c)\|'$ is

$$\exists v_0 \ldots v_0 v_{z_1+1} \ldots v_{z_1+1} v_{z_2+1} \ldots v_{z_n+1} \ldots \|\pi(y)\|'.$$

Statements (i) and (ii) now quickly follow (for (ii), note that v_0 does not occur in $\|y\|^*$), completing the proof.

Obviously, if $\mathfrak{A}$ is a formula of $\mathrm{L}_{\omega_1\omega_1}$ with one free variable, then A and A^* define exactly the same set of natural numbers. From this and from Theorems I and II, it follows that if $\mathrm{CL}'_{\omega_1\omega_1}$ is the set of formulas having notations in $N'_{\omega_1\omega_1}$, then $\mathrm{CL}_{\omega_1\omega_1}$ and $\mathrm{CL}'_{\omega_1\omega_1}$ define exactly the same sets of natural numbers.

To see that the foregoing results can be extended to the class $\mathscr{E}^4$ of [1], we first note that it is easy to verify that $\mathscr{E}^4$ is the class of functions determined by schemata (I)-(IV) of [3] together with two schemata

$$(\mathrm{V_L}) \begin{cases} \varphi(0, a_2, \ldots, a_n) = \psi(a_2, \ldots, a_n), \\ \varphi(a_1, a_2, \ldots, a_n) = \chi_1(a_1, \varphi(a_1, a_2, \ldots, a_n), a_2, \ldots, a_n) \quad \langle 5, n, g, h_1, h_2 \rangle, \\ \varphi(a_1, a_2, \ldots, a_n) \leqslant \chi_2(a_1, a_2, \ldots, a_n), \end{cases}$$

$$(\mathrm{VI}) \quad \varphi(a_1, a_2) = f_4(a_1, a_2) \qquad \langle 6, 2, 4 \rangle,$$

where χ_1, χ_2 and ψ are previously introduced functions with indices h_1, h_2 and g, respectively, and f_4 is as defined on p. 28 of [1]. Then we can define an $\mathscr{E}^4$-predicate $\mathscr{E}^4$-In(e) analogous to the predicate In(e) of [3]. Moreover, since the function sb_n^m of [3] is, in fact, elementary and hence also in $\mathscr{E}^4$, the proof of the Primitive Recursive Recursion Theorem easily adapts to prove the following

$\mathcal{E}^4$-RECURSION THEOREM. *If* $\varphi(a_0, a_1, \ldots, a_n)$ *is in* $\mathcal{E}^4$, *there exists a number e such that e is an* $\mathcal{E}^4$*-index of the function* $\lambda a_1, \ldots, a_n \varphi(e, a_1, \ldots, a_n)$.

Let $e4(z, n)$ be that function, analogous to the function pr of [3], which enumerates the class $\mathcal{E}^4$. The system O_4 of ordinal notations and its accompanying order relation $<_4$ are defined by adapting the definition of O' and $<'$ of [3]. Namely, all occurrences of O' and $<'$ are replaced by O_4 and $<_4$, respectively, and all occurrences of $\mathrm{In}(e)$ and $\mathrm{pr}(z, n)$ are replaced by $\mathcal{E}^4$-$\mathrm{In}(e)$ and $e4(z, n)$, respectively. Using the fact that $\mathcal{E}^4$ is closed under the operation Ω (cf. [1], p. 34), it is easy to see that the functions n_O and ψ ([3], p. 75) are in $\mathcal{E}^4$ by schema $(\mathrm{V_L})$, so that $+_4$ can be defined by the using the $\mathcal{E}^4$-Recursion Theorem. The analogues of properties (I)-(XXIV) for this system are easy to prove with simple modifications of the original proofs. Now the functions $\mathrm{lh}(n)$, $(n)_i$, $\dot{-}$ and U are in $\mathcal{E}^4$ as are the predicates $\mathrm{Seq}(n)$, $<$ and T. Then, again using the closure of $\mathcal{E}^4$ under Ω and schema $(\mathrm{V_L})$, it follows that not only the function ψ defined on p. 76 of [3] is in $\mathcal{E}^4$, but also the obvious analogues φ_4 and π_4 of the functions φ and π, respectively, are in $\mathcal{E}^4$. Then, adapting the methods of [3], one easily shows that, for all $c \in O$, if $a \leqslant_O c$, then $\pi_4(a) \in O_4$ and $|\pi_4(a)|^4 = |a|$, where the definition of $|a|^4$ is analogous to that of $|a|$. Hence O_4 has notations for as many ordinals as O. The converse is established as easily as its analogue in [3], so that we see that the $\mathcal{E}^4$-ordinals coincide with the recursive ordinals. The adaptation to $\mathcal{E}^4$ of the results above on notations for infinitary formulas is carried out in a similar manner.

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