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CONVERGENCE ALMOST EVERYWHERE OF SEQUENCES OF MEASURABLE FUNCTIONS

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In papers [2] and [5] there are two theorems concerning the convergence almost everywhere, which can be formulated in the following way:

THEOREM (Taylor [2]). *If* $(X, \mathcal{S}, \mu)$ *is a* σ*-finite measure space and* $\{f_n\}_{n\in N}$ *is a sequence of* μ*-a.e. finite measurable real functions defined on* X *which converges* μ*-a.e. on* X *to a* μ*-a.e. finite measurable real function* f*, then there exist a monotone sequence* $\{\delta_n\}_{n\in N}$ *of positive numbers tending to zero and a set* $E \subset X$ *such that* $\mu(X-E) = 0$ *and*

$$\lim_{n\to\infty} \frac{|f_n(x)-f(x)|}{\delta_n} = 0 \tag{1}$$

for every $x \in E$.

THEOREM (Yoneda [5]). *If* $(X, \mathcal{S}, \mu)$ *is a* σ*-finite measure space and* $\{f_n\}_{n\in N}$ *is a sequence of* μ*-a.e. finite measurable real functions defined on* X *which converges* μ*-a.e. to a* μ*-a.e. finite measurable real function* f*, then there exists a non-negative* μ*-a.e. finite measurable real function* d *defined on* X *such that for every* $\varepsilon > 0$ *there exists an integer* $n(\varepsilon)$ *such that for every* $n > n(\varepsilon)$ *and for every* $x \in X$ *we have*

$$|f_n(x)-f(x)| < \varepsilon d(x). \tag{2}$$

In this paper we shall study the convergence almost everywhere in the more general setting. Suppose that $(X, \mathcal{S})$ is a measurable space and $\mathcal{I} \subset \mathcal{S}$ is a proper σ-ideal of sets. We say that some *property* W *holds* $\mathcal{I}$*-a.e. on* X if the set of all points in X, which do not have the property W, belongs to $\mathcal{I}$. Now we introduce the following definitions: (For brevity, we shall say "convergence in the sense of Egoroff, Taylor or Yoneda" omitting "with respect to $\mathcal{I}$".)

Definition 1. We say that a sequence $\{f_n\}_{n\in N}$ of $\mathcal{I}$-a.e. finite $\mathcal{S}$-measurable real functions defined on X *converges to an* $\mathcal{I}$*-a.e. finite* $\mathcal{S}$*-measurable real function* f *in the sense of Egoroff* if there exists a sequence

$\{E_m\}_{m\in N}$ of sets belonging to $\mathcal{S}$ such that $X-\bigcup_{m=1}^{\infty} E_m \in \mathcal{J}$ and for every $m \in N$ the sequence $\{f_n \mid E_m\}_{n\in N}$ converges uniformly to $f \mid E_m$.

Remark 1. Observe that in Definition 1 we can assume that the sequence $\{E_m\}_{m\in N}$ is increasing.

Definition 2. We say that a sequence $\{f_n\}_{n\in N}$ of $\mathcal{J}$-a.e. finite $\mathcal{S}$-measurable real functions defined on X *converges to an $\mathcal{J}$-a.e. finite $\mathcal{S}$-measurable real function in the sense of Taylor* if there exist a monotone sequence $\{\delta_n\}_{n\in N}$ of positive numbers tending to zero and a set $E \subset X$ such that $X-E \in \mathcal{J}$ and (1) holds for every $x \in E$.

Definition 3. We say that a sequence $\{f_n\}_{n\in N}$ of $\mathcal{J}$-a.e. finite $\mathcal{S}$-measurable real functions defined on X *converges to an $\mathcal{J}$-a.e. finite $\mathcal{S}$-measurable real function f in the sense of Yoneda* if there exists a non-negative $\mathcal{J}$-a.e. finite $\mathcal{S}$-measurable real function d defined on X such that for every $\varepsilon > 0$ there exists an integer $n(\varepsilon)$ such that for every $n > n(\varepsilon)$ and for every $x \in X$ inequality (2) holds.

We shall study the connections between convergence $\mathcal{J}$-a.e. and convergence in the sense of Egoroff, Taylor and Yoneda.

THEOREM 1. *The following statements are equivalent:*

(a) $\{f_n\}_{n\in N}$ *converges to f in the sense of Egoroff.*

(b) $\{f_n\}_{n\in N}$ *converges to f in the sense of Taylor.*

(c) $\{f_n\}_{n\in N}$ *converges to f in the sense of Yoneda.*

Proof. (a) $\Rightarrow$ (b) (see [2]). Let $\{E_m\}_{m\in N}$ be a sequence of $\mathcal{S}$-measurable sets such that $X-\bigcup_{m=1}^{\infty} E_m \in \mathcal{J}$ and for every m the convergence is uniform on E_m. For each fixed m let us choose an increasing sequence $\{n_{m,r}\}_{r\in N}$ of natural numbers such that

$$|f_n(x)-f(x)| < \frac{1}{r+1}$$

for every $x \in E_m$ and for every $n \geqslant n_{m,r}$. Let $\{n_t\}_{t\in N}$ be an increasing sequence of natural numbers such that

$$\lim_{t\to\infty}(n_t/n_{m,t}) = \infty$$

(such a sequence always exists). Put

$$\delta_n = \begin{cases} 1 & \text{for } 1 \leqslant n \leqslant n_1, \\ 1/\sqrt{t} & \text{for } n_{t-1}+1 \leqslant n \leqslant n_t \quad (t = 2, 3, \ldots). \end{cases}$$

If $x \in E_m$ for some $m \in N$, then there exists t_0 such that $n_t \geqslant n_{m,t}$ for $t \geqslant t_0$. If $n \geqslant n_{t_0}$, then there exists $t \geqslant t_0$ such that $n_t \leqslant n < n_{t+1}$. So for such n we have

$$|f_n(x)-f(x)| < \frac{1}{t+1} = \delta_n^2.$$

Hence (1) holds for every $x \in \bigcup_{m=1}^{\infty} E_m$.

(b) $\Rightarrow$ (a). Let $\{\delta_n\}_{n\in N}$ be a sequence of numbers and let $E \subset X$ be a set appearing in the convergence in the sense of Taylor. Put

$$E_m = \left\{x\colon \text{ for every } k \geqslant m, \ \frac{|f_k(x)-f(x)|}{\delta_k} \leqslant 1\right\}$$

for $m \in N$. Then we have $\bigcup_{m=1}^{\infty} E_m = E$ and it is not difficult to see that on every E_m the convergence is uniform.

(a) $\Rightarrow$ (c). The proof is essentially the same as in [5]. It suffices only to observe that if the sequence $\{f_n\}_{n\in N}$ converges to f in the sense of Egoroff, then the sequence $\{f_n^*\}_{n\in N}$, defined by

$$f_n^*(x) = \max\left(\frac{1}{n}, \sup_{k\geqslant n} |f_k(x)-f(x)|\right) \quad \text{for } n \in N,$$

converges to zero in the sense of Egoroff.

(c) $\Rightarrow$ (a). Let d be a function appearing in the convergence in the sense of Yoneda. Put $E_m = \{x\colon d(x) \leqslant m\}$. Then we have $X - \bigcup_{m=1}^{\infty} E_m \in \mathscr{I}$ and it is not difficult to verify that on every E_m the convergence is uniform.

Thus the theorem is proved.

From Theorem 1 it follows that for every pair $(\mathscr{S}, \mathscr{I})$, where $\mathscr{S}$ is a σ-field of subsets of some non-empty set X and $\mathscr{I} \subset \mathscr{S}$ is a σ-ideal of sets, all three types of convergence are equivalent. Obviously, each type of convergence implies convergence $\mathscr{I}$-a.e. Now we study conditions under which the inverse implication holds. We start with the definition (see [3]).

Definition 4. We say that a pair $(\mathscr{S}, \mathscr{I})$ *fulfills the condition* (E) if for every set $D \in \mathscr{S} - \mathscr{I}$ and for every double sequence $\{B_{j,n}\}_{j,n\in N}$ of sets belonging to $\mathscr{S}$ and satisfying the conditions

$$B_{j,n} \subset B_{j,n+1} \text{ for } j, n \in N, \quad \bigcup_{n=1}^{\infty} B_{j,n} = D \text{ for } j \in N$$

there exist an increasing sequence $\{j_p\}_{p\in N}$ of natural numbers and a sequence $\{n_p\}_{p\in N}$ of natural numbers such that

$$\bigcap_{p=1}^{\infty} B_{j_p,n_p} \notin \mathscr{I}.$$

Remark 2. In [3] it is proved that the condition (E) is equivalent to the possibility of the topologization of the convergence with respect to $\mathscr{I}$, which is a generalization of the convergence in measure.

THEOREM 2. *Suppose that the pair* $(\mathscr{S}, \mathscr{I})$ *fulfills the countable chain condition* (*that is, every pairwise disjoint family of sets belonging to* $\mathscr{S} - \mathscr{I}$ *is at most denumerable*). *Then the convergence* $\mathscr{I}$*-a.e. of every sequence* $\{f_n\}_{n\in N}$ *of* $\mathscr{I}$*-a.e. finite* $\mathscr{S}$*-measurable real functions to an* $\mathscr{I}$*-a.e. finite* $\mathscr{S}$*-measurable* *function f implies the convergence of $\{f_n\}_{n\in N}$ to f in the sense of Egoroff (or Taylor, or Yoneda) if and only if the pair $(\mathscr{S}, \mathscr{I})$ fulfills the condition* (E).

Proof. Suppose that the pair $(\mathscr{S}, \mathscr{I})$ fulfills the condition (E). Let $\{f_n\}_{n\in N}$ converge $\mathscr{I}$-a.e. to f. Put

$$E_{j,n} = \left\{x \in X\colon \text{ for every } k \geqslant n,\ |f_k(x) - f(x)| < \frac{1}{j}\right\}.$$

Then $E_{j,n} \subset E_{j,n+1}$ for every $j, n \in N$ and $\bigcup_{n=1}^{\infty} E_{j,n} = E_0 = X - A$ for every $j \in N$, where $A \in \mathscr{I}$ is a set on which $\{f_n\}_{n\in N}$ does not converge to f or some f_n or f is not finite.

From the assumption it follows that there exist an increasing sequence $\{j_p\}_{p\in N}$ of natural numbers and a sequence $\{n_p\}_{p\in N}$ of natural numbers such that

$$\bigcap_{p=1}^{\infty} E_{j_p,n_p} = E_1 \notin \mathscr{I}.$$

It is not difficult to verify that the sequence $\{f_n | E_1\}_{n\in N}$ converges uniformly to $f | E_1$.

Suppose that for ordinal numbers $\alpha < \eta$, where $\eta < \Omega$, we have found sets $E_\alpha \in \mathscr{S}$ such that $\{f_n | E_\alpha\}_{n\in N}$ converges uniformly to $f | E_\alpha$ for every $\alpha < \eta$. If

$$D_\eta = E_0 - \bigcup_{\alpha<\eta} E_\alpha \in \mathscr{I},$$

then $\{f_n\}_{n\in N}$ converges to f in the sense of Egoroff, which completes the proof. Suppose that $D_\eta \notin \mathscr{I}$. Obviously, $D_\eta \in \mathscr{S}$. Put

$$E^{(\eta)}_{j,n} = \left\{x \in D_\eta\colon \text{ for every } k \geqslant n,\ |f_k(x) - f(x)| < \frac{1}{j}\right\}.$$

Then $E^{(\eta)}_{j,n} \subset E^{(\eta)}_{j,n+1}$ for every $j, n \in N$ and $\bigcup_{n=1}^{\infty} E^{(\eta)}_{j,n} = D_\eta$ for every $j \in N$. Again by assumption we can find an increasing sequence $\{j^{(\eta)}_p\}_{p\in N}$ of natural numbers and a sequence $\{n^{(\eta)}_p\}_{p\in N}$ of natural numbers such that

$$\bigcap_{p=1}^{\infty} E^{(\eta)}_{j^{(\eta)}_p,n^{(\eta)}_p} = E_\eta \notin \mathscr{I}.$$

Then the sequence $\{f_n | E_\eta\}_{n\in N}$ converges uniformly to $f | E_\eta$. Since the pair $(\mathscr{S}, \mathscr{I})$ fulfills the countable chain condition, there exists an ordinal number $\beta < \Omega$ such that $E_0 - \bigcup_{\alpha<\beta} E_\alpha \in \mathscr{I}$, so $X - \bigcup_{\alpha<\beta} E_\alpha \in \mathscr{I}$ and $\{f_n\}_{n\in N}$ converges to f in the sense of Egoroff.

Suppose now that the pair $(\mathscr{S}, \mathscr{I})$ does not fulfill the condition (E). We shall construct a sequence $\{f_n\}_{n\in N}$ of finite $\mathscr{S}$-measurable real functions converging $\mathscr{I}$-a.e. to zero which is not convergent in the sense of Egoroff.

From the assumption it follows that there exist a set $D \in \mathcal{S} - \mathcal{J}$ and a double sequence $\{B_{j,n}\}_{j,n\in N}$ of sets belonging to $\mathcal{S}$ such that $B_{j,n} \subset B_{j,n+1}$ for $j, n \in N$, $\bigcup_{n=1}^{\infty} B_{j,n} = D$ for $j \in N$, and for every increasing sequence $\{j_p\}_{p\in N}$ of natural numbers and for every sequence $\{n_p\}_{p\in N}$ of natural numbers we have $\bigcap_{p=1}^{\infty} B_{j_p,n_p} \in \mathcal{J}$. Put $\hat{B}_{j,n} = \bigcap_{k=1}^{j} B_{k,n}$ for $j, n \in N$. It is not difficult to see that $\hat{B}_{j,n} \subset \hat{B}_{j,n+1}$ for $j, n \in N$, $\bigcup_{n=1}^{\infty} \hat{B}_{j,n} = D$ for $j \in N$, and $\hat{B}_{j+1,n} \subset \hat{B}_{j,n}$ for $j, n \in N$. Also $\hat{B}_{j,n} \subset B_{j,n}$, so $\bigcap_{p=1}^{\infty} \hat{B}_{j_p,n_p} \in \mathcal{J}$ for every $\{j_p\}_{p\in N}$ and $\{n_p\}_{p\in N}$.

For every $n \in N$ we put

$$f_n(x) = \begin{cases} 1/j & \text{for } x \in \hat{B}_{j,n} - \hat{B}_{j+1,n},\ j \in N, \\ 2 & \text{for } x \in D - \hat{B}_{1,n}, \\ 0 & \text{for } x \in X - D. \end{cases}$$

Then we have $\{x\colon |f_n(x)| \leqslant 1/j\} = \hat{B}_{j,n}$ and, consequently,

$$\left\{x\colon \text{for every } k \geqslant n,\ |f_k(x)| \leqslant \frac{1}{j}\right\} = \bigcap_{k=n}^{\infty} \hat{B}_{j,k} = \hat{B}_{j,n}.$$

It is not difficult to verify that for every $x \in X$

$$\lim_{n\to\infty} f_n(x) = 0.$$

Suppose now that $A \in \mathcal{S}$ is a set such that $\{f_n | A\}_{n\in N}$ converges uniformly to zero. Then it is easy to see that there exists a sequence $\{n_j\}_{j\in N}$ of natural numbers such that

$$A \subset \bigcap_{j=1}^{\infty} \hat{B}_{j,n_j}.$$

Hence $A \in \mathcal{J}$ and from the fact that $\mathcal{J}$ is a proper σ-ideal it follows that the sequence $\{f_n\}_{n\in N}$ does not converge to zero in the sense of Egoroff.

The proof is completed.

Remark 3. In [1] there is an example showing that convergence except on a set of the first category does not imply convergence in the sense of Egoroff with respect to the σ-ideal of sets of the first category, so the pair (sets having the Baire property, sets of the first category) does not fulfill the condition (E). Another proof of the last fact is given in [3].

Remark 4. The countable chain condition in Theorem 2 is essential. The short note [4] includes an example showing that a sequence of functions defined on an uncountable set convergent everywhere to zero need not converge to zero in the sense of Egoroff with respect to $\mathscr{I} = \{\emptyset\}$. So for the pair $(\mathscr{S}, \mathscr{I})$, where $\mathscr{S} = 2^X$, $\mathscr{I} = \{\emptyset\}$ and $\operatorname{card}(X) = \mathfrak{c}$, convergence $\mathscr{I}$-a.e. does not imply convergence in the sense of Egoroff, though this pair fulfills the condition (E).

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